THE TEACHING OF MATHEMATICS
2008, Vol. XI, 1, pp. 35–40

WHY IS IT NOT TRUE THAT $0.999\ldots < 1$?

Petr Eisenmann

Abstract. The contribution describes three basic obstacles preventing students from understanding the concept of infinite series in teaching of mathematics and provides means to their removal.

ZDM Subject Classification: D70; *AMS Subject Classification*: 01A35.

Key words and phrases: Infinite series, obstacle, limit of a sequence, potential and actual perception of the infinite limit process, the relation of phylogenesis and ontogenesis.

1. Introduction

Good teaching requires clear exposition of main facts, as well as all those which are trivial but could be lasting prejudices of students.

In mathematics lessons at secondary schools, a frequently discussed issue is whether it is true that

$$0.999\ldots < 1 \qquad \text{or} \qquad 0.999\ldots = 1.$$

From my experience as a college teacher, I know that the vast majority of freshmen choose the first alternative without any hesitation. Their justification is almost always the same: *If a decimal number begins with a zero, it cannot equal one but it is smaller than one.* Similarly, Mundy and Graham mention students' frequent statement that *The number* $0.999\ldots$ *equals approximately* 1*, gets closer and closer to* 1*, but it is not exactly* 1 [9]. The students think that *The difference between* $0.999\ldots$ *and* 1 *is infinitesimally small, but there is one* or *The number* $0.999\ldots$ *is the last number before* 1 (see studies [3] and [12]).

In the subsequent discussion with students (we will come back to it), it might be appropriate to say that $0.999\ldots$ can be understood as the infinite sum

$$0.999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots. \tag{1}$$

This way we come to infinite series. The key issue now is the question: Are students capable of accepting the thesis that the sum of an infinite number of positive real numbers is a real number? At this stage, the most frequent answer is: No.

Bero states that the task of calculating the sum of infinite series requires a coordinated usage of more concepts related to the infinity: the number of terms in the infinite series, the infinite process and the sum of infinite series [1]. These concepts are not differentiated in the student's mind and the usage of each of the concepts separately causes difficulties. The situation mentioned above, however, additionally requires using the links between them as well.

2. The obstacles

In my view, three basic obstacles (see the theory of the epistemological obstacles in [2]) can be specified concerning students' understanding of the concept of infinite series.

First, it is the attitude of students that an infinite series cannot be summed up. This view follows students' experience with finite sums:

It cannot be determined if it goes up to the infinity. It has no end, after all. (Ivan, a male grammar school student, 16 years).

One cannot sum the numbers up to the infinity as we always add something more to that, after all. (Marta, a female grammar school student, 17 years).

Second, it is a common idea of the majority of students that the sequence of partial sums of an infinite series of positive terms grows above all limits:

But when I add one more number, it further grows and it keeps on growing until the infinity. (Petr, a male grammar school student, 16 years).

From the point of the relation of phylogenesis and ontogenesis, this idea corresponds to Zenon's belief (about 490-430 BC) that the sum of an infinite number of line segments must be infinite.

It is possible to overcome this idea of students by specific geometrical procedures. For example, we can mention Oresme's imaginative method (Nicole Oresme, 1323?-1382) of cutting up a unit square (Fig. 1) for equality proof:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

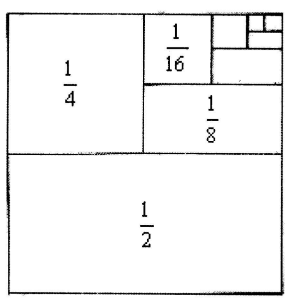

Fig. 1

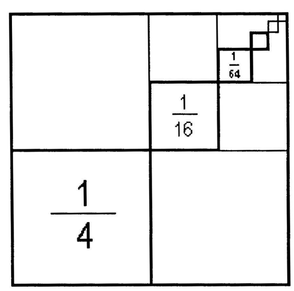

Fig. 2

The same can be shown with a line segment as well (Fig. 3).

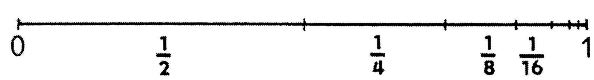

Fig. 3

Similarly, the illustrative Figure 2 can be used as a proof of the equality

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}.$$

After mastering the second obstacle, though, the third obstacle occurs (which is closely connected to the first one). For example regarding Figure 3, students often argue:

All the same, it never equals one, it only gets closer and closer to one, but it never gets there. (Marta, a female grammar school student, 17 years).

There is always a little bit missing to one—even if we keep on adding, there will always be a little bit missing. (Ondra, a male grammar school student, 18 years)

The above mentioned statements of students clearly show their potential understanding of the infinite process in the task (see also [10] or [8]). Students have not internalized the sum of a series as a limit of the sequence of its partial sums. However, even the majority of those who did this part in their lessons deal with non-standard tasks similarly. Hence, there is still a long way to go to get a deeper understanding of the limit and thus the infinite sum (see e.g. [11]).

How to deal with the third obstacle in lessons? Let us go back to the initial issue:

Is it true that $0.999\ldots < 1$ or $0.999\ldots = 1$?

An adequate method would be to explain to students the notion of the limit of a sequence and the sum of a series, and to give them the formula to calculate the sum of infinite geometrical series, by means of which to calculate the sum of series (1):

$$a_1 \cdot \frac{1}{1-q} = 0.9 \cdot \frac{1}{1-0.1} = 1. \tag{2}$$

However, it is advisable to present the above mentioned problem already before dealing with the corresponding parts. Besides, a great number of secondary schools never take the last step, namely to move from the geometrical sequence to infinite series. Even more significant drawback of this method is, however, that although students superficially master to give the sum of the geometrical series while using the formula (2), they rarely understand the heart of what they are doing and they cannot see the link to the statement $0.999\ldots = 1$.

Hence, teachers at secondary schools often adopt the 'equation' technique to justify the equality of $0.999\ldots = 1$:

$$\begin{array}{rrl} & x = 0.999\ldots & \quad \cdot 10 \\ - & 10x = 9.999\ldots & \\ \hline & 9x = 9 & \\ & x = 1 & \end{array}$$

Students are usually impressed by this smart solution. However, it involves a little bit of cheating as, in order to remove the infinite part of the decimal number, we multiply and subtract infinite decimal progressions term by term, without asking if we are justified in doing so.

At this point, it would be appropriate to use a simple task to show to the students that some cases of infinite sums cannot be dealt with by this mechanical method:

$$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\cdots = 1-1+1-1+1-1+\cdots \\ &= 1+(-1+1)+(-1+1)+(-1+1)+\cdots = 1. \end{aligned}$$

Given the students' actual perception of the infinite limit process, what I see as something yielding good results is the discussion stimulated by asking the supporters of the statement $0.999\ldots < 1$ the following question:

If $0.999\ldots < 1$*, then the difference* $1-0.999\ldots$ *must be a positive real number, the same way as it holds that for example*

$$0.9 < 1 \quad \textit{and} \quad 1-0.9 = 0.1 > 0.$$

What is then $1-0.999\ldots$ *equal to?*

In a very lively discussion students often reach the conclusion that the solution cannot be any number in the form of 0.0000000001, no matter how many 0's it might include (to be precise, arbitrarily, but finitely many). The reason is clear: the sum of such a number with the number $0.999\ldots$ is obviously a number greater than 1. Thus, the suggestions that might follow are: $0.000\ldots 1$ (explained by the statement *Infinitely many zeros and at the end* 1) or *Ten to the power of minus infinity* (which proves to be the same thing after the discussion). This discussion still concerns potential and actual perception of the infinite limit process and it is extremely valuable in terms of forming perceptions of the limit process.

The difficulty with $0.999\ldots = 1$ stands in contrast to student's understanding of the equation $0.333\ldots = \frac{1}{3}$ (see [4]). In a case study conducted with a real analysis in [5], a student stated that $0.333\ldots$ is equal to $\frac{1}{3}$ because one could divide 1 by 3 to get $0.333\ldots$. However, the student was adamant that the equation $0.999\ldots = 1$ is false, because *If you divide* 1 *by* 1*, you don't get* $0.999\ldots$*!* In the case of the equation $0.333\ldots = \frac{1}{3}$, the student might have been limited to see both $\frac{1}{3}$ and $0.333\ldots$ as processes. In the case of $0.999\ldots = 1$, the student may see $0.999\ldots$ exclusively as a process and $\frac{1}{1}$ as a process that does not result in $0.999\ldots$.

An adequate interpretation of Zenon's paradox of Achilles and Tortoise offers a strong argument for the supporters of the statement that $0.999\ldots = 1$. Let us use this paradox in our argument for the case that $0.999\ldots = 1$.

Achilles and Tortoise compete in a 100 m *race. Since Achilles runs ten times faster than Tortoise, Tortoise gets the initial advantage of* 90 m*. Who will win? Not Achilles, as everybody would think, but Tortoise. When Achilles reaches the starting point of Tortoise, Tortoise is* 9 m *ahead of Achilles. Before Achilles reaches that point, Tortoise is* 9 dm *ahead of him. And so on—Achilles will never catch up with Tortoise.*

For our purposes, let us adapt the parameters of the race: in the moment of start, Achilles and Tortoise will be separated by the distance $d = 90\,\text{m}$. Achilles will run at speed $v_A = 10\,\text{m/s}$ and Tortoise at $v_T = 1\,\text{m/s}$. In what time and in what distance from the start will Achilles reach Tortoise? Let us mark this point by C in Figure 4. Let us mark the distance $|AC|$ by x.

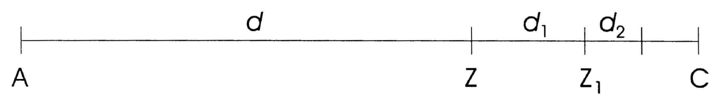

Fig. 4

How can we calculate the distance x? Achilles needs time t_1 to reach Tortoise's starting point Z. This is easy to calculate:

$$d = v_A \cdot t_1, \qquad 90 = 10 \cdot t_1, \qquad t_1 = 9\,\text{s}.$$

In this time, Tortoise covered the distance

$$d_1 = v_T \cdot t_1 = 1 \cdot 9 = 9\,\text{m}.$$

To reach this point (point Z_1), Achilles needs time $t_2 = 0.9\,\text{s}$ since it is true that

$$d_1 = v_A \cdot t_2, \qquad 9 = 10 \cdot t_2.$$

In this time, Tortoise covers the distance $d_2 = 0.9\,\text{m}$. We could proceed this way on and on. Therefore, we can represent the distance x as the infinite sum

$$x = 90 + 9 + 0.9 + \cdots,$$

thus $x = 99.99\ldots$ m. However, we can calculate this distance in a different way, as well. Achilles and Tortoise will reach the point C at the same time t. Comparing the track of Achilles with the track of Tortoise, we get

$$v_A \cdot t = v_T \cdot t + 90, \qquad 10 \cdot t = 1 \cdot t + 90, \qquad t = 10\,\text{s}.$$

So, $x = v_A \cdot t = 100\,\text{m}$. Hence, it is true that $99.99\ldots = 100$, or, equivalently, $0.999\ldots = 1$.

3. Conclusion

In conclusion, I would like to express my firm belief that a suitable procedure for teaching the discussed parts is the following sequence of steps: Motivation by presenting the problem of the sum of an infinite series (e.g., $0.999\ldots = 1$) $\to$ the limit of a sequence $\to$ the sum of the series. For giving a definition of the sequence limit I would be in favour for its gradual definition through the simpler case of a monotonous (e.g., decreasing) sequence limit—the definition of this limit does include the key idea of dependency with the addition of only two quantifiers.

An objection might be raised against the above mentioned traditional procedure (with the exception of the initial motivation step) as to the fact that this procedure does not respect the principle of congruence between the phylogenesis and ontogenesis. The sequence limit is viewed here as a simpler basic concept, the mastering of which is necessary for the understanding of the sum of an infinite series. Nevertheless, Archimedes, for example, summed up infinite series without having (and needing) the notion of limit. To be more precise, he did not have the definition of limit at his disposal—the limit process itself existed in similar lines of thoughts already before Archimedes (e.g., in Antifonos and his calculation of the area of a circle by means of gradual filling the circle with polygons (5th century BC), or in Eudoxus and his exhaustion method (4th century BC)—for details see, e.g., [6]). In their lessons, however, students do not encounter a sufficient number of models of the limit process before they go into the infinite series, and therefore they lack something to follow. Moreover, the principle of genetic parallel is not a universal principle in mathematics teaching (see, e.g., [8]), which is also a fact documented by, e.g., [7, p. 73] on the traditional procedure of Differential calculus → Integral calculus.

REFERENCES

1. Bero, P. (1985), *Fundamental concepts of differential calculus from the teaching aspect*, Dissertation, MFF UK, Bratislava.
2. Brousseau, G. (1997), *Theory of didactical situations in mathematics*, Eds. Balacheff, N. et al., Dordrecht/Boston/London, Kluwer Academic Publishers.
3. Cornu, B. (1991), *Limits*, in: D. O. Tall (Ed.), *Advanced mathematical thinking* (153–166), Dordrecht, Kluwer.
4. Dubinsky, Ed., Weller, K., McDonald, M. A., Brown, A. (2005), *Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis*, Part 2, Educational Studies in Mathematics, **60**, 253–266.
5. Edwards, B. (1997), *An undergraduate student's understanding and use of mathematical definitions in real analysis*, in: Dossey, J., Swafford, J.O., Parmentier, M., Dossey, A.E. (Eds.), *Proceedings of the 19th Annual Meeting of the North American Chapter of the In-ternational Group for the Psychology of Mathematics Education*, Columbus, OH, 17–22.
6. Hischer, H., Scheid, H. (1995), *Grundbegriffe der Analysis*, Heidelberg, Spektrum Akademischer Verlag.
7. Knoche, N., Wippermann, H. (1986), *Vorlesungen zur Methodik and Didaktik der Analysis*, Zürich, BI Wissenschaftsverlag.
8. Marx, A. (2006), *Schlervorstellungen zu "unendlichen Prozessen"*, Berlin, Verlag Franzbecker.
9. Mundy, F. J., Graham, K. (1994), *Research in calculus learning: Understanding of limits, derivatives and integrals*, MAA Notes **33**, 31–45.
10. Richman, F. (1999), *Is* $.999\ldots = 1$*?*, Mathematics Magazine, **72**(5), 396–400.
11. Tall, D. (1976), *Conflicts and catastrophes in the learning of mathematics*, Math. Education for Teaching, **2**, 2–18.
12. Tall, D., Schwarzenberger, R.L.E. (1978), *Conflicts in the learning of real numbers and limits*, Mathematics Teaching, **82**, 44–49.

Faculty of Science, Jan Evangelista Purkyně University, Ceske mladeze 8, 400 96 Usti nad Labem, Czech Republik

E-mail: eisenmannp@sci.ujep.cz