THE TEACHING OF MATHEMATICS
2010, Vol. XIII, 1, pp. 17–34

A SOCRATIC METHODOLOGICAL PROPOSAL FOR THE STUDY OF THE EQUALITY $0.999\ldots = 1$

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"When I use a word," Humpty Dumpty said in a rather a scornful tone,
"it means just what I choose it to mean—neither more nor less."
"The question is," said Alice, "whether you can make words mean different things."
"The question is," said Humpty Dumpty, "which is to be master—that's all."
Lewis Carroll, "Through the Looking Glass"

Abstract. Our objective is to produce a learning experience in the form of an interview which is structured in an introduction and three phases with the purpose of making the equality $0.999\ldots = 1$ acceptable providing the tail of dots "…" with a precise meaning. Essentially the interviewee will have to understand that a symbol is not what its aspect may suggest, but what we want it to be in a precise way. What we want will be the result of an evolutionary process of change of meaning, dictated by the context in which we move in each conceptual phase in which the experience is structured. In more detail, the Introduction will serve him/her to reflect on what a symbol is and to appreciate the usefulness of the positional system of numerical symbols being aware of the hidden character of the involved algebraic operations. The experience will run assigning different meanings to the symbol "…", each meaning reconciled with the previous one. Thus Phase 1 will extend the positional system of symbols to the rational numbers with the appearance of a new algebraic operation, the division. Phase 2 will state that the habitual algebraic operations are not sufficient to equip the symbol $0.999\ldots$ with a numerical meaning, which will force us to the introduction of a new algebraic operation in Phase 3: interpreting the tail of dots either as a dynamic process (movement) or as its stabilization in an end product (rest), we will choose the last one and deal with the non trivial problem of how to formulate algebraically what means that a dynamic process becomes stabilized. To attain this objective, logical quantifiers need to appear in scene and the use of a suitable mathematical assistant will encourage their understanding through visualization.

ZDM Subject Classification: I34; *AMS Subject Classification*: 97I30.

Key words and phrases: Meanings of symbols; meaning of "…"; equality $0.999\cdots = 1$; visualization.

1. Objective

Symbols in Mathematics usually do not provide a pictorial or mental image of what they represent. They are usually abstract or, if pictorial of some kind, they have evolved as we progress and refine our knowledge and notation, involving a series of hidden operations that, executed in a certain order, produce the concept or result that are meant to represent. If the student guides itself by what they may suggest or if its knowledge of what remains hidden is partial, problems may arise and give origin to interpretations that do not correspond with the intended purpose. The symbol 0.999 ... is a typical case and the literature in Mathematical Education contains several studies in which those dysfunctions are studied and remedy proposals are suggested (see [2], [6]).

Assuming basic knowledge acquired in High School, our objective is to implement a learning experience making the equality mentioned in the title convincing. The experience will run as a semi-structured clinical interview of around one hour of duration which we perform previous to the introduction of the real numbers.

2. Description of the experience

To carry out our study we elaborate a questionnaire for a clinic semi structured interview providing the general guidelines but allowing our intervention depending on students' answers, which gives us the chance to turn the interview into a learning experience for the student.

2.1 An informal description of the questionnaire

2.1.1 *Introduction.* Our aim is to engage in a dialog with students about the necessity of using symbols to represent whole amounts, distinguishing those symbols that visualize somehow what we are doing (for instance, "||||" for counting four objects) from those that visually do not suggest information on the represented amount (like "4"), i.e. abstract symbols. We point out that there is no harm using them due to the ability of our brain to recognize a form and assign a meaning to it. Further questions regarding operability of those symbols lead us to show the student the great evolutionary advance provided by the use of a positional system of numerical symbols, whose advantages could be summarized in their compactness, the few symbols that it requires, the facility with which is learned, the speed with which can be read or written and the simplicity of the calculation algorithms that it provides, insisting on the peculiarity of this system: the hidden execution of two algebraic operations (addition and multiplication) as well as the existence of a privileged and arbitrary number (ten).

2.1.2 *Phase 1.* Arguing on the necessity for the introduction of rational numbers, the naturalness of symbols such as 1/9 to represent them is apparent and the interviewee should be put to task into producing a positional system of symbols for those numbers. Without any special encouragement, the student executes the division that the symbol 1/9 suggests and arrives to expressions such as 0.1 or 0.111... which are recognized as representative of 1/9. We ask him to produce an explanation for in what sense both symbols represent the same number. Choosing 0.111... as the symbol, we ask him to assign a meaning to the tail of dots which at this stage he summarizes as "whenever a division is executed a '1' will appear".

Now we want him to analyze in what sense the symbol 0.111... can be considered as positional, suggesting he interprets 1/9 as a length of a segment giving him two options: either divide a unit length segment in nine equal parts (which will not produce any illumination to on what we are aiming at) or divide the segment in ten equal parts and superpose the previous segment. He locates the length represented by 1/9 in that partition and after some zooming, new partitions, mental reflection and calculation, it will take him to the indefinite sum $1/10 + 1/100 + 1/1000 + \cdots$ from where a positional interpretation (involving a further algebraic operation, division) is possible despite the fact he might not able to execute that indefinite sum directly, which lead us to request him to specify coincidences and differences with the positional system for whole numbers and to which degree a certain ambiguity still exists.

Finally, we request him to produce a procedure to recover the rational number from its symbolic decimal expression.

2.1.3 *Phase 2.* Confronted with the symbol $0.999\ldots$, we ask him to produce the rational number that supposedly generated it. After several attempts (the first clumsy and subsequent ones more delicate, following a certain procedure) he will be convinced that such a rational generator does not exist.

After clarifying that we wish to give a numerical sense to the symbol at issue, he will reach the conclusion that another interpretation to the symbol (not based exclusively on the three constituent algebraic operations of the positional system of symbols) has to be provided. We will point out the convenience that the new interpretation, whichever it might be, must be consistent with what we have done until now.

Basing ourselves on the positional interpretation that $0.999\ldots$ would have, we will propose to study that symbol progressively, that is to say, understanding it as a dynamic process 0.9, 0.99, 0.999, ... and with the help of the mathematical assistant (see Appendix 2) we will use a two-dimensional visualization making him distinguish between what is a term of the process, i.e. a number $(0.999\ldots9)$ and what, at the moment, is not $(0.999\ldots)$.

Returning to $0.111\ldots$ and to its visualization, the student will agree that it can be also understood as a process. To the question of where this process is aiming at, he will conclude that he is dealing with a process of approximation with a final end product (a concrete number 1/9), this specific end product guaranteed by the division procedure that generated the process. Thus, $0.111\ldots$ will be interpretable as a process, but also as the end product of that process and we will emphasize that last interpretation as the one in which we are interested when dealing with $0.999\ldots$.

2.1.4 *Phase 3.* Returning to $0.999\ldots$ as a process, we will ask him if it also would be possible to be interpreted as an approximation process as before and if there is an end product to that process or not. Confronted with its visualization he will conclude again that, supposing it stabilizes, either it will be the number 1 or a smaller number, suggested by the fact that each term of the process is strictly smaller than 1.

Visualization produces reasonable conjectures but hardly certainty. Since the impossibility to decide visually if there is end product or, supposing there is one, to determine which one is it, we will propose to analyze the situation algebraically, warning him that, if such an analysis is feasible, it will have to contemplate a novel algebraic operation, since our problem cannot be reduced to addition, multiplication and division.

Now the most delicate part of the experience takes place: the algebraic formulation of how to stabilize a dynamic process and that will run in several stages:

First, we will request him to explain what does it mean that a number B is larger than another number A arriving to "there is a backward movement from B that stays larger than A" and we will request him next to define, taking into account this formulation, what means that B and A are equal trying to obtain the verbalization: "any backward movement from B stays smaller than A".

Second, we will request him (i) to try to produce a tentative definition of when a process has a numerical end product and (ii) if it can be verified which one it is. Concerning (i) we look for a verbalization such as "the end product is a number which is larger than any member of the process but also the closest to all of them" and, regarding (ii), the desired verbalization is "if there are several possible candidates for end product, we choose the one (call it B) satisfying that, for any backward movement from B, there is a member of the process greater than the number produced by the backward movement".

Finally, in order to confirm that such a definition has been assimilated, we return to our visualization of the process (which until now was merely illustrative) to redesign it as a working tool allowing us to observe if the student is able to translate the previous algebraic definition to a series of linked images: the end product appears as a horizontal straight line, the backward movement is staged as a two-dimensional band and the terms of the process are points of the plane. The visual interaction between different bands and the number of terms located in them will produce the logical game of the quantifiers. Considering $0.999\ldots$ not as a process, but as the numerical end product of that process, the student will accept that the equality $0.999\ldots = 1$ is justified.

As a way of checking out if the student has interiorized the new sense that has been given to the tail of points, we will produce the paradox of Achilles and the turtle adapted to our situation.

If the educative experience has been successful, moving out of context the student will be prepared to understand the definition of the limit of a bounded monotonically increasing numerical sequence and, adapting the methodology, of a convergent numerical sequence (see [2]).

2.2 Content of the questionnaire

2.2.1 *Questions corresponding to the introduction*:

(1) What is a symbol for you?

(2) Our immediate goal is to count how many objects are in a certain collective, let us say, a flock of sheep, at an ancient time where no symbols for numbers had been developed. In order to be sure that no sheep is missing after a day of pasture, how would you proceed? Which would be a suitable system of symbols?

(3) The student suggests a simple system of symbols based on accumulation of sticks, like ||||, to which we point out the obvious disadvantages of such representations when dealing with large collectives; we invite him to produce new simplifying symbols, such as V, X, C, M. We point out that, as the collective increases in size, another difficulty may arise, such as the amount of symbols to memorize, suggesting that a certain "methodology" in the creation of symbols would be advisable.

(4) To counter this difficulty, we propose a way to tackle the problem of how to represent large amounts symbolically by using only a few different symbols, let us say, ten of them suggested by the anatomy of our hands. We proceed with the idea of the positional system. We comment on how many algebraic operations are involved implicitly in this symbolic system. We encourage the student to produce arguments on why such a system constitutes a huge advance in dealing with numbers.

2.2.2 *Questions corresponding to first stage*: We have already been able to develop a suitable system of symbols to count objects (whole numbers). It is positional in the sense that the same symbol means something different depending on the position where it is located and two algebraic operations are hidden and executed in a certain order.

(1) Suppose that you want to distribute a cake between nine friends. Which symbol would you attach to the amount of cake that corresponding to every one? Once the natural symbol is given, what does the bar in the symbol 1/9 suggest? Calling such expressions rational numbers, could such symbols be used to represent lengths?

(2) In the computer screen we make two windows available: a visual one with in and out zooming capabilities and a computational one. In the visual window, choose a conventional length as a unit length and divide it in nine equal segments; call L the first segment.

(3) Is 1/9 adequate to be assigned as the length of L? Does the bar in the symbol still mean division? Please proceed to execute algebraically the division: first in paper and later in the computational window (Fig. 1). Written in paper and depending on the student, different symbols for the same division procedure will appear, such as 0.1 and variants of 0.111 . . . . Depending on how many significant digits have been allowed to appear, the screen will show something like 0.11111111. Not an easy job to convince him that we want to settle for one and only symbol, for he sees no need for it. He will discard the symbol produced by the screen as incomplete. After some discussion on other alternatives, we ask him to settle for 0.111 . . . . He understands why "1" appears in the symbol but to avoid confusion with alternative expressions such as 0.11.., we point out that the election of precisely three "1" and dots is arbitrary. Just to be on the safe side we insist on which meaning does the student attach to the tail of dots at this stage of the experience (three more "1"?, indefinitely many "1?" . . .). Now we inform him that we are embarking in an argument to justify why such symbol might be considered as positional.

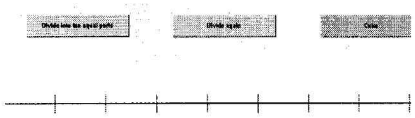


Fig. 1

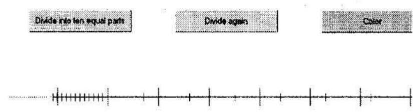


Fig. 2

(4) In the visual window, let us consider again a segment of unit length and proceed to divide it in ten equal parts and superpose our segment L. Locate its right endpoint in the subdivision, proceed to divide this subinterval en ten equal parts and locate again the right endpoint (Fig. 2).

Proceeding in that way and zooming as many times as desired to conjecture where to proceed with our next partition, we ask him to redirect his attention from the visual screen to the computational one to perform algebraically what is going on. The ever increasing length of the added segments constructed to match the segment L will translate algebraically in the "indefinite" sum $1/10 + 1/100 + 1/1000 + \cdots$ from where a positional interpretation for $0.111\ldots$ can be achieved. Four questions than are in order: Is it reasonable to condensate this addition process symbolically as $0.111\ldots$? In which sense one can interpret $0.111\ldots$ as a positional symbol? Which hidden operations are involved in this symbol? Although not physically possible to complete the aforementioned sum (just as the division was before), could we assign the symbol $0.111\ldots$ to the length $1/9$?

(5) Which are the differences (if any) between the positional symbols assigned to whole and rational numbers? We are mainly interested in his thoughts concerning the static versus dynamic interpretations.

(6) With our computational screen activated and considering several examples, we analyze operationally the positional symbols for rational numbers:

a) Is it certain that a period appears every time the division is executed in a rational number a/b?

b) Conversely, if a positional symbol with a period is given, how do you proceed to recuperate the rational number which generates this symbol? We do not proceed further if a procedure is not provided by the student (such as considering the period divided by as many nines as digits in the period).

2.2.3 *Questions corresponding to the second phase*:

(1) Now we have a look to the symbol $0.999\ldots$. We ask him to search for a rational number which by repeated division provides this symbol. After some painful work, what was first a strong feeling (namely $0.999\ldots$ does not appear to be the result of repeated division) becomes a certainty using the procedure developed in 2.2.2(6)b. If the tail of dots in $0.999\ldots$ stands for repeated division, $0.999\ldots$ has no meaning at all. But if a meaning to $0.999\ldots$ is going to be provided (let us say, a number or a length), the tail of dots has to stand

for something else.

(2) Turning to the positional interpretation of 0.111 explored in 2.2.2 which provided us with a dynamic process, we produce a window (see Appendix 2) showing $0.111\ldots$ now as a bi-dimensional dynamic process. Apart from his visual and symbolical meaning, we want the student to attach a numerical nature to what we are doing and thus we shall insist in what is the nature of such process, which the student will rightly identify as an *approximation* process with end product 1/9, this last affirmation guaranteed by the repeated division procedure mastered above.

Now we turn to the visual representation of $0.999\ldots$ understood as the process 0.9, 0.99, 0.999, ... Just to be sure that symbolic confusions do not arise, we ask the student to state the difference between $0.999\ldots9$ (which he identifies on the screen once a concrete number of nines is specified) and $0.999\ldots$ (unable to be identified in the screen).

(3) What differences and similarities do you see between both processes $0.111\ldots$ and $0.999\ldots$?

2.2.4 *Questions corresponding to the third phase*:

(1) Our aim is to provide the symbol $0.999\ldots$ with a numerical meaning, which clearly cannot be done neither using exclusively the three algebraic operations nor dealing with it as a process.

Have a look to our window and supposing for the time being that such a process is approximating some number, would it be a number near or equal to 1? Since visualization as we have it now is not enough to produce an exact answer, we try to convince the student that what is needed is a tool to force the dynamic process to rest and for that purpose we shall develop a new algebraic operation.

Let us take a detour.

(2) How do you check that a number B is bigger than another number A? After several attempts and trying to be precise with the use of the common language in questions and answers, we will instigate the formulation "there is a backward movement from B that stays greater than A".

(3) Next we will request him to define, taking into account the former formulation, what means B and A being equal. A bit of patience is required to arrive to a verbalization such as: "any backward movement from B stays smaller than A" with special emphasis in what "any" means. It takes some time to arrive to such formulation for inequalities logically conditioned appear and that is usually the first time for him.

(4) We will ask him if, in the event of having to apply this definition of equality, is it reasonable to limit ourselves to the set of backward movements exemplified by 0.1, 0.01, 0.001, ... to verify the equality of numbers.

(5) Now we return to the process 0.1, 0.11, 0.111, which he knows approaches 1/9. One candidate for end product is obviously 1/9. Imagine you have another

one; could you adapt the former verbalization of equality of two numbers to both possible candidates to end product of the process in question in terms of backward movements from the candidate to the members of the process to decide which one is it? Exhibiting our window, we mark the end product as a horizontal line, the backward movements as two-dimensional bands and the terms of the process as dots, and we encourage him to play with different backward movements and zooming to make clear to him what happens.

(6) The visual play should encourage him to produce a definition of when a process has a numerical end product and how can he verify which one is it, looking for the verbalization “a process has numerical end product B if for any backward movement from B, there is a term of the process greater than the number produced by the backward movement”.

(7) To check that such a definition is in his armoury, we request answers to the questions: Taking bands of smaller amplitude, how many points of the process are located within them? Is there any relation between band and points lying in it? What has to happen to be sure that your selected end product is really the one you are looking for? Is it enough to check a few bands or the condition has to be fulfilled for any band?

(8) Turning to the process 0.9, 0.99, 0.999, . . . we ask him to select possible candidates for end product: numbers larger than 1 will immediately discarded. Numbers smaller than 1 will fall once definition (6) is applied. The candidate 1 satisfies the definition and we are done.

(9) Now to Zeno’s paradox: $A\chi\iota\lambda\lambda\varepsilon\upsilon\sigma$ (Achilles) is challenged to a footrace with a turtle (suppositions: constant speed; Achilles runs at 10 footsteps/second; the turtle at 1 footstep/second). They start running at the same time, but Achilles allows the turtle a head start of 900 footsteps. The respective positions of Achilles and the turtle are shown step by step:

Simultaneously the seconds taken by Achilles to reach each of the positions that the turtle just left are indicated: 90 (P1), 99 (P2), 99.9 (P3), 99.99 (P4), . . .

2.3 Development of the clinical interview: different levels of reasoning.

During our study twenty clinical interviews were carried out. Students of the first year of Technical Architecture and Telecommunications Engineering of the University of Seville participated voluntarily (2006, 2007 and 2008). They agreed to the audio recording of the interviews and also to the use of their corresponding transcriptions in our analysis. Their previous formation in Mathematics was the habitual one of students completing the Baccalaureate in Spain.

The questionnaire shown in the previous section is designed to be used in clinical interviews as a reasoning and learning tool allowing the interviewer to detect progresses and/or deficiencies in their ways of reasoning. It is semi-structured so that the interviewer can intervene when the line of reasoning of the student is way off the intended purpose. Not all questions and answers carry the same weight in our study: several of them are crucial indicating unequivocally which line of reasoning is being pursued by the student. The analysis of the answers obtained through interviews allowed us to classify students in three generic types which we call A, B and C type students according to their progress in understanding. About one third of our students could finally be classified as C type.

2.3.1 *Type A students.* Type A students present very serious deficiencies either when manipulating concepts related to potential and/or present infinite or when dealing with what a symbol means. Frequently they respond to what Sierpinska [5] classifies as "finitist": those which do not admit the existence of infinity or those that when they say "infinite", they really think "very big". Type A students begin having difficulties at the first stage of the interview, when we introduce the tail of dots identifying 0.111 . . . as the result of the repeated division of 1 by 9 (see 2.2.2(3)). Being 0.1 their preferred symbol for the result of division of 1 by 9, it takes some struggle to make them accept that both symbols stand for the same process and they do not accept kindly that the tail of dots in 0.111 . . . may stand an unending succession of "1".

Professor: What does it mean the bar in the symbol 0.1?

Student: It means that a "1" will appear every time we perform the division.

Professor: Would it represent the same as 0.111 . . . ?

Student: Well, only if you add "n" to it (writes 0.111 . . . n)

Professor: What does n mean?

Student: Any number? For instance 0.111 . . . 1. Many "1"? I do not know: As many "1" as necessary?

Professor: As necessary for what?

Student: As many "1" as the number of times I proceed to division.

Professor: But how many times do you perform the division?

Student: As many times as desired, but I only can perform it a certain number of times.

Professor: Thus, a never ending tail of "1" is not what is meant with the symbol 0.111 . . . ?

Student: Never ending tail . . . ? There is nothing of that sort, it has to end. Isnt it? I do not know what 0.111 . . . means, unless it stands for 0.111 . . . 1; then, all right, but I prefer $0.\underline{1}$, is more compact. When you write 0.111 . . . I think of $0.\underline{1}$.

We try to break our deadlock using the computer screen to deal with the equality $1/9 = 0.111\ldots$ (see 2.2.2(4)). Contradictions are unavoidable.

Professor: Look at the image in the screen. What interpretation can be given to this equality?

Student: 0.111 . . . understood as 0.1? Hmm, the length increases and increases. It never stops. Does it tend to infinity? I do not know.

Professor: What is tending to infinity?

Student: The lengths 0.1, 0.11, etc . . .

Professor: 0.111 . . . becomes infinitely large?

Student: Sure because there are infinitely many "1" since it never stops. Now I understand: For you 0.111 . . . is something infinite because there are infinitely many "1" but, of course, it is not 0.1. That is!

Professor: And that means that 0.111 . . . is infinitely large but 0.1 is not?

Student: Yes

Professor: Then, 0.111 ... does not equal 1/9?

Student: Sure, 1/9 is 0.1. But, on the other hand, 0.1 is one divided by nine but ... since it is not possible to divide exactly in nine parts, the length is not going to be exact. We cannot divide a straight segment in nine exact parts, hence one has to stop when writing 0.111 ... .

When those kinds of arguments appear, it makes no sense to proceed further with our script concerning the meaning of 0.999 ... since we havent been even able to overcome their resistance to the symbol 0.111 ... . Students having those types of obstacles are able to conduct the operations of algebraic nature without trouble, rounding off infinite decimal tails and their difficulties and deficiencies will only be detected when dealing with infinity in any of its manifestations. The idea of real number is off limits for them.

Some students accepting the equality $1/9 = 0.111\ldots$ but having an obvious lack of computational maturity have also been classified as type A, namely those refusing to admit that a decimal expression such as 0.999 ... does not arise from division: confronted with the symbol 0.999 ... and after laboriously searching for a fraction which by division gives 0.999 ... and failing to find one, are incapable of producing convincing procedures on how to deal with this situation. Even if we provide the algebraic arguments for the non-existence of such a fraction, they stubbornly refuse to acknowledge our arguments and remain convinced that such fraction exists and hence there is no intellectual need for looking further for other interpretations of 0.999 ... .

A type students are not equipped to proceed to the second phase of our questionnaire and we thought that, once detected, the most advisable thing to do was to try to overcome their misconceptions in the classroom, but no through the interview.

2.3.2 *Type B students.* B type students are able to perceive and handle potentially infinite processes and admit equality between 1/9 and 0.111 ..., because they observe that, when dividing 1 by 9, there is a never ending step by step process which provides the infinite tail and that a valid short cut representation for what is happening is what the tail of dots stands for. Nevertheless, when we raise the idea of considering the process complete and look for what its final end product is, difficulties begin to arise since they do not admit the idea that a process can be completed. For them the tail of dots indicates a process on the move but without end. If we cannot induce them to change this perception, concepts such as limit of a sequence/function will not be in their reach. What are we dealing with is what Sierpinska [5] classifies as "potentialist". In our experience, these students do not complete successfully the second phase of our questionnaire.

During interviews B type students sail calmly through our first phase. Mainly they start using the symbol $0.\underline{1}$ but change to 0.111 ... with no trouble. They accept that 0.999 ... is not the result of any quotient of whole numbers and some even got to perceive 0.999 ... as an approximation process and start to direct their thinking in that sense:

Professor: Is it possible to express 0.999 ... as a quotient of whole numbers?

Student: No. Professor: What meaning can we attach to 0.999 ...?

Student: Something which approaches a number close to one. A bigger decimal expression.

Professor: Something? Do you mean that $0.999\ldots$ in itself is not a number?

Student: Surely it is not.

Students classified as B type interpret the tail of dots as an indication of a potentially infinitely process but do not admit the idea that the process can be considered complete in some sense and therefore that the symbol may need some reinterpretation. Let us see the reasoning of a student of this type:

Professor: Turning to our visualization of $0.999\ldots$ (as we did with $0.111\ldots$), can you provide some value attached to it?

Student: If you insist in $0.999\ldots$ having a value, it would be one (looking at the screen) or perhaps some smaller number.

Professor: Is it one or not?

Student: (uncomfortable) I do not see anything convincing in the screen. It may be one or not. Clearly it is not one, because there is no fraction generating the expression.

Professor: Thus, $0.999\ldots$ does not represent a number.

Student: Not an exact number but something approximating one or something smaller.

Professor: What is the difference between the symbols $0.999\ldots9$ and $0.999\ldots$?

Student: (after some reflection) $0.999\ldots9$ is a number, depending on how many nines are presentif there are seven nines, it should be 9999999/10000000 (points out a dot in the screen). But $0.999\ldots$ is not a number.

Professor: $0.999\ldots$ is then an approximation process?

Student: Yes.

Professor: With no numerical value attached?

Student: None.

Professor: And when all nines are present?

Student: It cannot happen.

In order to have a student of B type advancing in his level of reasoning it is necessary to remind him that $0.111\ldots$ was seen as a number and clarify him that we want a numerical value also attached to $0.999\ldots$ and therefore a new interpretation of the symbol is mandatory, namely how to proceed from a process to its final product and which algebraic operations will be needed to go from a moving process to its rest. B type students are not able to proceed to our third phase.

2.3.3 *Type C students.* Type C students are those reaching a higher level of reasoning. Throughout the interview they show a flexible way of reasoning enabling them to attach different meanings to the same symbol in order to attain the objective: after rejecting the symbol as a division process, they accept the approximation process which gives a clue where the process is going to and then proceed, with our guidance, to reformulate the symbol as the final product of this process, understanding the algebraic and logical considerations involved.

These students exhibit the same difficulties as B type students, but are able to overcome them:

Professor: Is the symbol $0.111\ldots$ meant to describe an infinity of “1”, even if you cannot write them all?

Student: It is not possible to write all the ones.

Professor: But can we agree that three dots stand for all of them?

Student: Yeah, all ones are needed to represent 1/9. Since it is not possible to write all of them, 0.111 . . . will do as a representation

Professor: But are all there?

Student: Yes. In our last phase we will concentrate in the interpretation of the symbol 0.999 . . . as the final product of the corresponding process of approximation and how to insure which this final product should be.

The reasoning of C type students evolves through several stages.

Professor: How could we interpret the symbol 0.999 . . . visually?

Student: The process is of an approximative nature 0.9, 0.99, 0.999 and so on. These are the approximations. It comes closer and closer to some numerical value.

Professor: You just mentioned three stages in the approximation process, Can we have more?

Student: (Sure) As many as you want. If I were able to write them all, I would reach the desired number.

Professor: But since you are unable to write them all . . .

Student: The symbol stands for all of them.

Professor: Also for its end product?

Student: Yes, if you want to change its meaning again (laughs).

Professor: What meaning would it be?

Student: The end product of the process.

Professor: Can you be more specific?

Student: Something like "the number which is closest to all approximations and larger than all of them"?.

Professor: Very good, indeed!

At this moment the majority of students reaching C type assume that 0.999 . . . has a numerical value. The visualization suggests that this number cannot be larger than one. Now is time to reflect on what does it mean that two numbers are different.

Professor: How do you understand that a number is smaller than other one?

Student: When . . . the larger number marks a bound not reached by the smaller one.

Professor: Do you mean that there is what we may call a certain distance between them?

Student: Yes.

Professor: Would it be possible to rephrase this by saying that it is possible to back down from the larger one without reaching the other?

Student: It can be seen that way.

Using the idea of the backward movements, students are able to verbalize conditions of equality and inequality between two numbers.

Professor: How many backward movements do you need to ensure their inequality? Is it enough to assume that there is one backward movement of this type to ensure that two numbers are different?

Student: Yes.

Professor: Is it possible to do this when both they are equal?

Student: No, when carrying out a backward movement from one of them, we shall land behind the other.

Professor: How could we formulate the equality of two numbers using backward movements? Say the numbers are x and y.

Student: Each backward movement from x produces a number smaller than y.

Professor: By "each" you mean ... ?

Student: If one backward movement from x produces a number larger than y, then they are different. So you have to check all backward movements to be sure.

Professor: Suppose now that you are dealing with a process (like $0.111\ldots$ or $0.999\ldots$) with an end product x. As you stated before, x should be the number closest to all approximations. According to our considerations on inequality involving backward movements and so on, can you provide a definition of "closest"?

Student: (after some tries) Any backward movement from x is smaller than a certain approximation of the process.

Now the student looks the screen where the process $0.111\ldots$ is shown and applies this condition to deduce the equality between $0.111\ldots$ and $1/9$:

Professor: On one hand you have a visualization of a process, namely $0.111\ldots$. On the other you know the equality $0.111\ldots = 1/9$ to hold.

Student: Do you mean $1/9$ is the end product of the process?

Professor: You tell me. Since $1/9$ is larger than any approximation $0.111\ldots1$ is it fair to assume that $1/9$ is the closest number to all approximations and hence can be considered as the end product of the process $0.111\ldots$?

Student: Yeah, if we check all backward movements away from $1/9$ ...

Professor: In order to check them, would it be enough to settle with backward movements of amounts 0.1, 0.01, 0.001, ... ?

Student: (after some hesitation) Yes, it seems reasonable.

Professor: Why? Suppose you want to consider a backward movement such as 0.05.

Student: 0.05 is larger than 0.01 and if what we want happens for 0.01 it also happens for 0.05.

Professor: Right. Using your definition of equality and assuming $0.111\ldots$ has a numerical value, is it true that $0.111\ldots$ equals $1/9$?

Student: (manipulating the assistant) Yes. If the numerical value of $0.111\ldots$ is supposed to be the closest number to all approximationsit seems that $1/9$ is the one, because considering a larger number and making a backward movement produces an interval without approximations and considering a smaller one, ... , no, it does not work.

We continue our interview relating the condition of equality that was produced by the student with the amount of points (approximations) that are inside and outside the intervals that determine backward movements:

Professor: Please, try to state a definition of what an end product of an approximation process would be.

Student: Can I use $0.111\ldots$ to formulate it?

Professor: Yes as long as it applies to other processes as well.

Student: Taking several backward movements away from $1/9$ ...

Professor: Why $1/9$?

Student: Because I know the end product to be $1/9$.

Professor: What if you did not know it?

Student: I have to guess.

Professor: So, you need a candidate to check upon.

Student: (hesitating) Yes, without a candidate there is not much I can do.

Professor: And considering other possible approximation processes, how would you guess?

Student: Recovering the rational number generating the process: thats my candidate.

Professor: Yes. But, if you are unable to recover ...

Student: Why should I be unable ... ?

Professor: 0.999 ..., remember?

Student: You are right. Well, performing a visualization ... ? Not a very exact procedure but for starters ...

Professor: Fair enough. Lets go back to 0.111 ...

Student: (manipulating the assistant) Right. I know that 1/9 is our candidate (draws a line) — Any backward movement away from 1/9 produces a band ... I select one band of width 0.1, another of width 0.01 and so on.

Professor: Let us look at one of them. How many approximations of the process inside and outside the band?

Student: Some inside and some outside.

Professor: Please, be more precise.

Student: The band contains infinitely many approximations and a finite number of them lie outside.

Professor: That would happen for a certain backward movement?

Student: No, for all of them. Otherwise 1/9 would not be the right choice for end product.

Professor: The number of approximations outside depending on ...

Student: The scope of the backward movement, the width of the band. The smaller it is, the larger the number of approximations lying outside.

Professor: So, there is a connection between width and outside lying approximations?

Student: Yes, as I said the smaller the width, the bigger the number of approximations outside.

Professor: But always an infinity of them lying inside.

Student: Yes, but a smaller decreasing infinity (laughs).

Now is time to turn to 0.999 ... .

Professor: Is 0.999 ... a process with an end product?

Student: (looking at the screen) 0.999 ... equals 1.

Professor: Why?

Student: Its pointless to check on numbers different from 1. I am backing down away from 1 and the approximations lying outside ... are ... a finite number and an infinite number of them inside.

Professor: Does it happen for a certain backward movement or for anyone?

Student: For anyone.

Professor: Your conclusion is ... ?

Student: Positive, 0.999 ... is 1.

When we ask what happens with a symbol like 1.999 ... their answers gives us a more extensive idea of the mental image used:

Student: It equals 2, because if we start making backward movements away from 2, the distance between numbers of the type 1.9, 1.99, 1.999 and 2 is becoming smaller and smaller.

There will be a moment when this distance is negligible in the sense we mentioned before. Once the student has clarified his ideas and expressed the condition that must be fulfilled to ensure that 0.999 ... equals 1, we raise Zeno's Paradox regarding the race between Achilles and the turtle analyzing the successive positions reached by Achilles and the turtle and the time needed to reach them.

In the following fragment it can be observed how students of C type apply their definition to solve the paradox:

Professor: Are we here in the presence of an approximating process?

Student: Sure, exactly as we did before.

Professor: How could we express the end product of that process?

Student: End product, hmm ... one hundred seconds.

Professor: Why so?

Student: Considering $99.999\ldots$ the end product is 100, the argument the same as before.

Professor: How much time needs Achilles to reach the turtle?

Student: Positive, ... one hundred seconds.

Once the interview is finished, type C students commented on their own surprise on what has been achieved.

Appendix 1. Some commentaries on the literature on the subject

Several authors have considered the question of how to deal with $0.999\ldots$ Richman [6] analyzes the meaning of decimal numbers and brings to light the problems that arise when students operate with infinite tail decimals in the same way as if the tails were finite.

Tall [9] describes factors of conflict and tendencies that frequently lead to catastrophes and that arise in the minds of students when confronted with infinite processes, as in the interpretation of $0.999\ldots$ In this sense he points out the strong held belief that real numbers have an unique decimal expression and hence $0.999\ldots$ has to be different from 1. On the other hand, the verbal definition of limit causes conflicts due to the colloquial meaning of some of the expressions that it involves. Tall indicates that resolution of those conflicts sometimes produces higher levels of reasoning in students.

In the context of APOS theory, Dubinsky, Weller et all [1] study the errors and difficulties related to the understanding of the symbol $0.999\ldots$. According to their analysis, 1 it is an object that the students have encapsulated whereas $0.999\ldots$ is seen by many individuals as a process. Until $0.999\ldots$ is not seen as an object and compared to 1 difficulties cannot be resolved.

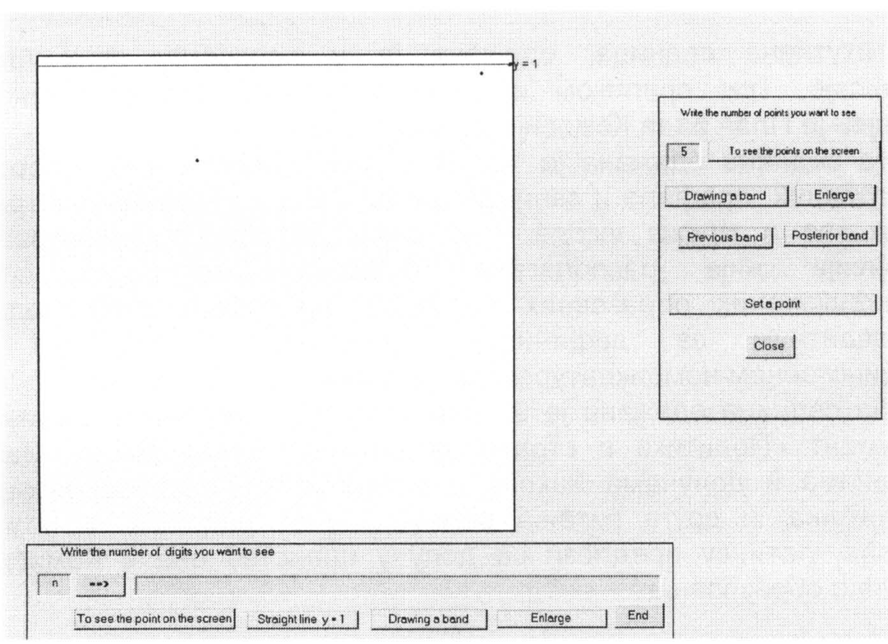
Sierpinska [7, 8] indicates that the factors that seem to determine the attitudes of students towards the equality $0.999\ldots = 1$ are their attitudes towards the mathematical knowledge and the infinite. In this sense she classifies the conceptions of infinite of students in "finitist" (if something is infinite then it is limitless, not bounded) and "infinitist" (they reject the idea that if something is infinite then it must be limitless) and within each one of these groups further distinctions are possible.

In the "finitist" camp there is a common characteristic: the non acceptance of infinity: either they reject the idea that something can be infinite, since in this world everything is finite or infinity is only admitted as a mental construction in mathematical theories. With regard to the equality $0.999\ldots = 1$, in both cases the equality is unacceptable; either is an abstract idea and the existence of a infinite decimal tail is inadmissible.

In the "infinitist" camp, students are shown to believe in two types of infinite, the bounded and the unbounded infinite and classify the subtypes that detect in this group. Students called "potentialist" see $0.999\ldots$ as an infinite process that is being constructed in time approaching 1 but never reaching it. Students classified as "actual potentialists" admit that after an infinite time the succession will be completed and all their terms will be available. Thus $0.999\ldots$ is seen as arising in time and when all terms are present it will reach 1 or the last number before 1. Sierpinska denominates "actual actualist" to the third subtype within this second group. For its members $0.999\ldots = 1$ is a kind of agreement reached through a mathematical proof based on conventions established previously. Sierpinska indicates that attitudes towards the mathematical knowledge can constitute a serious obstacle and that the appearance of mental conflicts related to those obstacles can be a departure point to overcome them.

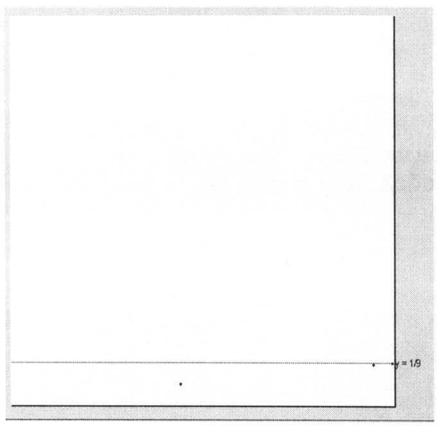
Appendix 2. Visualization and mathematical assistant

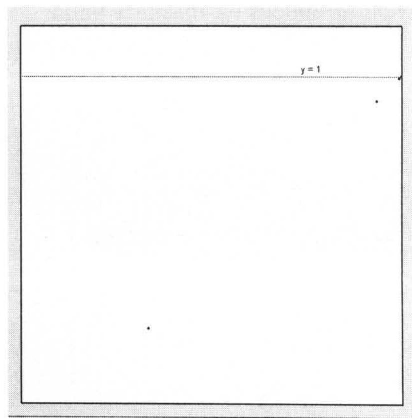
Thinking in $0.999\ldots$ as an approximation process and eventually understand it as the final product of this process as well as having a suitable candidate for this approximation is helped by the use of a mathematical assistant producing a screen where as many approximations as desired can be visualized. For this purpose we propose a visualization similar to the one described in Navarro and Pérez Carreras [3] (see figure below):



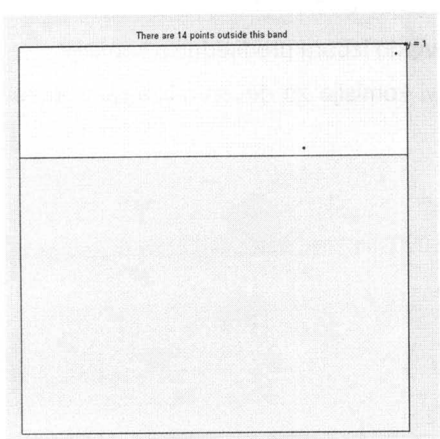
In order to help the student when dealing with processes we have used an interactive screen (see figure) designed with a mathematical assistant (Matlab 5.3) whose use does not entail familiarity with the program.

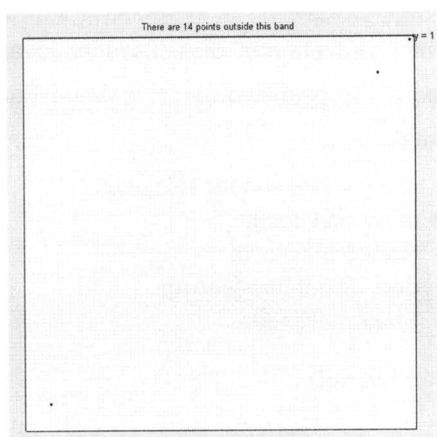
The members of the process under study appear in the upper left window and correspond to the points $(1 - \frac{1}{n}, a_n)$. A horizontal line shows the candidate chosen by the student as end product of the process. The next two figures show processes $0.111\ldots$ and $0.999\ldots$.



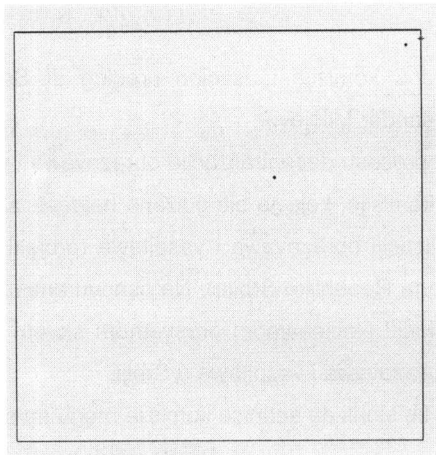


The screen is interactive in the sense of allowing the student to use it as a tool. Just with mouse and educated guesses the student can select how many members of the process are to be seen as well as select different candidates for end product; zooming is also a possibility in order to have a better view of where the process is heading to. Horizontal bands can be drawn (see figure) and isolated as squares whose side has the same length as the width of the band (see figure):





On those squares different candidates of end products can be selected: the figure shows the process $0.999\ldots$ and an erroneous candidate which can be discarded drawing more members of the process until the right candidate is considered.



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