The Teaching of Mathematics
2009, Vol. XII, 2, pp. 57-72

UNDERSTANDING ALGORITHMS OF VERTICAL ADDITION AND SUBTRACTION

Milosav M. Marjanović and Aleksandra N. Mandić

Abstract. In this paper, we split into all intermediate steps and illustrate algorithms of vertical addition and subtraction of two-digit and three-digit numbers. We consider the relating of symbolic codes of arithmetic to their iconic representations to be a basic way of supplying them with meaning. Hence, our didactical maxim is: I see, therefore I understand.

Writing this paper, our intension is deepening of teachers' understanding of these classical algorithms of arithmetic and, as our analysis shows it clearly, a right understanding is more a matter of good knowledge of didactics of mathematics than of mathematics itself.

ZDM Subject Classification: F32; AMS Subject Classification: 00A35, 97F30.

Key words and phrases: Decimal notation, decimal grouping, algorithms of vertical addition and subtraction, regrouping and the Cantor Principle of Invariance of Number.

1. Introduction

The meaning of the verbs "to see" and "to understand" overlap when the former is used in the examples as they are "to see reason (the point)", "to see why", "Yes, I see", etc. In mathematics, the meaning of the verb "to see" is twofold – to see figuratively and to become aware of some iconically represented facts, using eyes and interpreting the seen by the mind. This process of seeing and interpreting is the exclusive way of learning mathematics with understanding in the case of primary school students. It has its historical reflection in the ancient Hindu manuscripts, where some mathematical facts are represented by drawings followed by a "Behold!" just to stimulate the acquirement of the onlooker.

Let us recall that when a number is written in decimal place value notation, then it evokes in our mind an image of a set whose elements are nicely grouped. Namely, such a set is imagined as an arrangement of groups of its subsets, each group consists of at most nine subsets so that those in the first group (from the right) are 1-element, in the second 10-element, in the third 100-element, etc. When the elements of a set are grouped in that way, then we say that such a grouping is *decimal*. When a sum or difference of two numbers in decimal notation is written, then each component evokes a nice (decimal) grouping but the whole expression does not. That is why we tend to transform such an expression into the unique decimal notation of the number it stands for. The process and the result of transforming a numerical expression so that the decimal notation of its value is obtained, we call *decimalization*. Let us say that, throughout this paper, we will be using the terminology fixed in [3].

According to our point of view, the four beautiful classical algorithms of decimalization of sums, differences, products and quotients of two numbers in decimal notation have a high developmental value. They show how, by abstracting, a problem reduces to and is solved by performing operations at the level of syntax. These operations are always feasible, no matter how large are the involved numbers, relying only on the knowledge of addition (subtraction) table and multiplication (division) table.

Teaching children to apply these algorithms with full understanding of the involved operations, "we teach them far beyond the routine itself". Thanks to the hand-held calculators, "children no longer need to acquire a high degree of speed and accuracy in doing long" calculations. We use here the quotation marks to indicate cuttings from the text of R. Skemp's book [5] (and we still think that long division should be taught to children if it is approached in the proper way). The interesting book [2] of L. Ma displays some examples and some statistics showing how even teachers' understanding of these algorithms could be quite poor.

Our aim in this paper is to visually support all steps involved in performing algorithms of vertical addition and subtraction. Seeing in images the way how an operation is performed is the best what can be done at this level to let children grasp its rationale.

2. Counting – the introductory theme of arithmetic

When a child says that the number of chairs in the classroom and the number of students in that class are equal, then he/she expresses a very deep and abstract fact. Expressing it in the language of set theory, what he/she says is a common property of the set of chairs and the set of students which is called the *cardinality* of these sets. And when children have a good grasp of this idea of equivalence of sets, they have certainly made a significant progress in their cognitive development.

As a phylogenetic reflection, let us recall that the dual terms still existing in some contemporary languages are reliable evidence of a small number system that consisted of "one", "two" and "many" and that was created by our ancestors from the very remote past. Further development of these abstract ideas came only slowly and in the modern European languages (say in English), the first ten numbers are labeled, each by another word, while almost all other higher numbers are formed on additive and multiplicative principles. For instance, 25 is 20 and 5, 300 is 3 times one hundred, etc. Semantics of the number names led to their ordered reciting. This piece of the language develops spontaneously and majority of preschool children will have learnt to recite number names (at least up to 10) before they start going to school. Gaining of this skill we state as the first didactical task attached to this theme.

- *Children learn to recite in order number names up to 10, going forwards and backwards*.

Of course, mere reciting of number names does not develop their meaning. But when a set of objects is fixed and when number names, in their natural order, are attached each time to another object, then lastly pronounced number name indicates the cardinality of that set. This process of finding out the cardinality of a set of objects is called *counting*. Now we state the second didactical task attached to this theme.

- *Children practice counting, finding so the number of elements of various concrete sets. Such counting should be performed starting with arbitrary elements of a set and following different orders*.

Counting is an activity which does develop the meaning of number concepts invariantly of the nature of elements of sets and any way how they are ordered. (Here we see a special aspect of the Cantor Principle which will be explained in more detail in the following section of this paper). In the same way how the idea of a set precedes that of number, the ideas of *addition scheme* (union of two sets) and *subtraction scheme* (difference of two sets) precede operations of addition and subtraction, respectively.

In the presence of concrete sets of different things or illustrations presenting them, children count and answer typical questions as “How many of these are there and how many of those”, “How many are there, altogether”, “How many of them were there and how many are left”, etc. All such exercises are done orally, being confined to the range of numbers up to 10. This practice of counting and answering questions is a prelude to the themes of addition and subtraction. Having this in mind, we formulate the third didactical task attached to this theme.

- *In the presence of visual material, children count and answer questions which are intended to help development of ideas of addition and subtraction schemes*.

Let us note that this section is somewhat longer than needed for the purpose of this paper. As it is missing in [3], we consider this part of the text as a concise complement to that paper.

Counting can be seen as the most primitive form of calculation. Relating it to operations, counting forwards can be seen as continual addition of 1’s and counting backwards as subtraction of 1’s. To represent iconically numbers up to 10, we use arrangements of sticks (straight lines)

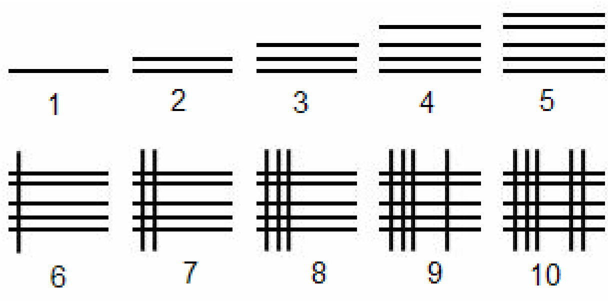


Fig. 1

Number images in the case of larger numbers are formed on additive principle. For instance, thirteen and sixteen are represented by arrangements in Fig. 2.

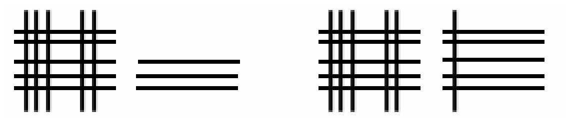


Fig.2

Knowing that "teen" is a modification of "ten", the number names suggest sums: 10 + 3 and 10 + 6, respectively. (See also [4]).

3. Regrouping – a special aspect of Cantor Principle

Given a set A, the symbol $\overline{\overline{A}}$ was used by G. Cantor (1845–1918) to denote its cardinal number. Two overbars suggest two kinds of ignoring (abstraction) – the nature of elements of the set *A* and the way how they are ordered. This process of abstracting, leading from the idea of a set to the idea of its cardinal number was somewhat more generally formulated in [3] and called *Cantor Principle of Invariance of Number*: *Cardinal number of a set is formed independently of the nature of its elements and any kind of their organization.*

The desired generality consists of replacing the word "order" by the word "organization" which is taken here to denote any way how the elements of a set may be ordered, arranged, grouped, etc. (Mathematicians express this meaning saying that the set is endowed with one or more structures).

Different groupings of elements of a set may lead to different expressions denoting its cardinality. Since the cardinality does not depend on the way of grouping, such expressions, when equated, result in equalities that hold true in general. When letters are used instead of specific numbers, such equalities represent general rules of arithmetic and some of them, which are more basic, are taken to be laws of arithmetic from which, then, all other rules of arithmetic can be derived.

When, for instance, the addition of two numbers is performed by adding units to units, tens to tens, etc. then, there exist several intermediate steps. Understanding each of these steps makes a thorough understanding of that procedure. Let us demonstrate it in the case of a simple addition.

$$
\begin{aligned}
25 + 18 &= (20 + 5) + (10 + 8) = (20 + 10) + (5 + 8) = (20 + 10) + 13 \\
&= (20 + 10) + (10 + 3) = ((20 + 10) + 10) + 3 = (30 + 10) + 3 \\
&= 40 + 3 = 43
\end{aligned}
$$

Here, as the intermediate steps, we see decomposition of numbers into sums and several cases of application of (generalized) association law.

Let us note that properties expressed by commutative and associative laws had been taken as a matter of course until the beginning of the 19^{th} century, when they were stated formally (F.–J. Servois, Ann. de Math., 5, 1814/15).

Formal explanation to children of calculation procedures would be nonsense and it is not even of much help for teachers' understanding of such matters. Proceeding further we will keep in mind the following didactical maxim:

I see, therefore I understand.

4. Addition – cases of two-digit numbers

In this section, we will split into all intermediate steps the algorithm of vertical addition of two-digit numbers, illustrating these steps by number images. And as it is well known, writing summands one above the other is particularly convenient because, in that way, columns of units, tens, etc. are formed.

The way how addition of multi-digit numbers is carried out can be roughly expressed in the form of the following narrative rule: units are added to units, tens to tens, hundreds to hundreds and so on. The manner of narrative expressing is very suggestive but it often happens that some essential details are left out. (Here, it is the case with carryings from column to column). A subtle treatment of significance of narrative expressing can be found in the Bruner's book [1].

Before proceeding further, we fix the way how groups of tens are represented. iconically (see Fig. 3).

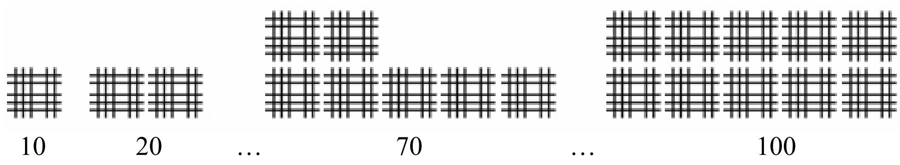


Fig.3

As for addition scheme, it will be represented by two sets A and B, with decimal grouping of their elements and which are situated one above the other (Fig. 4). The arrow suggests formation of the third set which is their union and whose elements are also in decimal grouping.

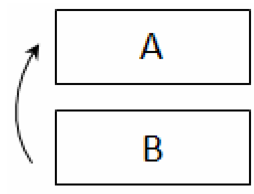
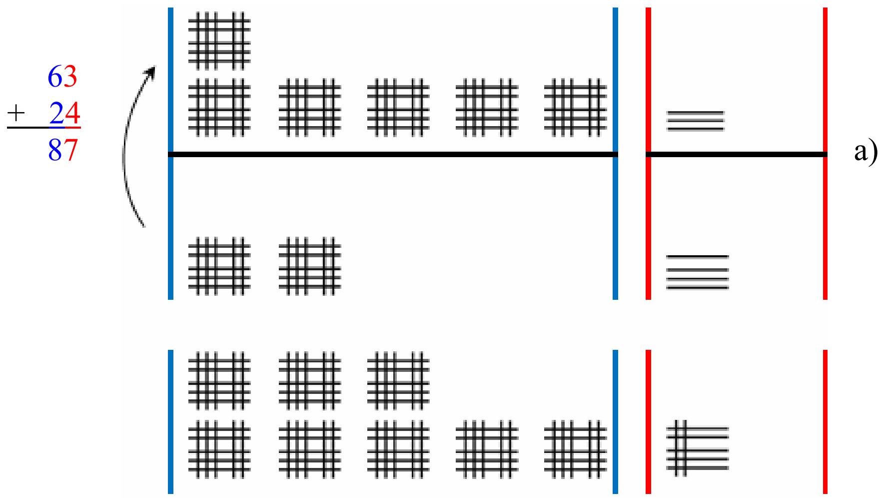


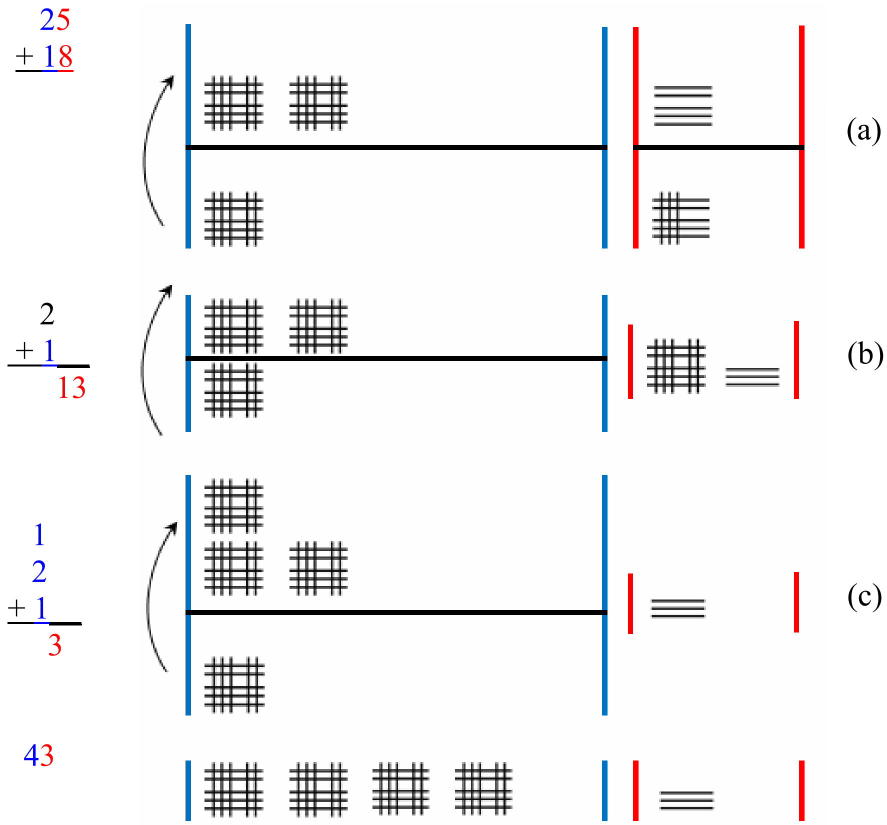
Fig. 4

Example 1. (No carrying occurs)



The color of digits is in accordance with the color of columns.
Comments: *(*a) We add units to units and tens to tens.

Example 2. (The case when carrying occurs)



Comments: (a) We add units to units. (b) We carry over one ten to the tens column. (c) We add tens to tens.

The teacher should elaborate similarly a number of examples, asking children to give comments which describe each step. As an intermediate stage, the teacher display to his/her students the series of steps denoted symbolically (without the accompanying illustrations), asking them to comment. For instance,

```
                              1
  36          3               3
+ 48 ,     + 4 ___,         + 4 ,          , etc.
              14              4    84
```

But as soon as the understanding of the procedure is evident, the teacher thinks of suppressing intermediate steps. In the compressed form the addition runs as follows:

```
  1
 25
+18
---
  3
```

Comments: We add units to units and we get 13, we write 3 in the units column and we carry over one ten to the tens column. Finally, we add tens to tens.

```
  1
 25
+18
---
 43
```

5. Subtraction – cases of two-digit numbers

The way how subtraction of multi-digit numbers is carried out can be again expressed as a narrative rule: units are subtracted from units, tens from tens, hundreds from hundreds and so on. But when, for instance, the number of units of minuend is smaller than the number of units of subtrahend, then this rule cannot be applied without carrying over one ten from the tens column to the units column. (When we say that a larger number cannot be subtracted from a smaller one, we mean that such subtraction is not feasible in the set N of natural numbers).

In this context, the subtraction scheme is seen as the process of taking away the elements of a set A (in the role of minuend) as many times as the set B (in the role of subtrahend) has elements. Such a scheme is represented in Fig. 5, where the arrow suggests that taking.

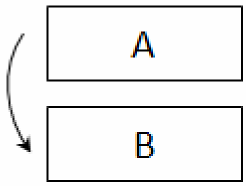
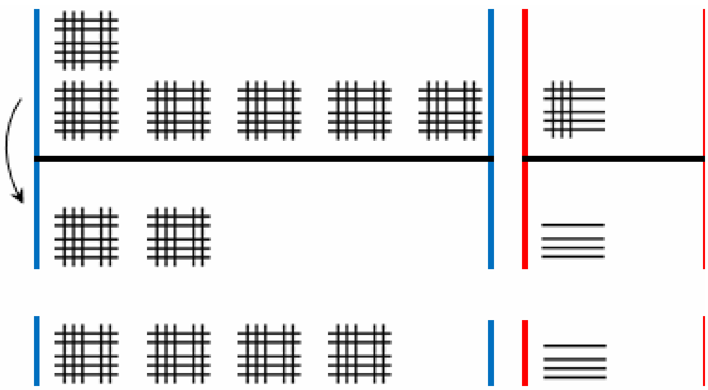


Fig.5

Example 3. (No carrying occurs)

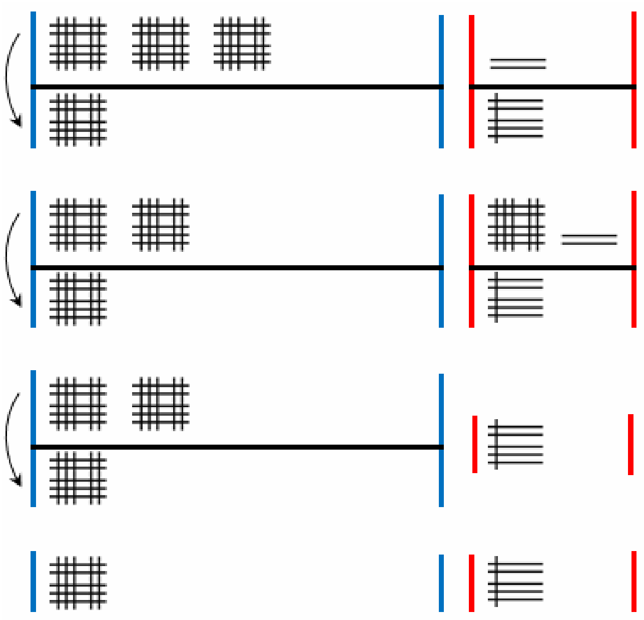
```
 68
-24
---
 44
```



(a)

Comments : (a) We subtract units from units and tens from tens.

Example 4. (The case when carrying occurs)



32
- 16 (a)

212
- 1 6 (b)

2
- 1
6 (c)

16

Comments: *(*a) To subtract 6 from 2 is not feasible. We carry over one ten to the units column. (b) We subtract units from units. Finally, (c) we subtract tens from tens.

When a number of similar examples accompanied with illustrations have been worked out, examples without illustrations should be displayed to students to comment them. For instance,

$$\begin{array}{r} 75 \\ -\ 28 \\ \hline \end{array}, \quad \begin{array}{r} 615 \\ -\ 2\ 8 \\ \hline \end{array}, \quad \begin{array}{r} 6 \\ -\ 2\ \ \\ \hline 7 \end{array}, \quad 47 \qquad \text{etc.}$$

Once students have a firm grasp of this procedure, subtraction in the compressed form is practiced. For example,

$$\begin{array}{r} 1 \\ 32 \\ -\ 16 \\ \hline 16 \end{array}$$

Comments: To subtract 6 from 2 is not feasible. We carry over one ten to the units column and we write 1 over the first digit of minuend. Then, we subtract 6 from 12 and 1 from 2 (one ten has been carried over.)
carried over)

6. Addressing teachers' knowledge

Children gain understanding through activities that they are exposed to. On the other hand, teachers' understanding is based on the knowledge of processes through which concepts of numbers and arithmetic operations are formed. Strictly speaking, this understanding does not depend as much on a good knowledge of the subject matter as it depends on a thorough understanding of the corresponding didactical procedures by which abstract concepts are related to their intuitive interpretations and activities with concrete material.

When the didactical aim is establishing of the meaning of addition and subtraction schemes, sets of a great variety of objects are intentionally used. But when formal calculation is the subject of elaboration, once chosen, these representations should not be varied much. In this paper, all elements of the used sets are represented by a uniform sign (a straight line, image of a stick). Addition scheme consists of two sets with decimal grouping of their elements. Taken together, these two sets form the third one and, when needed, its decimal grouping is obtained by carrying one ten. Subtraction scheme consists of two sets, each being decimally grouped and the first of them representing minuend and the second subtrahend. The role of the second set is to show how many elements of the first set have to be taken away. This taking is done column by column, and when not feasible, the set representing the minuend is regrouped by carrying one ten.

Understanding of a teacher consists of appreciation of the used didactical material – a uniformly chosen sign to denote the unit of counting and the role of nice (decimal) grouping. Both addition and subtraction are performed as column by column adding and taking away, respectively (simulating a real activity) and parallel to it, formal operations are attached and performed. Carrying one ten is a regrouping of sets with the final aim to get resulting sets represented in the form of decimal groupings. According to the Cantor Principle, neither the chosen way of grouping nor regroupings, when they happen, affect the cardinality of the involved sets. Therefore, for teachers, a good understanding is comprehension of the way how formal operations are related to their visual interpretations, what is a matter of didactics of mathematics rather than of mathematics itself.

7. Addition – cases of three digit numbers

The iconic sign representing one hundred will be the arrangement of 10 tens framed by a line. With such a frame, the arrangement is taken as a whole representing a higher value unit. In Fig. 6, we see how hundreds are represented as plates arranged one behind the other.

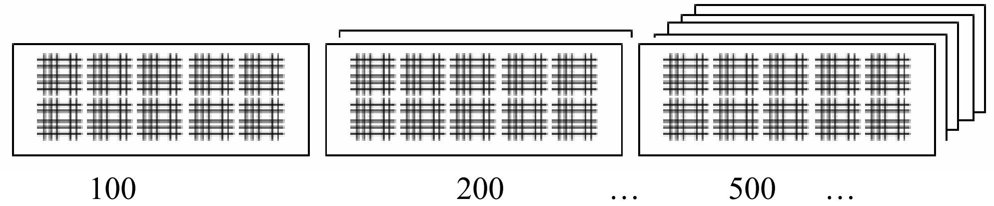
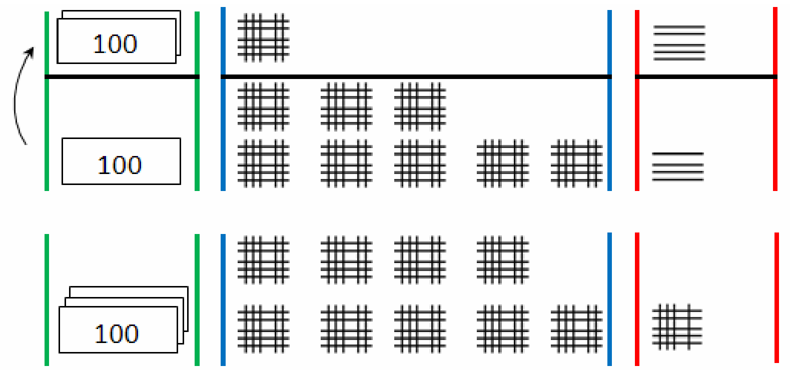


Fig. 6

Example 5. (No carrying occurs)

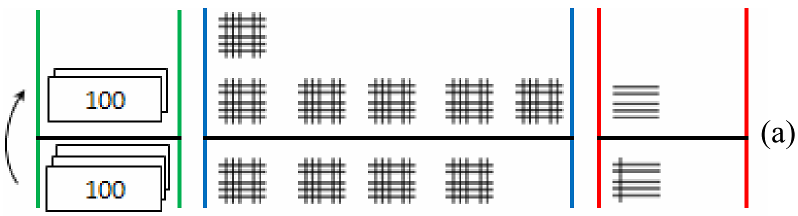
215
+ 184
399



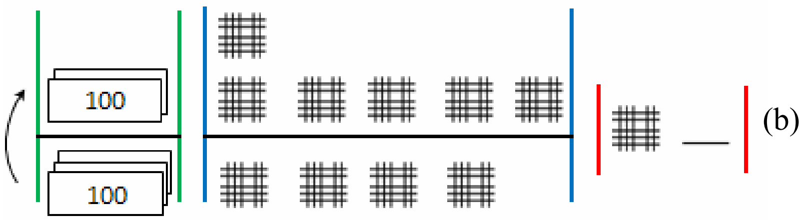
Here, we add units to units, tens to tens and hundreds to hundreds.

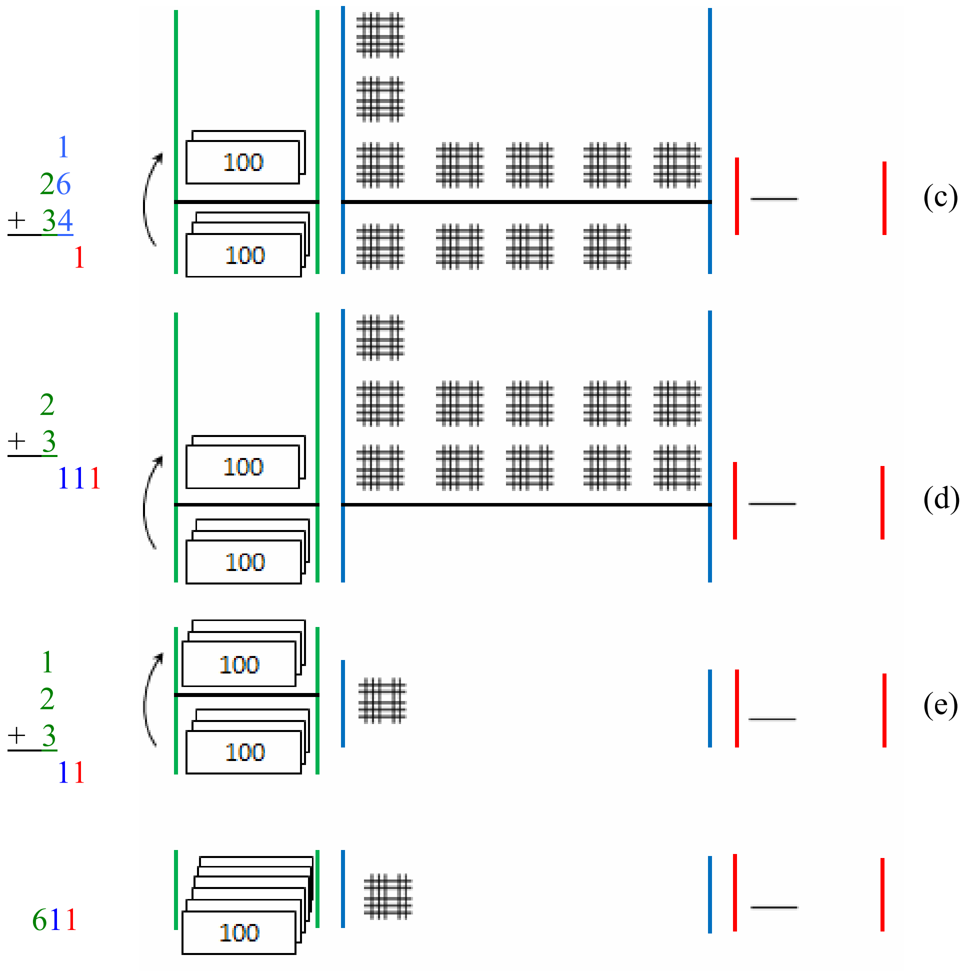
Example 6. (The case when carrying occurs)

265
+ 346



26
+ 34
11





Comments: (a) We add units to units. (b) We carry over one ten to tens column. (c) We add tens to tens. (d) We carry over one hundred to the hundred column. (e) We add hundreds to hundreds.

Before suppressing intermediate steps, the teacher should ask children to comment steps when they are symbolically designated but not illustrated. For instance,

		1		1	
385	38	38	3	3	
+ 497 ,	+ 49	+ 49	+ 4	+ 4 ,	, etc.
	12	2	182	82	882

The compressed form of this addition is:

```
   11
   385
+  497
   882
```

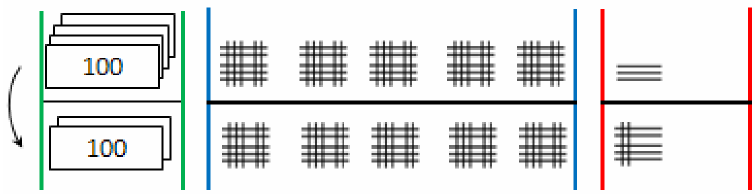
8. Subtraction – cases of three-digit numbers

Example 7. In this example all three subtractions: units from units, tens from tens and hundreds from hundreds are directly feasible.

```
  467
-  245
  222
```

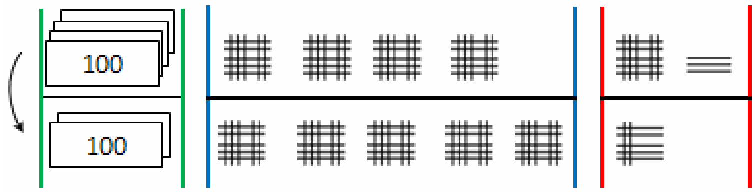
Example 8. (The case when carrying occurs)

```
  453
-  257
```



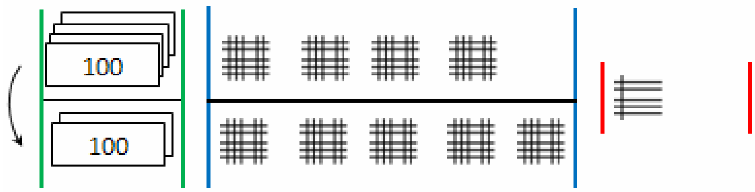
(a)

```
  4413
-  25 7
```



(b)

```
   44
-  25
     6
```



(c)

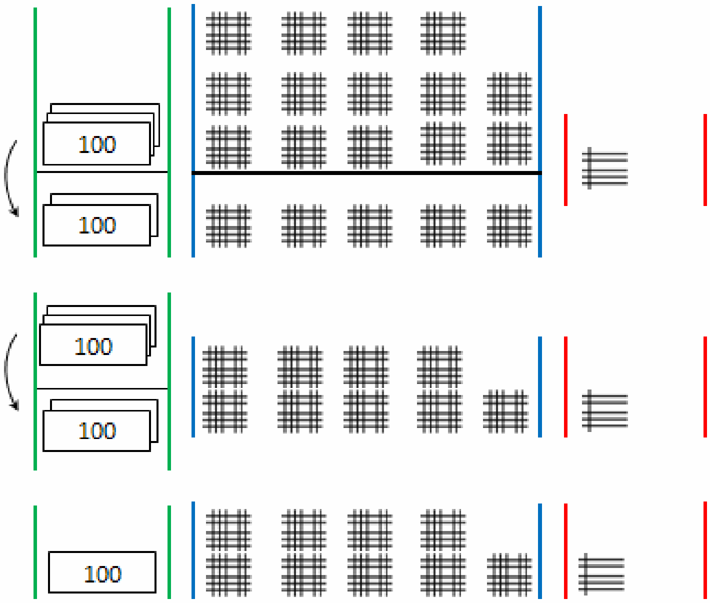
314
- 2 5
6

(d)

3
- 2
96

(e)

196



Comments: (a) Subtraction of units from units is not feasible. One ten is carried over to the units column. (b) Units are subtracted from units. (c) One hundred is carried over to the tens column. (d) Tens are subtracted from tens. (e) Hundreds are subtracted from hundreds.

Example 9. (Minuend ends with zeros)

300
- 195

(a)

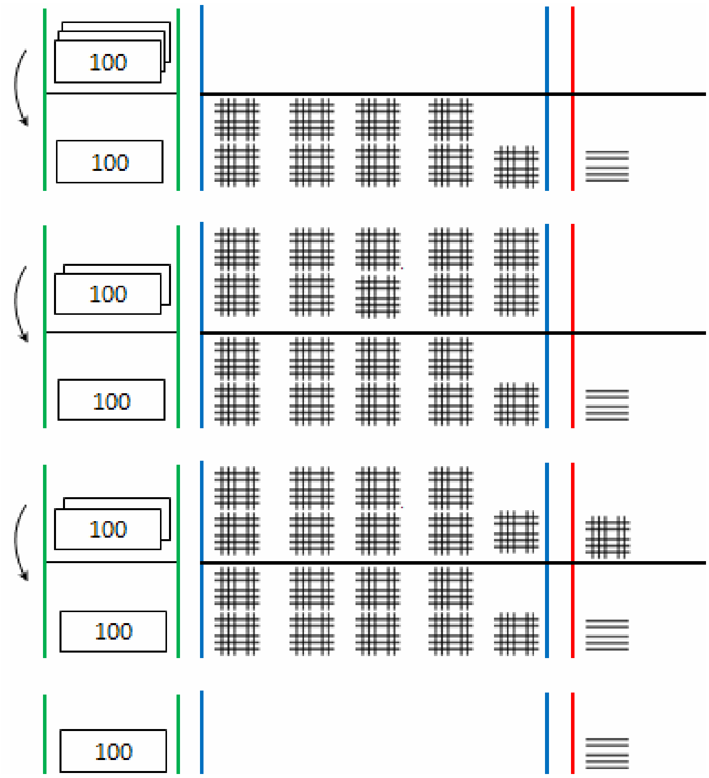
2 100
- 1 95

(b)

2910
- 19 5

(c)

105



Comments: (a) One hundred is carried over the tens column. (b) One ten is carried over the units column. (c) We subtract units from units, tens from tens and hundreds from hundreds.

And again, a number of exercises is done, where all intermediate steps are denoted and displayed to children to comment them without the help of illustrations. For instance,

703	6103	6913	69		
- 426	- 4 2 6	- 42 6	- 42		etc.
			7	277	

When addition and subtraction are done in the compressed form, at the first stage, children should comment speaking of carrying one ten, one hundred etc. Finally, when they learn to calculate in the most automatic way, they use number names only.

The skill that children have gained dealing with two-digit and three-digit numbers is spontaneously extended to the case of multi-digit numbers. Firstly, of course, numbers should be seen as sums of products of higher units as, for example,

$$2437 = 2\cdot1000 + 4\cdot100 + 3\cdot10 + 7$$

$$32854 = 3\cdot10000 + 2\cdot1000 + 8\cdot100 + 5\cdot10 + 4, \text{ etc.}$$

Then the intermediate steps are displayed only symbolically, when children should comment them speaking of carrying one thousand, one ten thousand, etc. until, finally, the stage of the most automatic operations is reached.

REFERENCES

1. Bruner, J., The Culture of Education, Harvard University Press, 1996.

2. Ma, Liping, Knowing and Teaching Elementary Mathematics, Lawrence Erlbaum Associates, London, 1999.

3.Marjanović M. Milosav, A Broader Way Trough Themas of Elementary School Mathematics, The Teaching of Mathematics, vol. II, 1, 41 – 58; vol. II, 2, pp. 81 – 103; vol. III, 1, pp. 41 – 51; vol. V, 1, pp. 47 – 55; vol. VI, 2, pp. 113 – 120; vol. VII, 1, pp. 35 – 52; vol. VII, 2, pp. 71 – 93.

4. Marjanović M. Milosav and Mandić N. Aleksandra, Numbers as Visible Shapes, The Teaching of Mathematics, vol. XII, 1, pp. 45 – 49.

5. Skemp, Richard, Mathematics in the Primary School, Routledge Falmer, 2002.

E-mail: milomar@beotel.rs
edusoft@nadlanu.com