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THE FIBONACCI SEQUENCE AND THE GOLDEN QUADRATIC

Abstract. The fact that the golden mean ($\Phi = 1.61803\ldots$) appears both as the limit of the ratio of consecutive Fibonacci numbers, as well as one of the solutions of the golden quadratic, prompted us to conduct a graphical analysis of this equation in order to ascertain what kind of connection its geometry has with the Fibonacci sequence. Our results indicate that the following are all subsumed by the geometry of this equation: the Fibonacci sequence, a sequence of powers of Φ, Division in Extreme and Mean Ratio, Φ, as well as the golden rectangle.

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Analysis

That a connection exists between the golden quadratic and the Fibonacci sequence can be seen in the fact that the golden mean appears as one of the solutions of the former, and as the limit of the sequence of ratios of consecutive terms of the latter. In order to investigate if any connection could be established between the geometry of this equation and the Fibonacci sequence, we started by recognizing that when expressed as $x^2 - x - 1 = 0$, the golden quadratic can be read as the first step in the solution process of system $y = x^2 - x - 1$, $y = 0$. The fact that this is the simplest of all the possible systems into which this equation can be re-expressed [1], made it our first choice to guide our inquiry. Its graph is shown in Figure 1.

Its solution is represented there by the intersections of the parabola and the x-axis: $A(1.6180, 0)$ and $B(-0.6180, 0)$. The parabolas y-intercept is given by point $C(0, -1)$. The coordinates of the remaining points shown in Figure 1 are as follows: $D(1/2, -5/4)$, $E(1, -1)$, $F(1.618, -1)$, and $G(-0.618, -1)$.

By looking at these intersections, it is easy to realize that segments OC and OB are the golden sections of line OA. It can also be seen that segments OA and OC correspond to the sides of a golden rectangle with an area of 1.618, while segments OB and OC correspond to the sides of a smaller golden rectangle with an area of 0.618. These golden rectangles are made apparent by drawing segments AF, BG, and FG. These considerations prove here, as done in our earlier work for system $y = 1/(x - 1)$, $y = x$, that the geometry of the golden quadratic subsumes its alternative interpretations [1]. As Figure 1 indicates, the intersections of these golden rectangles with the parabola bring forward a number of geometric figures. It was the geometric richness made apparent by this construction that suggested
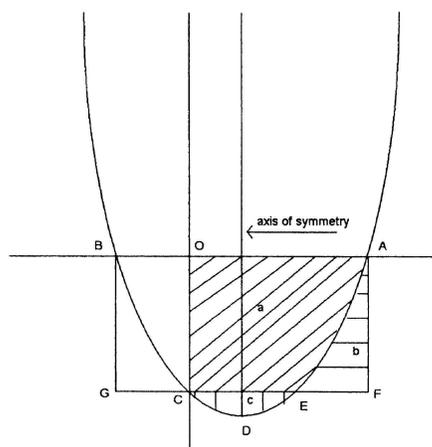
Figure 1. The graph of the golden quadratic in the form of system and is shown here. In it, intersections A, B, and C bring forward, via auxiliary lines AF, BG, and FG, golden rectangles $OAFC$ and $OCGB$. The intersections of the former of these rectangles with the parabola define, among others, the three figures which have been represented as 'a', 'b', and 'c'. The question motivating this work is to be answered through the evaluation and comparison of the areas of these three figures.

the connection between the geometry of the golden quadratic and the Fibonacci sequence.

Our first intuitive effort, which proved to be a fruitful one, consisted of the evaluation and further comparison of the areas of the figures defined by the intersection of the parabola and golden rectangle $OAFC$; that is, of regions a, b, and c. These areas, respectively designated as S_a, S_b, and S_c, were evaluated by integration [2], and the results obtained shown in Table 1.

Region	Area integral	Result
a	$\int_0^\Phi [0 - (x^2 - x - 1)]\, dx - \text{Area}\, c$	$S_a = 5\Phi/6$
b	$\int_1^\Phi [x^2 - x - 1 - (-1)]\, dx$	$S_b = \Phi/6$
c	$\int_0^1 [-1 - (x^2 - x - 1)]\, dx$	$S_c = 1/6$

Table 1. The area integrals and their respective evaluations are here shown for the highlighted geometrical figures appearing in Figure 1.

Out of these three numbers, the one corresponding to region a was taken as the basis of comparison. A number of ratios were then taken and the results obtained shown in Table 2, which is the center of the following discussion.

Discussion

As previously stated, our original goal was to unveil a connection between the geometry of the golden quadratic and the Fibonacci sequence. Once Figure 1 was

drawn and the areas of regions a, b, and c were evaluated, our goal narrowed to that of exploring the possibility of extracting the Fibonacci sequence from some kind of interplay between these areas. We decided to take the largest area S_a, and keep it unchanged in the numerator of a series of ratios r_n whose denominators were to be determined through a trial and error procedure via the combination of areas S_b and S_c, in a way such as to make every ratio approximate a Fibonacci number. The original criterion guiding this effort consisted in comparing the number coming out of a given ratio via substitution of Φ by 1.618034, with the Fibonacci numbers. The ratio selected was the one producing the closest match with a given Fibonacci number. Inspection of the area values of Table 1, allowed us to realize from the very beginning that number 5 could be precisely reproduced by the quotient

$$\frac{S_a}{S_b + 0S_c} = \frac{5\Phi/6}{\Phi/6} = 5.$$

Both areas S_b and S_c are shown in the previous quotient to stress the fact that it was the addition or subtraction of multiples of these areas, what was to allow for the production of lower or higher valued ratios. With the previous quotient in sight, it seemed only natural to try the following ratios: $S_a/(0S_b + S_c)$, and $S_a/(S_b + S_c)$. Upon the indicated substitution of Φ for 1.618034, the former reduced to 8.1, while the latter to 3.1. The fact that the denominators of these two fractions were logical extensions of the denominator of the ratio precisely reproducing number 5, combined with the close match between these values and Fibonacci numbers 8 and 3, convinced us that these were the ratios sought, even if no other candidates were brought forward.

Our next move was directed to find the ratio reproducing number 2. Continuing with the procedure started before consisting in methodically increasing the coefficients of areas S_b and S_c, we decided to try the following quotients: $S_a/(2S_b + S_c)$, $S_a/(S_b + 2S_c)$, and $S_a/(2S_b + 2S_c)$. In the order given, the ratios produced the following numbers: 1.9, 2.2, and 1.5, According to the 'close-match' criterion previously mentioned, we decided in favor of the first of these ratios. Having produced 'close-match' ratios for Fibonacci numbers 2, 3, 5, and 8 we decided to make a pause in the calculations in order to take a close look at the selected ratios. Inspection of the denominators of these ratios allowed us to realize that the coefficients of the area terms there appearing were Fibonacci numbers themselves. Some further trial and error was needed in order for us to become aware of the connection existing between the coefficients of neighboring ratios. This connection, to be described below, allowed us in turn to write the corresponding ratios for numbers 1, 1, 13, 21, 34 and 55. All of these ratios, either in terms of S_a, S_b, and S_c, or in the corresponding re-expressions in terms of Φ (to be referred from now on as the 'Φ version' of the area ratio expression), are shown in column 3 of Table 2. The approximate Fibonacci number to which each of those ratios reduces upon the indicated substitution is given in column 4. The percentage difference between each of those numbers and their respective Fibonacci counterparts is given in column 5.

A simple down-the-table inspection of the coefficients of the two terms constituting these ratio's denominators makes it evident that they reproduce, in a

crisscross pattern, the elements of two Fibonacci sequences unfolding backwards into the domain of negative Fibonacci numbers. Looking at the coefficients of the S_b terms, as they transit from $n = 1$ to $n = 10$, we find that they produce the sequence 5, 3, 2, 1, 1, 0, 1, -1, 2, and -3. The coefficients of the S_c terms, along the same transit, produce on their part, the following sequence: 3, 2, 1, 1, 0, 1, -1, 2, -3, 5.

n	f_n	r_n	$r_n(\Phi)$	$\%$	Φ'_n	$r_n(\Phi'_n)$	R_n
1	1	$\dfrac{S_a}{5S_b + 3S_c} = \dfrac{5\Phi}{5\Phi + 3} = 5\Phi^{-4}$	0.7	27			
2	1	$\dfrac{S_a}{2S_c + 3S_b} = \dfrac{5\Phi}{2 + 3\Phi} = 5\Phi^{-3}$	1.2	20	1	1	$\dfrac{5\Phi + 3}{2 + 3\Phi} = \Phi$
3	2	$\dfrac{S_a}{2S_b + S_c} = \dfrac{5\Phi}{2\Phi + 1} = 5\Phi^{-2}$	1.9	5	2	2	$\dfrac{2 + 3\Phi}{2\Phi + 1} = \Phi$
4	3	$\dfrac{S_a}{S_c + S_b} = \dfrac{5\Phi}{1 + \Phi} = 5\Phi^{-1}$	3.1	3	3/2	3	$\dfrac{2\Phi + 1}{1 + \Phi} = \Phi$
5	5	$\dfrac{S_a}{S_b + 0S_c} = \dfrac{5\Phi}{\Phi + 0} = 5\Phi^{0}$	5	0	5/3	5	$\dfrac{1 + \Phi}{\Phi + 0} = \Phi$
6	8	$\dfrac{S_a}{S_c + 0S_b} = \dfrac{5\Phi}{1 + 0\Phi} = 5\Phi^{1}$	8.1	1	8/5	8	$\dfrac{\Phi + 0}{1 + 0\Phi} = \Phi$
7	13	$\dfrac{S_a}{S_b - S_c} = \dfrac{5\Phi}{\Phi - 1} = 5\Phi^{2}$	13.1	1	13/8	13	$\dfrac{1 + 0\Phi}{\Phi - 1} = \Phi$
8	21	$\dfrac{S_a}{2S_c - S_b} = \dfrac{5\Phi}{2 - \Phi} = 5\Phi^{3}$	21.2	1	21/13	21	$\dfrac{\Phi - 1}{2 - \Phi} = \Phi$
9	34	$\dfrac{S_a}{2S_b - 3S_c} = \dfrac{5\Phi}{2\Phi - 3} = 5\Phi^{4}$	34.3	1	34/21	34	$\dfrac{2 - \Phi}{2\Phi - 3} = \Phi$
10	55	$\dfrac{S_a}{5S_c - 3S_b} = \dfrac{5\Phi}{5 - 3\Phi} = 5\Phi^{5}$	55.5	1	55/34	55	$\dfrac{2\Phi - 3}{5 - 3\Phi} = \Phi$

Table 2. Ten members of the Fibonacci sequence are given in column 2 in reference to the position index 'n' shown in column 1. A number of ratios r_n in terms of the magnitudes of area regions 'a', 'b' and 'c', alongside their respective re-expression in terms of Φ, are shown in column 3. As described in the text, these ratios were used in a series of arguments producing the results shown in the remaining columns.

The fact that the 'Φ version' of these denominators could be reduced to the form $f_{6-n}\Phi + f_{5-n}$, $n \geqslant 1$ allowed us to arrive at two interesting results. The first is embodied in the following two equivalent expressions for r_n:

$$(1) \qquad r_n = \frac{S_a}{f_{6-n}S_b + f_{5-n}S_c}, \quad n \geqslant 1,$$

$$(2) \qquad r_n = \frac{5\Phi}{f_{6-n}\Phi + f_{5-n}}, \quad n \geqslant 1.$$

The second consisted in recognizing that all of the said denominators, being constituted by the sum of an integer and an integer multiple of the golden mean, could be

replaced by a power of Φ [2]. This property of the golden mean, in the form it takes for the case under consideration, has been algebraically represented in equation

$$(3) \qquad \Phi^{6-n} = f_{6-n}\Phi + f_{5-n}.$$

The indicated replacement of the said denominators by the corresponding power of Φ dictated by equation (3), followed by simplification, transformed each of the 'Φ version' area ratio expressions into a constant multiple of Φ^{n-5}, $n \geqslant 1$, as shown by the rightmost term of column 3. A down-the-table inspection of these terms shows their conformance to a sequence of powers of Φ. This leads to the following version of r_n, equivalent to those given by equations (1) and (2):

$$(4) \qquad r_n = 5\Phi^{n-5}, \qquad n \geqslant 1.$$

These interesting results prompted us to keep tinkering with the area ratio expressions. When the ratio $R_n = r_n/r_{n-1}$, $n \geqslant 2$ was formed using the 'Φ version' of these expressions, we witnessed the crisscross replication of the Fibonacci sequence extending to the numerators of the resulting expressions. Further application of equation (3) to these ratios reduced all of them to Φ. This result is in agreement with that obtained when the ratio r_n/r_{n-1}, $n \geqslant 2$, is formed using the corresponding elements of the power sequence described above. R_n is shown in the eighth column of Table 2.

Even if the results shown in column 4 could have been taken as evidence for a positive response to the question motivating this work, we considered the question of what the requirement was for any given r_n to precisely produce its corresponding Fibonacci number f_n.

What we did was to equate each of the 'Φ version' ratios with the Fibonacci number they were expected to produce, and solve for Φ. In doing this we found that in order to precisely reproduce the expected Fibonacci number, each ratio r_n required the replacement of Φ by its approximation Φ'_n, quantified by the following quotient:

$$(5) \qquad \Phi'_n = \frac{f_n}{f_{n-1}}, \qquad n \geqslant 2.$$

We decided to take a deeper look at this matter in order to explore the possibility of generalizing this finding via a proof to the effect that:

$$(6) \qquad \text{if} \quad \Phi = \Phi'_n \quad \text{then} \quad r_n(\Phi'_n) = f_n, \qquad n \geqslant 2.$$

What we did was to substitute Φ by f_n/f_{n-1} in equation (2). This equation was selected as the starting point of this argument because in it r_n is given explicitly as a function of Fibonacci numbers. The indicated substitution, followed by simplification, led us to:

$$(7) \qquad r_n(\Phi'_n) = \frac{5f_n}{f_{6-n}f_n + f_{5-n}f_{n-1}}.$$

The fact made evident by this expression, that if $f_{6-n}f_n + f_{5-n}f_{n-1} = 5$ is true, $r_n(\Phi'_n) = f_n$ is also true, opens a way to rephrase the tentative proof statement previously advanced in (6), as follows:

(8)   if $f_{6-n}f_n + f_{5-n}f_{n-1} = 5$ then $r_n(\Phi'_n) = f_n$, $n \geqslant 2$.

The validity of statement (6) will be established by proving that

$$(9) \qquad\qquad f_{6-n}f_n + f_{5-n}f_{n-1} = 5, \quad \forall n.$$

Notice that the proof of statement (9) will be attempted for all integers n.

The direct proof offered below makes use of the following identities concerning the Fibonacci sequence, all of them true for all integers n [3].

$$(10) \qquad\qquad f_{n+1} = f_n + f_{n-1},$$

$$(11) \qquad\qquad f_{-n} = (-1)^{n+1} f_n,$$

$$(12) \qquad\qquad f_{1-n} = (-1)^n f_{n-1}.$$

$$(13) \qquad f_n^2 - f_{n+1}f_{n-1} = (-1)^{n+1},$$

$$(14) \qquad\qquad f_{1-n} = (-1)^n (f_{n+1} - f_n).$$

With these preliminaries in place, the proof starts with the repeated application of formula (10) to f_{6-n} and f_{5-n} in order to produce their respective f_{1-n} and f_{-n} versions, as follows:

$$(15) \qquad f_{6-n} = 8f_{1-n} + 5f_{-n}, \qquad f_{5-n} = 5f_{1-n} + 3f_{-n}.$$

Substitution of relations (15), and using formulas (11), (12) and (14) leads to:

$$f_{6-n}f_n + f_{5-n}f_{n-1} = 5f_nf_{-n} + 5f_{n-1}f_{1-n} + 8f_nf_{1-n} + 3f_{-n}f_{n-1}$$
$$= 5(-1)^{n+1}f_n^2 + 5(-1)^n(f_{n+1} - f_n)f_{n-1} + 8(-1)^n f_nf_{n-1} + 3(-1)^{n+1}f_nf_{n-1}$$
$$= 5(-1)^{n+1}(f_n^2 - f_{n+1}f_{n-1}).$$

Upon substitution of equation (13), the previous expression finally reduces to (9). Being this so, then

$$(16) \qquad\qquad r_n(\Phi'_n) = f_n, \qquad n \geqslant 2 \quad \blacksquare$$

The proof's conclusion shown in (16), a confirmation of our suspicion that the Fibonacci sequence could be obtained through the manipulation of the area ratio expressions derived from the geometry of the golden quadratic represents the perfect colophon to our efforts. The process of reaching this conclusion gave us, again, the opportunity to witness first hand the multilayered concatenation of order inherent to this field. Here we transited from a quasi-Fibonacci sequence obtained replacing Φ by 1.618034 in the area ratio expressions, to a perfect replication in the denominators of these ratios. After this we witnessed the unfolding of a sequence of powers of Φ, and then a perfect replication of the said sequence in both the numerators and denominators of ratios R_n. This was followed by their eventual

reduction to Φ, and from here, finally, to the confirmation of the declared goal of this effort, represented by (16).

The remaining problems

It should be mentioned here that it is reasonable to expect similar results when the geometry of algebraically equivalent re-expressions of the golden quadratic are analyzed this way. These yet to be explored systems, written below, can be used by those teachers/tutors interested in exposing their freshman or sophomore calculus students to the research experience.

1. $y = x^2 - x$, $y = 1$.
2. $y = x^2$, $y = x + 1$.
3. $y = x^2 - 1$, $y = x$.
4. $y = x - 1$, $y = 1/x$.
5. $y = x$, $y = 1/(x - 1)$.

REFERENCES AND NOTES

1. José C. Íñiguez et al., Nexus Network Journal **8**, 2 (2006), 54–61.

2. The procedure used in the reduction of the area integrals of Table 1 is the same used in the reduction of ratios r_n and R_n to powers of Φ. It makes use of the property of the golden mean described in the text in reference to equation (3). The alternative form $\Phi^n = f_n\Phi + f_{n-1}$ facilitates the integral's reduction. An interesting discussion on the subject can be found at: `http://www.friesian.com/golden.htm`

3. S. Rabinowitz, *Algorithmic Manipulation of Second Order Linear Recurrences*, The Fibonacci Quarterly, **37** (1999), 162–177.

Identities I and II written below are simple re-expressions of Rabinowitz' Theorem 2 (the addition formula for w), and Theorem 10 (D'Ocagne's identity for w), both true for all integers m and n.

$$\text{I.} \quad f_{m+n} = f_{m+1}f_n + f_m f_{n-1},$$
$$\text{II.} \quad f_{m-n} = -(-1)^n f_{m+1}f_n + (-1)^n f_m f_{n+1}.$$

Our equations (10)–(14) were obtained from I or II via the following substitutions:

a) Identity I becomes equation (10) for $m = 1$.

b) Identity II becomes the first equation (11) for $m = 0$. Equation (12) is in turn obtained from (11) via an index change.

c) Identity II becomes equation (13) for $n = m-1$. Here use is made of the fact that $(-1)^{1-m} = (-1)^{m+1}$.

d) Identity II becomes equation (14) for $m = 1$.

Upon their re-expression for particular values of m, identities I and II become true for all integers n.

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