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PHASES OF MATHEMATICAL MODELLING AND COMPETENCE OF HIGH SCHOOL STUDENTS

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Abstract. Competence of mathematical modelling are these days an important part in person's competence. This paper deals with student's problems with mathematical modelling as a part of problem solving. Three phases of mathematical modelling are suggested and some types of models are described in this paper. Presented results are based on a pedagogical survey which is characterized here too.

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Key words and phrases: Mathematical modelling, phases of mathematical modelling, types of models, competence of mathematical modelling.

Introduction

Due to technological progress the world has become more and more complicated. We stand in front of new and heavy problems that need to be solved. These are the problems the solutions of which are not universal and are not possible to be learned. Many solutions require specific information that cannot be learned by heart due to information progress. Even these reasons show that it is more important to focus on identifying solution conditions and not results themselves when solving problems. Teaching the pupils should focus more on improvement of competence that could become a key and not only to enrich knowledge structures. These facts became obvious and that is why they should be included in curriculum changes.

In this article we focus to competence of mathematical modelling as a part of competence that is necessary for problems solving. We deal with these competence in the 2nd chapter. In the 1st chapter we describe the process of mathematical modelling which is divided into three periods. In the last chapter we briefly describe the research and we interpret gained results.

1. Mathematical modelling

Mathematical modelling is a cognitive method where original object or situation is substituted by a model and by examining this model we gain the information we would gain by examining the original object or situation. Modelling is an inseparable part of solutions of problems that need not be necessarily mathematical. That is why we include competence of modelling among one of the key competence.

The core of mathematical modelling is a careful examination of all information presented in the problem and according to given information, building of mathematical (abstract) model in the way of mathematical, and logical relations and other presentations that authentically describe given situation. Modelling process can be divided into three phases:

1. identification of model situation starting points,
2. construction of a mathematical model,
3. verification of the built model.

Identification of model situation starting points is the initial phase of modelling. In this phase, the relations between starting points are mainly characterized. First of all it is necessary to decide which input information is relevant and it is important to include them in the model and on the other hand omit the not important one. In this phase competence dealing with information plays an important role.

The next phase is construction of a mathematical model, i.e. conversion of gained information from the first phase to mathematical language—mathematization. The results are various mathematical presentations: various types of equations and inequalities, propositional functions, graphs, geometric figures and similar. This phase in modelling process is the most important one and it seems, according to experience of various researches realized in this field, like it is the most difficult one.

The last phase is the phase of the verification of built model. In this phase a suitability of the model is verified, i.e. whether the model corresponds to the given situation. Model cannot be in controversy with real situation and everything in the model must be held within mathematical rules and it must adequately describe original situation. In this phase, where mathematical model is interpreted retrospectively, demathematization is necessary. It is important even when interpreting the results of the solution of the model—explanation of gained solutions in the language in which original problem was presented. We do not consider model solution and results interpretation to be a part of modelling process itself.

Types of mathematical models

Separating a process of model solving from a process of modelling itself gives us the opportunity to divide models according to mathematical apparatus by which they were made, and according to presentations that were used by modelling. According to this criterion, we can simply talk about five types of models:

1. *Arithmetic model* - model is represented by, e.g. operation table, ordered tuplet, vector and similar.
2. *Algebraic-analytic* - model is represented by some type of equation or inequality, or equation or inequality system elements of which are variables, sets, functions, vectors, matrixes and similar.
3. *Graphical* - model is represented by graph expressing certain functional dependence.

4. *Geometrical* - model is represented by geometrical figures.
5. *Combined.*

Selection of model type (type of mathematical representation) depends on features of model situation—on input information.

2. Competence of mathematical modelling

In this part of the article we will deal with competence of mathematical modelling. First of all we will explain how do we interpret the terms *competence* and *key competence* and we will focus on the difference between them.

The term *competence* can be explained as unification of all knowledge, skills, capabilities and attitudes that one acquires during his whole life. Single competence enable their holder to act adequately in certain situation in certain field of human activity. *Key competence* are those competence that are usable in various fields of activities; they represent only a part of acquired knowledge, skills and attitudes. Resulting from this we can take key competence as multifunctional complex of knowledge, skills, capabilities and attitudes; understand them as an individual's potential to show knowledge, skills, capabilities and attitudes in various practical activities. To gain the attribute "key", the competence must be inevitable and useful.

According to studies of some publications (see [1], [6]) and researches (see [5]) that were realized in this field, and knowledge of mathematical modelling process, we suggest the following competence of mathematical modelling:

- focus on starting points of model situation,
- structuralize areas and situations that has to be modelled,
- "mathematization" (transferring "reality" to mathematical structures)—expose quantitative or three-dimensional relations and regularity of real situations,
- produce mathematical models,
- prove the model from perspective of real situation,
- think, analyze, present the model (including its borders or limitations),
- "demathematization" (interpretation of mathematical models in respect of "reality"),
- track and control the process of modelling.

3. Research

According to research, the inability of pupils to solve problems is mostly connected with their problem with modelling. In the research we performed in March 2007, and in which 398 3rd and 4th grade pupils of various high schools took part, we concentrated to individual phases of mathematical modelling. The goal was to find out in which phase students make the most of their mistakes, and this way identify the most problematic competence of mathematical modelling. In this research we focused to algebraic-analytical type. For this purpose we used following the problems.

Problem no.1. Water basin has two outlets which parameters are described in the following table.

	Outlet no.1	Outlet no.2
Outlet diameter	6 m	9 m
Turbine wattage	1250 W	1875 W
Absolute power	basin/45 hod.	basin/30 hod.

Using both outlets at the same time the basin would be empty in 18 hours. Emergency situation occurred, and it was necessary to empty the basin within 24 hours. Therefore, both of the outlets were opened simultaneously. However, the basin was empty only in 22.5 hours. It was discovered after analysis that outlet no.2 was jammed during the process and no water was flowing through this outlet. Calculate the number of hours for which this outlet was functioning after being open. Comment your steps.

Problem no.2. Water basin has two outlets. It would be empty in x hours by the first outlet and in y hours by the second one.

a) While emptying the basin various situations can arise. What situation expresses the following equation: $9\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{4.5}{y}=1$.

b) Count in how many hours will the basin be empty using the first outlet only and in how many hours will it be empty using the second one if the basin is empty in 12 hours using both outlets at the same time. Use the information from case a). Comment your steps.

In Problem no.1 we monitored how pupils are able to select significant information that is related to their competence, to focus *on the starting points of the model situation*. This competence is together with the one "*to structure the areas or situations that are supposed to be modelled*", and together with competence related to work with information, the most important one in the first phase of mathematical modelling. In this problem there are also other competence of mathematical modelling to be identified.

Problem no.2 a) was aimed to diagnose competence like *proving the model from real situation point of view, "demathematization" (interpretation of mathematical models in meaning of "reality")*. These competence are mostly applied in the last phase of mathematical modelling. However, demathematization is important when interpreting the result we achieve by solving created problem.

Problem no.2 b) was aimed to diagnose competence: "*mathematization*" (transfer of "reality" into mathematical structures)—*reveal quantitative or three-dimensional relations and regularity of real situations, create mathematical models.* It concerns the most significant competence exercised in the second phase of mathematical modelling.

Evaluation of research results

The level of competence for individual phases of mathematical modelling was judged according to students' successfulness (see Fig. no.1). Binary scoring was used for evaluation.

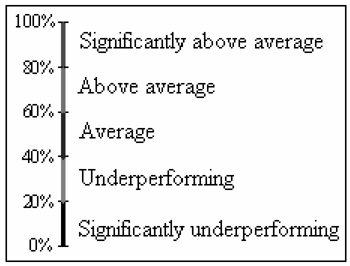

Fig. no. 1

For example students' competence used in the first phase of mathematical modelling are average if successfulness in this phase is higher than 40% but lower than or equal to 60%.

In the tests, students had the biggest problems with solving the problem no.1. Only 19,85% (79) out of all of the pupils created correct mathematical model for that. In this group 93,67% (74) of students correctly solved the created model. The rest of them did not deal with it at all (63) or were not able to solve incorrectly created mathematical model (256). As we mentioned before, this problem was related to competence *focus on the starting points of the model situation* and competence related to handling of information. In general we can say that students have big problems with tasks that are related to achieving data from more sources (table, information in text). They cannot fully focus on those data; they read them but did not understand them. This is closely connected to their disability to interpret given information correctly and hence they cannot work with them later on. They cannot link the given information; logical conclusion is what is causing the problems. The mistakes in models were caused by trying to use all of the information given in the table, or by recklessly combining that information (see Fig. no.2). That is why students were getting more equations than unknowns, or other way around, and even if they tried to solve them, no one came to any solution. According to these results it is clear that students do not analyze the situation, they do not think about hidden relations that lead to creation of mathematical model. It is obvious that students did not understand the core of the problem and thats the reason why they were not successful in selecting of relevant information and hence they were not able to solve the problem. Competences of students that took part in the research that were used in the first phase of mathematical modelling are evidently below-average.

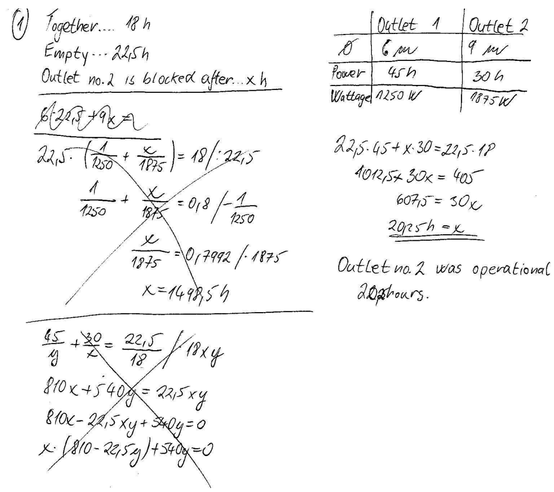

Fig. no.2. Student's solution of problem no.1.

We reviewed competence used in the second phase of mathematical modelling with the help of problem no.1, but we gained most of the information from the solutions of problem no.2 b) itself. The reason is that in this problem there were only useful data given and they were specified easily. So students could not fail on a fact that they were not able to focus on important starting points of model situation. We assume from the solutions of problem no.1 that if students understand the base of the problem they focus on right starting points and realize the relations between them, so creation of mathematical model is relatively successful (71,54%). Likewise the mathematization of the situation in problem no.2 b) was successful (74,37% – 296). However, coming to actual solution was not that optimistic. This problem was successfully solved only by 23,12% (92) of all the students. The base of the failure was that students did not know how they could use the information from case a) of that problem because they did not understand the equation. That was their mostly used written explanation (see pic.no.3). 92,39% (85) of students that solved this problem were successful also in interpretation of the equation given in case a) of problem no.2. Other successful solvers used the equation from case a) without being able to interpret it. It is clear form information gained from solutions of problem no.2 b) that students have competence like: "mathematization" (*transfer of "reality" into mathematical structures*)—*reveal quantitative or three-dimensional relations and regularity of real situations, create mathematical models* and they have these competence exceptionally developed.

Competence of proving *the model from real situation point of view, "demathimatization" (interpretation of mathematical models in meaning of "reality")* are the manifestation of high level of mathematical modelling. Mostly the first one mentioned is being omitted. The proof for that is solving of the models that have no relation with given situation, or even solving of the models that are insolvable. As to *the interpretation of mathematical models*, students encounter lot

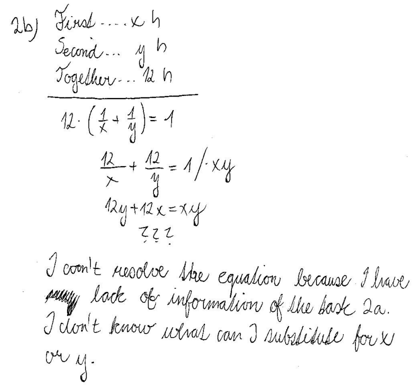

Fig. no.3. Student's written solution of problem no.2 b

of problems with this competence (see Fig. no.4). Only 23,12% (92) of students meaningfully interpreted given equation. Those students that managed to create mathematical model in problem no.1 had a big advantage. Huge relation was discovered between competence of *interpretation of mathematical models* and *focus on the starting points of the model situation.* However there is a low level of both of those competence. Competence *interpretation of mathematical models*, is below-averaged. One of the reasons could be the fact that students usually do not solve problems like these at math lessons.

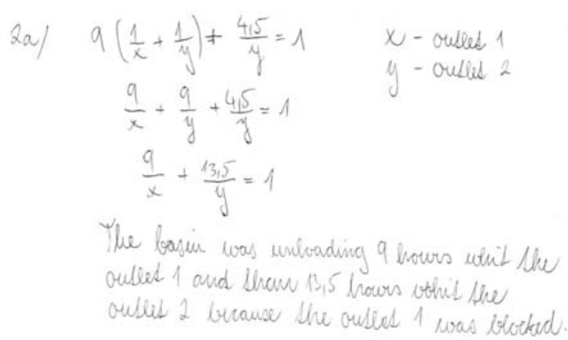

Fig. no.4. Student's solution of problem no.2a

Students encounter most of their problems in the first phase of mathematical modelling. The most problematic competence is to *focus on the starting points of the model situation.* Right after that comes *proving of the model from the real situation point of view, "demathematization" (interpretation of mathematical models in meaning of "reality").* This means that students can model situations which are easy specified (whereas they do not prove the model) and interpretation of the solution is easy. These situations are mostly made up and fake what is not useful for the students. To make mathematics studying more useful, and not for students only, it is necessary to re-evaluate its goals and methods. In our opinion mathematics study should focus to the fact that students should learn the following:

- understand features and see connections,
- model,
- form questions, problem solutions,
- form hypotheses, create conclusions,
- argue, explain and prove,
- listen, discuss and critically consider opinions,
- organize and plan the work,
- search information and know how to note, order and handle them,
- present and understand information in tables and graphs,
- cooperate and present results of the work as a team.

Conclusion

As we mentioned in the first chapter, mathematical modelling is significant cognitive method. It develops the ability of inductive and deductive thinking of students, competence like *problems solving, forming and proving of hypotheses, revealing of causal relations and connections between features.* Sure, mathematics study should not be only about making students learn how to solve daily problems but should focus on so called affective goals that are equally important. However this was not the subject of international testing like PISA, TIMSS etc., in which we did not come up well. That is why it is important to re-think the change of mathematics education and move from quantity to quality. Nowadays, it is not really useful to know a lot of things if you cannot use it practically. We can search for different information in a split of a second, but we have to know how we can work with that information later on. This is what the biggest problem for our students is.

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