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A CONTRIBUTION TO THE DEVELOPMENT OF FUNCTIONAL THINKING OF PUPILS AND STUDENTS

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Abstract. This contribution provides an idea of a suitable presentation which can contribute to motivation, illustration and deeper understanding in secondary school mathematics of fundamental concepts and ideas of the themes Function, Differential and integral calculus, as well as further development of students' functional thinking.

ZDM Subject Classification: I20; *AMS Subject Classification*: 97I20.

Key words and phrases: Function, graph, differential equation, functional thinking.

1. Introduction

The main goal of this contribution is to provide material that can serve for motivation, illustration and deeper understanding of basic concepts and ideas of the differential and integral calculus, as well as further development of functional thinking in teaching mathematics.

What is meant by the concept of *functional thinking*? The concept was first used at the turn of 19th/20th century by a German mathematician Felix Klein (1849–1925), a leading personality in the movement for reformation of mathematical education in Europe. It was the functional thinking that Klein considered as the central idea of teaching mathematics.

The phylogenetic development of the concept of function is analyzed in detail, e.g. in [5] and [7] or [2].

Functional thinking starts developing in an individual much earlier than the concept of function is defined at the second stage of primary school. As early as preschool, children encounter a variety of event causes and dependencies in their everyday lives, and thus they begin to develop their sense of causality. At the first stage of primary school they work with a variety of dependency tables, they prepare for the coordination system and they draw various graphs and diagrams. Despite the fact that the concept of *function* is not mentioned at all during this so called motivational or preeducational stage, the emergence and gradual reinforcement of functional thinking has a significant impact on it. Bero found out on the basis of his experiment that a considerable shift in the sense of causality comes up at the age 11–12 (see [1]).

Teaching at primary and secondary schools should help pupils to identify certain types of changes and dependencies, which are part of common events in the real world, as well as to become familiar with their representations. Pupils are supposed to analyze those dependencies in tables, diagrams and graphs, and express some simple cases by means of mathematical rules. Examining these dependencies hence leads to understanding of the concept of function. The concept of function is built gradually by working with a large number of isolated models of function (see [3]).

The concept of functional thinking is analyzed in detail, e.g. in [8] or [4].

A minor reference to drawing and interpretation of graphs of real functions in [6] has been the main impulse for writing of this paper.

The filling of a vessel with liquid can be considered as our initial situation. At the very beginning let us suppose that there is a solid in the shape of a *circular cylinder* with base of radius R and height H. Let us put liquid with the constant inflow rate Q into this vessel from above and ask the following question: What does the dependence of the altitude of the liquid level on time look like?

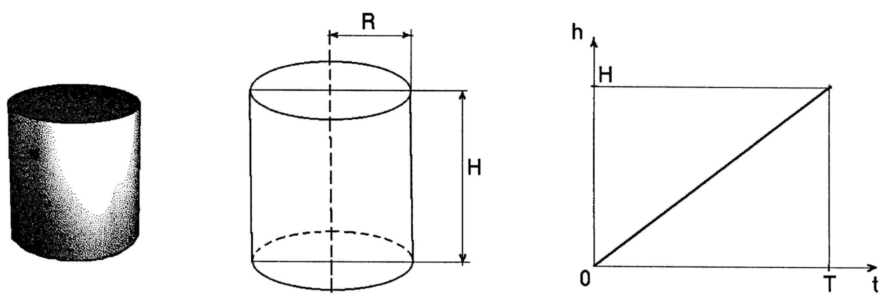

Fig. 1

It is evident that this is a linear function (see Fig. 1), the graph of which goes through the origin of the system of coordinates. Let the cylinder be filled within time t up to height h. The volume $V(h)$ of this part is obviously equal to

$$V(h) = \pi R^2 h = Qt.$$

From this equation we get the desired linear function

$$h(t) = \frac{Q}{\pi R^2} \cdot t,$$

which meets both the initial condition $h(0) = 0$ and the fact that at the moment of filling the whole cylinder

$$T = \frac{\pi R^2 H}{Q}$$

it holds that $h(T) = H$.

It is worth to concentrate our attention to the fact that the ratio of the inflow rate Q and the area of the perpendicular section πR^2 expresses the speed of the level increase, which is constant in this case. The domain of the function $h(t)$ is the closed interval $[0, T]$, and the closed interval $[0, H]$ is the range of its values.

From the methodological point of view it is important to let students draw pictures of graphs of these functions for two cylinders that differ by a single parameter (see Fig. 2) in one and the same system of coordinates.

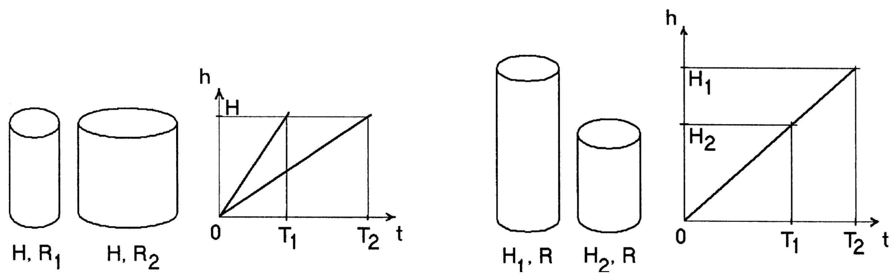

Fig. 2

Now turn our attention to the *right circular cone* (see Fig. 3). What will the course of the filling of the vessel look like here? What elementary function will be describing it?

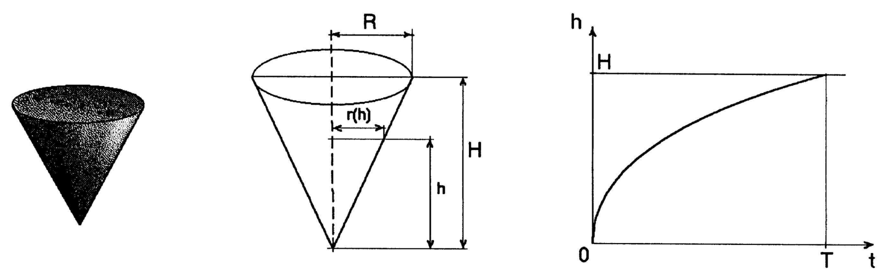

Fig. 3

It will evidently hold that:

$$V(h) = \frac{1}{3}\pi(r(h))^2 \cdot h = \frac{1}{3}\pi\left(h \cdot \frac{R}{H}\right)^2 \cdot h = \frac{1}{3}\pi\frac{R^2}{H^2} \cdot h^3.$$

From the equation $V(h) = Q \cdot t$ we get the desired function

$$h(t) = \sqrt[3]{\frac{3H^2Q}{\pi R^2} \cdot t}$$

expressing the dependence of the level height h on time t. This function evidently suits the boundary conditions $h(0) = 0$ and $h(T) = H$, where the filling period is

$$T = \frac{\pi R^2 H}{3Q}.$$

According to our expectation this is an increasing function. We should not even be surprised by the fact that it is concave—since the cone placed on the tip like this becomes wider with the growing height and therefore the level will be going up more and more slowly. Such considerations may contribute positively to the development of functional thinking and students should always have a chance of carrying out such tasks in learning.

Let us stop for a while and think about the behaviour of this function in the neighbourhood of the origin. What is the tangent to the graph of function $h(t)$ at this point from the right hand side? From Fig. 3 it is evident that the vertical axis h is its tangent. More precisely, it holds that $h'_+(0) = \infty$. This consideration can be used in teaching for propaedeutics of the infinitesimal calculus. The improper derivative of the function $h(t)$ at the origin does correspond with the speed of the level increase and is infinitely big in the "infinitely small tip" of the cone.

Here it is also possible to draw pictures of graphs of functions $h(t)$ again in one system for various cones, the frustum of a cone, the cylinder, the "inscribed cone", etc.

The reader is already feeling that there is high time for the general solution of the whole problem.

2. General solution of the problem

Therefore let $r(h)$ be a continuous function defined on the closed interval $[0, H]$, which is positive on the interval $(0, H)$. Let us call it the *profile function*. By rotating the graph of this function around the axis of independent variable h, a solid of revolution with the volume of $Q \cdot T$ and height H (see Fig. 4) will be generated.

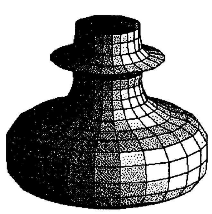
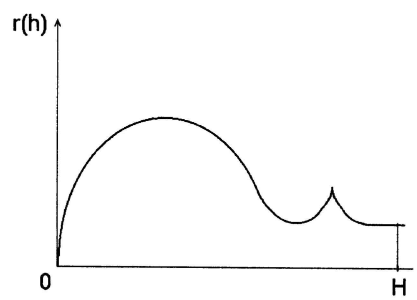

Fig. 4

If in the course of filling this solid by constant inflow rate Q at height h for elementary time dt, there is an elementary growth with volume $Q\,dt$ (see Fig. 5), it is possible to approximate it by means of the volume of a little elementary cylinder having radius $r(h)$ and height dh.

From the equation

$$Q\,dt = \pi r^2(h)\,dh \tag{1}$$

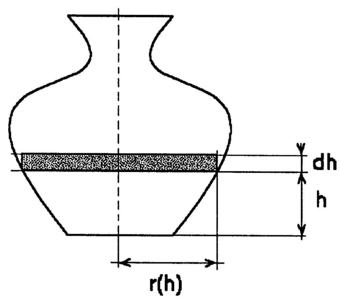

Fig. 5

we get the following differential equation

$$\frac{dh}{dt} = \frac{Q}{\pi r^2(h)}, \tag{2}$$

one of the particular solutions of which is the desired function $h(t)$, which will be called the *height filling function.* The so called *time filling function* $t(h)$ is defined as the inverse function to $h(t)$ on the interval $[0, H]$, which shows the period of time for which the solid is filled to height h.

The reader himself can test whether the differential equation (2) leads to correct results already in the case of the cylinder and the cone solved in an elementary manner.

What are the qualities of the filling function in general? As this is the case of mutually inverse functions, we will continue talking, for instance, only about the height filling function $h(t)$.

Firstly. Function $h(t)$ is *differentiable* and therefore it is also *continuous.*

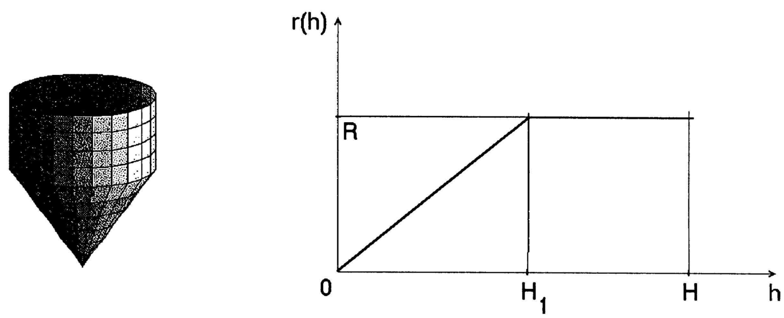

Fig. 6

This conclusion results from the continuity of the function on the right-hand side of equation (2) and therefore also the continuity of function $h'(t)$. Let us have a look at the solid in Fig 6. For students the profile function $r(h)$ is a precious example of a non-elementary function.

$$r(h) = \begin{cases} \dfrac{R}{H_1} \cdot h, & \text{for } h \in [0, H_1), \\ R, & \text{for } h \in [H_1, H]. \end{cases}$$

The solid itself is composed of a cone of height H_1 and a cylinder with radius R linked to it. From the point of view of functional thinking it will be a contribution if students try to estimate the shape of the filling function before their calculation starts. Even function $h(t)$ will evidently be "composed" of the above determined filling function for both the cone and the cylinder. With regards to the time of filling the cone T_1 with the level at height H_1, it is necessary to place the initial part of the graph of the linear filling function for the linked cylinder to point (T_1, H_1). The function $h(t)$ will also be non-elementary, but differentiable on the whole interval $(0, H)$ (see Fig. 7).

$$h(t) = \begin{cases} \sqrt[3]{\dfrac{3H_1^2 Q}{\pi R^2} \cdot t}, & \text{for } t \in [0, T_1), \\ H_1 + \dfrac{Q}{\pi R^2}\left(t - \dfrac{\pi R^2 H_1}{3Q}\right), & \text{for } t \in [T_1, T], \end{cases}$$

where

$$T_1 = \frac{\pi R^2 H_1}{3Q} \quad \text{and} \quad T = \frac{\pi R^2 H_1}{3Q} + \frac{\pi R^2 (H - H_1)}{Q}.$$

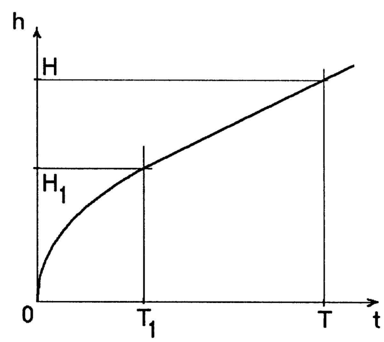

Fig. 7

It will be useful for students if they check whether function $h(t)$ really meets everything that we expect this function to bring. It holds that

$$h(0) = 0, \quad h(T_1) = H_1, \quad h(T) = H, \quad \lim_{t \to T_1^-} h(t) = H_1.$$

The equality

$$h'_-(T_1) = h'_+(T_1) = \frac{Q}{\pi R^2}$$

further confirms the fact included in equation (2), which means that derivative of the height filling function $h(t)$ (speed of the level increase) depends only on the radius of the solid at the given height $r(h)$, which therefore lead to the surprising finding that the circular level of the given radius grows quickly as in the solids of arbitrary shapes. The same equality expressing the differentiability of function $h(t)$ at point T_1 meets our expectations: With continuous change of the radius even the speed of the level increase is changed continuously.

SECOND. Function $h(t)$ is *increasing*. It results from equation (2), the right-hand side of which is greater than zero on the interval $(0, H)$. The derivative of the height filling function $h(t)$ is therefore positive on the interval $(0, T)$ and function $h(t)$ is increasing.

The THIRD and the FOURTH conclusions can be deduced from the equation

$$\frac{d^2h}{dt^2} = -\frac{2Q^2}{\pi^2}\frac{1}{r^5(h)}\frac{dr}{dh},$$

which we can get derivating equation (2) with respect to time. Because the sign of the second derivative of the height filling function with respect to the time is influenced only by the sign of derivative of the profile function with respect to the height, the following statement can be made:

If the profile function $r(h)$ is increasing (or decreasing), the height filling function $h(t)$ is concave (or convex).

This theorem is in compliance with our expectation. If the solid becomes wider in the course of filling ($r(h)$ is increasing), the speed of the level increase is smaller ($h(t)$ is concave). If, on the contrary, the solid becomes narrower with the growing level of the liquid (the decreasing $r(h)$), the level will increase more and more quickly (the convex $h(t)$).

This theorem directly implies the following statement:

If the profile function $r(h)$ at the height h_0 has a local extremum, the time filling function $t(h)$ has a point of inflection at this height h_0.

Draw a graph of the height filling function $h(t)$ of the two solids placed in the middle of Fig. 8 in connection with this statement.

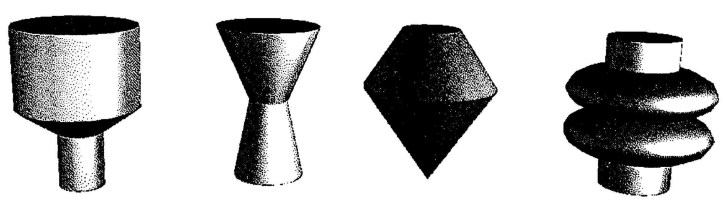

Fig. 8

The four sentences, which have just been referred to, enable us to draw pictures of graphs of filling functions for different solids without any difficulties. Therefore, will you be able to draw pictures of graphs of filling functions of various solids for all solids in the picture above (Fig. 8)?

It will be a great contribution for the development of our students' functional thinking if we proceed even the other way round, i.e. if we make a draft of shapes of the solids which have the filling functions there are drawn for instance in Fig. 9.

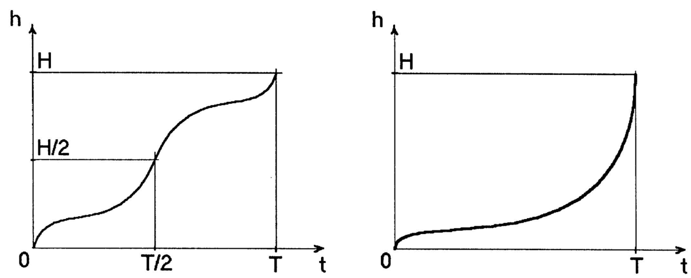

Fig. 9

A great number of further connections between the qualities of the profile function $r(h)$ and the behaviour of filling functions $h(t)$ and $t(h)$ can certainly be deduced or discovered from pictures of the prepared graphs. For instance, it is evident that solids symmetrical according the plane normal to the axis of revolution have the filling function that is centrally symmetrical.

3. Conclusion

In conclusion, I wish to express my belief that the presentation of the above stated problem in teaching can contribute to motivation or deeper understanding of such concepts as the function, the inverse function, convex and concave function, continuity and differentiability of the function, the local extremum, the parametric system of functions. It is a great advantage that this problem can be presented nearly in an arbitrary width—it can go through the stage of a mere intuitive drawing of pictures of graphs of the mutually corresponding filling and profile functions up to the stages of creating hypotheses and proving them. In addition to that, it can positively influence the development of functional thinking of students.

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