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Sanjay K. Khattri

FOURIER SERIES AND LAPLACE TRANSFORM THROUGH TABULAR INTEGRATION

Abstract. This article is intended for first year undergraduate students. In this work, we explore the technique of tabular integration, and apply it for evaluating Fourier series and Laplace transform.

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Introduction

Integration by parts and tabular integration are used to integrate product of two functions. The technique of tabular integration is very well known [1, 2]. But still it has not find its way into textbooks [3]. Traditionally we teach integration of product of two functions by technique of integration by parts. It is our experience that students take more time to learn and successfully apply this technique. Especially when we need to repeatedly apply the technique of integration by parts. We find that students make mistakes. On the other hand, we find that students easily apply the tabular integration technique even in complex situations. Classically the technique of integration by parts is given as follows:

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

In the above equation, it is preferred to choose as u a function which is easy to differentiate, and whose derivative may vanish. Such as polynomial functions. In many situations, we may need to repeatedly employ the integration by parts until the differentiation of u vanishes. Let us implement the technique of integration by parts for evaluating the integral

$$I = \int x^2 \sin x\,dx. \tag{2}$$

Since the third derivative of x^2 is null, it is appropriate to pick x^2 as u. Thus,

$$\begin{aligned} u &= x^2 \quad &\text{and}\quad & dv = \sin x\,dx,\\ du &= 2x\,dx \quad &\text{and}\quad & \int dv = \int \sin x\,dx,\\ du &= 2x\,dx \quad &\text{and}\quad & v = -\cos x. \end{aligned}$$

Substituting u, v, dv and du in the equation (1), we obtain

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2 \int x \cos x\, dx. \tag{3}$$

Now for evaluating $\int x \cos x\, dx$, we again need to use the integration by parts. Let us now choose:

$$\begin{aligned} u &= x && \text{and} && dv = \cos x\, dx, \\ du &= dx && \text{and} && \int dv = \int \cos x\, dx, \\ du &= dx && \text{and} && v = \sin x. \end{aligned}$$

Substituting u, v, dv and du in the equation (1), we obtain

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x.$$

(We omit the integration constant.) Now substitution the above integral in the equation (3) gives the desrired integral:

$$\begin{aligned} \int x^2 \sin x\, dx &= -x^2 \cos x + 2(x \sin x + \cos x) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x. \end{aligned}$$

Let us now evaluate the same integral by using tabular integration. In this technique, we form a table consisting of two columns. The first column contains successive derivatives of the function which is easy to derivate or whose higher order derivatives may vanish. After that, all entries of the first column are alternately appended with plus or minus signs. While, the second column contains successive integrals (antiderivatives). The first column is designated by D and the second one by I. Now the last step is to find the successive terms of the integral. They are given by multiplying each entry in the first column by the entry in the second one which lies just below it. Finally the integral is equal to the sum of the terms obtained. For evaluating the integral (2), we can form Table 1. Thus by the tabular integration:

$$\begin{aligned} \int x^2 \sin x\, dx &= -x^2 \cos x + 2x \sin x - 2 \cos x + 0 \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x. \end{aligned}$$

The number of terms is finite only if the higher order derivatives of the function u vanish. For example, the third order derivative of the function x^2 is null (see Table 1). Otherwise at any level the process of integration can be terminated by forming a remainder term defined as the integral od the product of the entry in the first column and the entry in the second column that lies directly across it.

D	I
$+x^2$	$\sin(x)$
$-2x$	$-\cos(x)$
$+2$	$-\sin(x)$
-0	$\cos(x)$

Table 1. Table for integrating $\int x^2 \sin x\, dx$

D	I
$+f(x)$	$g(x)$
$-f^1(x)$	$g^{[1]}(x)$
$+f^2(x)$	$g^{[2]}(x)$
$-f^3(x)$	$g^{[3]}(x)$
$\vdots$	$\vdots$
$(-1)^n f^n(x)$	$g^{[n]}(x)$
$(-1)^{n+1} f^{n+1}(x)$	$g^{[n+1]}(x)$

Table 2. Table for integrating $\int f(x)g(x)\, dx$

Let us now understand the tabular integration through the integration

$$\int f(x)g(x)\, dx.$$

Table 2 is the integration table for this integral. In this table, $f^i(x)$ denotes the i^{th} derivative of the function $f(x)$. While, $g^{[i]}$ denotes the i^{th} integration (antiderivative) of $g(x)$. From Table 2, the integration is:

$$\begin{aligned}\int f(x)g(x)\, dx &= f(x)g^{[1]}(x) + f^2(x)g^{[3]}(x) - \cdots \\ &\qquad + (-1)^n f^n(x) g^{[n+1]}(x) + (-1)^{n+1} \int f^{n+1}(x) g^{[n+1]}(x)\, dx \\ &= \sum_{i=0}^{n} (-1)^i f^i(x) g^{[i]}(x) + (-1)^{n+1} \int f^{n+1}(x) g^{[n+1]}(x)\, dx.\end{aligned}$$

Let us now apply the tabular integration technique for finding the Fourier series of the following even function:

$$f(x) = x^{2k}, \qquad -\pi \leqslant x \leqslant \pi.$$

The Fourier series is given as:

$$f(x) = a_0 + \sum_{n=0}^{\infty} [a_n \sin nx + b_n \cos nx]. \tag{4}$$

Here,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Let us first evaluate a_0.

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{i\pi}^{\pi} x^{2k}\, dx = \frac{2}{2\pi} \int_0^{\pi} x^{2k}\, dx \qquad [\text{since } x^{2k} \text{ is even}] \\ &= \frac{1}{\pi} \left[\frac{x^{2k+1}}{2k+1} \right]_0^{\pi} = \frac{\pi^{2k}}{2k+1}. \end{aligned} \tag{5}$$

D	**I**
$+f(x)$	$\cos(nx)$
$-f^1(x)$	$\dfrac{\sin(nx)}{n}$
$+f^2(x)$	$-\dfrac{\cos(nx)}{n^2}$
$-f^3(x)$	$-\dfrac{\sin(nx)}{n^3}$
$+f^4(x)$	$\dfrac{\cos(nx)}{n^4}$
$\vdots$	$\vdots$
$(-1)^{2k-1}f^{2k-1}(x)$	
$(-1)^{2k}f^{2k}(x)$	$(-1)^k\dfrac{\cos(nx)}{n^{2k}}$
$(-1)^{2k+1}f^{2k+1}(x)$	$(-1)^k\dfrac{\sin(nx)}{n^{2k+1}}$

Table 3. Table for integrating $\int f(x)\cos nx\,dx$. Function $f(x)$ is differentiated $2k+1$ times, while $\cos nx$ is integrated $2k1$ times.

$b_n = 0$ since $x^{2k}\sin nx$ is odd. Let us evaluate a_n.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad [f(x)\cos nx \text{ is even}].$$

Table 3 is our integration table for the above integral. Integrating by parts $2k+1$ times gives:

$$\begin{aligned}\int_0^{\pi} f(x)\cos nx\,dx = \Bigg[& f(x)\frac{\sin nx}{n} + f^1(x)\frac{\cos nx}{n^2} - f^2(x)\frac{\sin nx}{n^3} \\ & - f^3(x)\frac{\cos nx}{n^4} + \cdots + (-1)^{k-1}f^{2k-1}(x)\frac{\cos nx}{n^{2k}} \\ & + (-1)^k f^{2k}(x)\frac{\sin nx}{n^{2k+1}}\Bigg]_0^{\pi} + (-1)^{k+1}\int_0^{\pi} f^{2k+1}(x)\frac{\sin nx}{n^{2k+1}}\,dx.\end{aligned}$$

Since $\sin n\pi = \sin 0 = 0$, the previous integral is equal to

$$
\begin{aligned}
&= \left[f^1(x)\frac{\cos nx}{n^2} - f^3(x)\frac{\cos nx}{n^4} + \cdots + (-1)^{k-1} f^{2k-1}(x)\frac{\cos nx}{n^{2k}}\right]_0^\pi \\
&\quad + (-1)^{k+1}\int_0^\pi f^{2k+1}(x)\frac{\sin nx}{n^{2k+1}}\,dx \\
&= \left[\sum_{i=1}^{k}(-1)^{i-1}\frac{f^{2i-1}(x)\cos nx}{n^{2i}}\right]_0^\pi + (-1)^{k+1}\int_0^\pi f^{2k+1}(x)\frac{\sin nx}{n^{2k+1}}\,dx.
\end{aligned}
$$

Now for the function $f(x) = x^{2k}$,

$$
\begin{aligned}
f^{2i-1}(x) &= 2k\cdot(2k-1)\cdot(2k-2)\cdot\ldots\cdot(2k-2i+2)x^{2k-2i+1} \\
&= \frac{(2k)!}{(2k-2i+1)!}\,x^{2k-2i+1}
\end{aligned}
$$

and $f^{2k+1} = 0$, and thus

$$
\int_0^\pi f(x)\cos nx\,dx = \left[\sum_{i=1}^{k}(-1)^{i-1}\frac{(2k)!}{(2k-2i+1)!}x^{2k-2i+1}\cdot\frac{\cos nx}{n^{2i}}\right]_0^\pi.
$$

Since $\cos n\pi = (-1)^n$, we have

$$
\int_0^\pi f(x)\cos nx\,dx = \sum_{i=1}^{k}(-1)^{i-1}\frac{(2k)!}{(2k-2i+1)!}\pi^{2k-2i+1}\cdot\frac{(-1)^n}{n^{2i}}.
$$

Substituting the above integral in the equation gives a_n:

$$
a_n = \frac{2}{\pi}\sum_{i=1}^{k}(-1)^{i-1}\frac{(2k)!}{(2k-2i+1)!}\pi^{2k-2i+1}\cdot\frac{(-1)^n}{n^{2i}}.
$$

Substituting a_0, a_n and b_n in the equation (4) we finally obtain

$$
(6)\qquad x^{2k} = \frac{\pi^{2k}}{2k+1} = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\sum_{i=1}^{k}(-1)^{i-1}\frac{(2k)!\,\pi^{2k=2i+1}}{(2k-2i+1)!}\cdot\frac{(-1)^n}{n^{2i}}\right]\cos nx,
$$

for $-\pi \leqslant x \leqslant \pi$. Substituting $x = 0$ and $k = 1$ in the equation (6) gives:

$$
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.
$$

Similarly, substituting $x = \pi$ and $k = 1$, resp. $x = 0$ and $k = 2$, gives:

$$
\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \quad \text{resp.} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.
$$

$$
\begin{tikzcd}
\mathbf{D} & \mathbf{I} \\
+\sin(\omega\, t) \arrow[dr] & e^{-st} \\
-\omega\, \cos(\omega\, t) \arrow[dr] & -\dfrac{e^{-st}}{s} \\
-\omega^2\, \sin(\omega\, t) \arrow[r] & \dfrac{e^{-st}}{s^2}
\end{tikzcd}
$$

Table 4. Table for evaluating Laplace transform $\mathcal{L}\{\sin \omega t\}$.

Let us now apply the technique of tabular integration for finding Laplace transforms. Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = \lim_{T\to\infty}\int_0^T f(t)e^{-st}\,dt. \tag{7}$$

Let us find out the Laplace transform of $\sin \omega t$. For evaluating it, we can form Table 4.

$$
\begin{aligned}
\mathcal{L}\{\sin \omega t\} &= \int_0^\infty e^{-st}\sin\omega t\,dt \\
&= \left[\frac{-e^{-st}\sin\omega t}{s} - \frac{\omega e^{-st}\cos\omega t}{s^2}\right]_0^\infty - \frac{\omega^2}{s^2}\underbrace{\int_0^\infty e^{-st}\sin\omega t\,dt}_{\mathcal{L}\{\sin\omega t\}}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{L}\{\sin\omega t\}\left(1+\frac{\omega^2}{s^2}\right) &= \underbrace{\lim_{T\to\infty}\left[\frac{-\sin\omega T}{se^{sT}} - \frac{\omega\cos\omega T}{s^2e^{sT}}\right]}_{0} - \left[0 - \frac{\omega}{s^2}\right], \\
\mathcal{L}\{\sin\omega t\} &= \frac{\omega}{s^2+\omega^2}.
\end{aligned}
$$

Let us now find the Laplace transform of $t^n e^{at}$. Table 5 presents tabular integration technique for evaluating this transform. Using the table, we obtain

$$
\begin{aligned}
\mathcal{L}\{t^n e^{at}\} &= \int_0^\infty t^n e^{(a-s)t}\,dt \\
&= \left[\frac{t^n e^{(a-s)t}}{a-s} - \frac{nt^{(n-1)}e^{(a-s)t}}{(a-s)^2} + \frac{n(n-1)t^{n-2}e^{(a-s)t}}{(a-s)^3} - \cdots \right. \\
&\qquad \left. + (-1)^n\frac{n!\,e^{(a-s)t}}{(a-s)^{n+1}}\right]_0^\infty \\
&= \frac{(-1)^{n+1}n!}{(a-s)^{n+1}} = \frac{n!}{(s-a)^{n+1}}.
\end{aligned}
$$

D	I
$+t^n$	$e^{(a-s)t}$
$-n\,t^{n-1}$	$\dfrac{e^{(a-s)t}}{(a-s)}$
$+n(n-1)t^{n-2}$	$\dfrac{e^{(a-s)t}}{(a-s)^2}$
$-n(n-1)(n-2)t^{n-3}$	$\dfrac{e^{(a-s)t}}{(a-s)^3}$
$+n(n-1)(n-2)(n-3)t^{n-4}$	$\dfrac{e^{(a-s)t}}{(a-s)^4}$
$\vdots$	$\vdots$
$(-1)^n\,n!$	$\dfrac{e^{(a-s)t}}{(a-s)^n}$
0	$\dfrac{e^{(a-s)t}}{(a-s)^{n+1}}$

Table 5. Table for evaluating Laplace transform $\mathcal{L}\{t^n e^{\alpha t}\}$.

For the purpose of exposition, we may see that for $s > a$:

$$\lim_{T\to\infty} \frac{T^n e^{(a-s)T}}{a-s} = \lim_{T\to\infty} \frac{T^n}{(a-s)e^{(s-a)T}} = 0$$

(applying L'Hospital rule n times).

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Stord Haugesund Engineering College, Norway

E-mail: `sanjay.khattri@hsh.no`