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THE MAXIMUM NUMBER OF RECTANGULAR ISLANDS

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Abstract. In this paper we consider the combinatorial problem of rectangular islands by elementary means. The topic of islands and the methods for its investigation is suitable also for high school students, although some of the corresponding results are quite new. The arising questions need no advanced mathematical knowledge. Because most of the problems are of finitary type, experimental mathematics with computer support proves to be useful for the formulation of general conjectures related to the bounds of the number of islands in particular configurations.

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1. Introduction

1.1. Historical background

Recently the notion of Czédli-type islands caught attention of several mathematicians. The notion comes from information theory: it appeared first in [5]. Several generalizations of this notion gave interesting combinatorial problems. In two dimensions, Gábor Czédli [3] has determined the maximum number of rectangular islands: on the $m \times n$ size rectangular board for the maximum number of rectangular islands he obtained $f(m,n) = \lfloor (mn+m+n-1)/2 \rfloor$. His proof is based on a result in lattice theory [4], but now by [2] two elementary ways are also known to prove the same result. The topic of islands is still developing, and already many branches of mathematics are involved. The reader can find a complete synthesis in [13] with the most important results, which can be approached in elementary ways, so can be used in elementary and grammar schools too. The technique of gradual increase of the water level is applied successfully in medical image processing [1]. This outlook may also motivate our students to consider mathematics not as an abstract sterile world, but rather as the general technique of systematic problem solving with real applications.

The reader might find further details in the References, but for understanding the present paper, the reader need not read anything in advance.

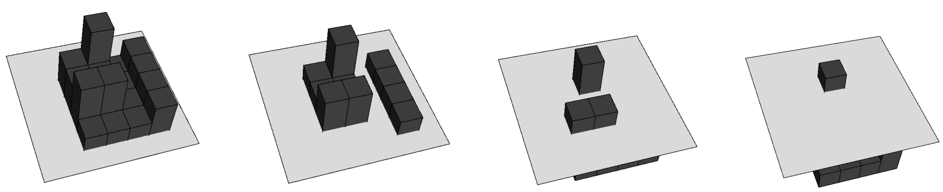
1.2. Didactic

The literature considers the maximum number of islands in the case, when the heights of the islands are not bounded. In this paper we try to find the maximum number of islands in the bounded height case. We will not solve the very general case, but we will be able to give the number in the $2 \times n$ and $3 \times n$ case, and point out some problems on larger boards.

The use of calculators and computer algebra systems is gaining greater and greater importance in education today. We should not be scared to use them. We apply here a computer program to get a hint on what the maximum number is, what is the construction giving this number; in short what we need to prove. We will be able to prove the conjectures by induction. Naturally, we may omit mentioning the use of a computer to get a *deus ex machina* type proof, but one of our goals here is to show the usefulness of using computers. The computers are playing an ever increasing role in our lives, their use in experimental mathematics [8] is nowadays inevitable. We do not replace proof by computer generated results, but rather use this result to find conjectures, to help the proofs.

2. Sequences and islands

Let us consider sequences of length n ($n \in \mathbf{N}$) containing numbers between 1 and k ($k \in \mathbf{N}$). We search for connected subsequences of this sequence having the property that the minimum number in the subsequence is larger than the numbers at the sides of the subsequence. Why are we interested in these "islands" and why are they called "islands"? Suppose that these numbers denote heights, we put these in a board containing $1 \times n$ cells and that water floods. First the water level is 0, but then it increases continuously. Then the found subsequences are exactly those, for which we can find a water level where the cells containing this subsequence is an island in the classical sense.



Example of the water flood in the case of a 4×4 board

For example (here $n = 12$, $k = 3$ and the islands are exactly the underlined subsequences):

$$2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 1,\ 2,\ 3,\ 1,\ 3,\ 1$$

In order to make it possible to ask questions about these subsequences, we need a mathematically correct definition of these islands, which can be the following:

Definition 1. We call *islands* those connected subsequences of a sequence of length n ($n \in \mathbf{N}$) containing positive integer numbers, that have the property that the minimum number in the subsequence is larger than the numbers at the sides of the subsequence. We call the minimum number in the subsequence the height of the island. We always assume 0's at the ends of the complete sequence, i.e. the complete sequence is always an island.

We can now formulate our first question.

Exercise 1. What is the maximum number of possible islands in a sequence of length n if the height is at most h?

This question seems hard at first, so we try to answer first some special, easier questions, which will also help us estimate the answer of this question. We can get an upper estimation if we do not bound the heights of the islands.

Exercise 2. What is the maximum number of possible islands in a sequence of length n?

We can construct n islands easily:

$$\underline{1,\ \underline{2,\ \underline{3,\ \underline{\dots,\ \underline{n}}}}}$$

But can't we have more than n islands? Checking the cases $n = 1, 2, 3$ we conjecture, that the answer is no. So we can state our first theorem and we will even try to prove it.

Theorem 1. *The maximum number of possible islands in a sequence of length n is n.*

Proof. Suppose, that we are standing on a cell of height k and the water level is just below k, say $k - \frac{1}{2}$. Then we are standing on an island. This island's height cannot be larger than k as we are standing on a point of height k not covered by the water. But the island's height cannot even be smaller than k, since the water level is larger than $k - 1$, all cells of height at most $k - 1$ are covered by the water and there are no island of height less than k above the water level. Therefore the island's height is k.

Let us do this for all cells in the $1 \times n$ board. Since we have n cells, this method gives us n, not necessarily different islands. On the other hand, if we have an island of height k, then standing on the cell of height k of the island we will get exactly this island by the above process. Therefore we can have at most n islands. ∎

This proof may seem a bit fuzzy, but it is mathematically correct. We will give a detailed proof later in the general $m \times n$ case. But this proof also shows, that if the numbers are different in the cells then we will always get n islands, no matter in what order the numbers are. This is easy to see: if the numbers in the cells are different, then the method gives us islands of different heights, therefore different islands and we have exactly n islands.

As we have already solved the case when the height is unbounded, let us continue with some special bounded case. If the maximum height $h = 1$, then we can have only one island, the whole sequence:

$$\underline{1,\ 1,\ \dots,\ 1}$$

What if $h = 2$? Since we want the maximum number of islands, it is easy to see that repeating a number in the sequence is not optimal: they are always in the very same islands, they can be collapsed together in one number and the number of islands does not change.

$$\underline{1,\ \underline{2,}\ 1,\ 1,\ \underline{2,}\ 1,\ \underline{2,\ 2,}\ 1} \quad \to \quad \underline{1,\ \underline{2,}\ 1,\ \underline{2,}\ 1,\ \underline{2,}\ 1}$$

Then the number of islands is determined by the first number in the sequence. For even n we will always get $\frac{n}{2}+1$ islands, while for odd n we will get more islands if we choose the first number to be 2, and in this case we have $\frac{n+1}{2}+1$ islands. With this we proved the following

THEOREM 2. *If $h=2$ then on a $1 \times n$ board $(n>1)$ we can have at most*

$$I_2(n) = \left[\frac{n+1}{2}\right] + 1$$

islands, where $[\cdot]$ denotes the greatest integer function.

Let us experiment with the $h=3$ case. Doing it by "hand" quickly proves that this is now a hard enough case to ask the computer for help: if we had $n=10$ then the number of possible cases is $h^n = 3^{10} = 59049$. Also, the still small enough $n=30$ is too large for even the computer to check all cases: $h^n = 3^{30}$ is about 206 thousand billion. But we can help the computer a little.

We write the program for the general case, i.e. the maximal height is h. First note that the number 1 must appear among the numbers in the sequence. If the minimum number in the sequence is $k \geq 2$, then subtracting $k-1$ from all numbers does not change the number of islands, but then there is at least one 1 among the numbers. We then search by recursion: let us try to put the number 1 on all positions from $1,\dots,n$ and assuming that this is the first 1 in the sequence, let us try the numbers $2,\dots,h$ on the positions before the 1 and the numbers $1,2,\dots,n$ after it. Also we keep track of the solved problems: how many islands we can construct if we are given the length and the bounding numbers of the subsequence, and can write some numbers in the subsequence between a minimum and a maximum height. Then if we run into the same problem during the recursion, then we do not need to recompute the result. We can simplify the computation even more, and the final algorithm has a running time linear in h and cubic in n. This is (naturally) much better than a running time h^n and the algorithm is even simple to code: it is less than 80 lines in C programming language. It is good enough to get conjectures: it can compute the maximum number of islands for $h=10$ and $n=1000$ in less than a second.

So, let us run the program for $h=3$. We get the following for different n's:

n	the sequence	the number of islands
2	1, 2	2
3	1, 2, 3	3
4	1, 3, 2, 3	4
5	2, 1, 3, 2, 3	5
6	2, 3, 1, 3, 2, 3	6
7	3, 2, 3, 1, 3, 2, 3	7
8	1, 3, 2, 3, 1, 3, 2, 3	7
9	2, 1, 3, 2, 3, 1, 3, 2, 3	8
10	2, 3, 1, 3, 2, 3, 1, 3, 2, 3	9
11	3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	10
12	1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	10
13	2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	11
14	2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	12
15	3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	13
16	1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	13
17	2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	14
18	2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	15
19	3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	16
20	1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3	16

The first thing to note is that the number of islands is not always n, for $n \geq 8$ it is less than n. This is not much of a surprise: we bounded the height and we would expect something like this. Looking at the number of islands more closely we find that they do a "hickup" for all n's divisible by 4: it is not increasing compared to the previous n. We know from the theorems above that the number of islands is n for $n \leq h = 3$ and we get the following:

THEOREM 3. *If $h = 3$ then on a $1 \times n$ board we can have at most*

$$I_3(n) = n + 1 - \left[\frac{n}{4}\right]^+$$

islands where $[\cdot]^+ = \max\{1, [\cdot]\}$.

We will not prove this theorem separately, as it will be included in the general case, but let us look at the construction the computer gave us for this case. Reading the numbers in the sequence backwards in the $n = 20$ case (for example) we see that every second number is 3, then every second number (on the remaining empty places) is 2 and the rest is 1. Although the computer sometimes did not follow this construction, it is easy to check, that this construction always gives the maximum number of islands (given by the computer, of course, which is not proved as yet). A little explanation (not a proof) why this might be the best construction: all 3's are island in themselves, all 2's with their neighboring 3's are islands (these islands are different so far), and the only island of height 1 is the whole sequence (as always). So the number of islands is $n+1$ minus the number of 1's in the construction, which is exactly $[n/4]$ (the $n \leq 3$ case must be handled separately, of course, as there is no 1 according to the construction).

Using the computer for $h \geq 4$ one will find the conjecture, that the maximum number of islands if $n+1-[n/2^{h-1}]^+$. And the construction is very similar to that of the case $h=3$: every second number is h, then every second number (on the remaining empty places) is $h-1$, then every second number (on the remaining empty places) is $h-2$, etc, and the rest is 1.

So let us prove our conjecture.

Theorem 4. *On a $1 \times n$ board using the maximal height $h \geq 1$ we can have at most*

$$I_h(n) = n + 1 - \left[\frac{n}{2^{h-1}}\right]^+$$

islands.

Proof. The main idea comes from the computer program: the recursion in programming is called induction in mathematics. So we prove by induction, actually by two inductions: first by h and second by n.

For $h=1$ and all n the statement is true:

$$n + 1 - \left[\frac{n}{2^{h-1}}\right]^+ = 1$$

and we can have only one island.

Let us suppose for induction by h that we proved the statement for heights less than a given h and for all n, and we want prove the theorem for h. We first note that the above construction gives n islands for $n \leq 2^h - 1$ (since only one 1 will appear in the construction), which is the absolute maximum, and of course

$$n + 1 - \left[\frac{n}{2^{h-1}}\right]^+ = n$$

in this case. Now we are supposing for induction by n that we proved the statement for numbers smaller than n using the maximal height h. Then we consider a sequence of $n \geq 2^h$ numbers.

We do know that the number 1 must appear somewhere in the sequence. Let us cut off a subsequence of length 2^{h-1} from one side of the sequence so that in the remainder part of the sequence there is a number 1. Then we have two cases.

Case 1. If there is a 1 in cut off part of length 2^{h-1}, then the number of islands is at most

$$\underbrace{n - 2^{h-1} + 1 - \left[\frac{n-2^{h-1}}{2^{h-1}}\right]} + \underbrace{2^{h-1}} - 2 + 1 = n + 1 - \left[\frac{n}{2^{h-1}}\right]$$

where the braced additives are coming from the induction, the -2 because the two subsequences are not islands (they contain cells with 1's), and the $+1$ is there because the whole sequence is an island.

Case 2: If there is no 1 in cut off part of length 2^{h-1}, then the number of islands is at most

$$\underbrace{n - 2^{h-1} + 1 - \left[\frac{n-2^{h-1}}{2^{h-1}}\right]} + \underbrace{2^{h-1} + 1 - \left[\frac{2^{h-1}}{2^{h-2}}\right]} - 1 + 1 = n + 1 - \left[\frac{n}{2^{h-1}}\right]$$

where the first brace comes from induction, the second because we must generate maximum number of islands in a subsequence of length 2^{h-1} with heights $2, 3, ..., h$ (there is no 1 in that part), the -1 is there because the first part is not an island (it contains a 1), and $+1$ stands for the whole sequence.

In both cases we proved that the number of islands is at most the stated number. To end the proof, we just check that the above construction gives the desired number of islands. ■

A natural generalization of this problem is to consider a rectangular board of size $m \times n$, and ask the maximum number of rectangular islands when we can use the heights $1, 2, \ldots, h$ only. This is a much more complicated question so let us start with some easier cases.

If $h = 2$, then one can easily find the answer: the maximum number of islands is

$$\left[\frac{m+1}{2}\right] \cdot \left[\frac{n+1}{2}\right] + 1$$

To prove this, we note first that the whole board is always an island, so in the following we are not concerned with it, if we say "island" we will always mean islands that are not the whole board. Then obviously, all islands contains only cells with height 2. The more cells an island contains, the less space is left for more islands, so the best way to obtain the maximum number of islands is that all islands contain only one cell, and we pack the cells as close as possible. Then the distance of these cells are at least 2, so in the best packing we have islands in every second cell both horizontally and vertically, like this (in the case 4×5):

2	1	2	1	2
1	1	1	1	1
2	1	2	1	2
1	1	1	1	1

Vertically there are $[(n+1)/2]$ columns in which we can write the height 2 and in those columns we can write 2 in $[(m+1)/2]$ cells. Multiplying these together and adding 1 for the whole board as an island, we get the said result.

If $h \geq 3$ then the problem is much harder. So, instead of "going higher" in the general case let us first consider a smaller board. Suppose, that our board's size is $2 \times n$ or $3 \times n$. We will be able to solve these cases by building on the $1 \times n$ case. To make the proofs easier to understand, let us start with some definitions, the first of which is a simple generalization of the $1 \times n$ case.

Definition 2. The *height of an island* is the minimum height of the cells within the island.

Definition 3. The *coast of an island* are the cells neighboring the island.

Definition 4. The *height of a coast* is the maximum height of the cells within the coast.

Definition 5. A *half-island* is a non-rectangular island, i.e. a set of connected cells with minimal height larger than the maximum height of the cells in the coast.

In the following we will call rectangular islands simply islands. We will always keep in mind that our goal is to find the maximum number of islands, so if we say we can change the heights of some cells without problems, we mean that this change does not decrease (or even increases) the number of islands.

Theorem 5. *The maximum number of islands in a rectangular $m \times n$ grid is at most mn.*

Proof. Let us start from a cell C of the grid having the height k. The connected subsets of cells with height at least k are half-islands with heights at least k. Let M be the half-island containing the selected cell C (we call this *the half-island implied by C* in the following), then the height of M is exactly k. If M is not a rectangle (i.e. it is not an island), then it means that M as a whole can only be part of islands of height less than k. Therefore we can decrease the heights of all cells in M by 1 without decreasing the number of islands.

Let us apply the above method for all cells of the grid and (as the height of the cells might change during the process) repeat it as long as we can find half-islands implied by some cell. If we find a half-island then at least two cell's height will decrease (the smallest non-rectangle contains a minimum of two cells), so the repetition will end in finite number of steps. At the end we arrive at some height setting, where all cells imply islands and not half-islands, and the number of islands did not decrease during the process. (Actually: the number, size and position of the islands does not change at all, only their heights might decrease.)

We have mn cells, they imply mn not necessarily different islands. On the other hand if M is an island of height k, then any cell with height k within M will imply M, and hence all islands are implied by some cell. Therefore we have at most mn different islands, and the proof is complete. ■

Note that if $m = 1$, then the maximum number of islands is equal to n. One can simply put all numbers from 1 to n in the cells. The above process will not change the heights, since the half-islands must be islands: all are of the form $1 \times k$. A cell of height h will imply an island of height h, and since h will run from 1 to n, we must have n different islands, as their heights are different.

It is also easy to see, that if an island M of height at least $h+2$ has a coast of height h, then the height of all cells in M can be decreased so that M's height will be $h+1$: this way M remains an island, all islands within M remain islands, and all islands containing M remain islands. In other words, with a coast of height h there must be a cell of height $h+1$ in the island otherwise the height of the island can be decreased.

Let us first consider the $2 \times n$ case. If we have an island of height 2, then it contains either $1 \times k$ or $2 \times k$ cells. The first case is not optimal: then the island's coast contains a $1 \times k$ grid under or over the island, which (naturally) can have cells of height 1 only like here in the left-hand side 2×3 grid:

3	2	3	1	3	2	3
1	1	1	1	4	2	4

If we simply repeat the island in that part of the coast, then the number of islands does not change:

3	2	3	1	3	2	3
3	2	3	1	4	2	4

but we may be able to increase the number of islands by changing the heights of those cells previously in the coast:

3	2	3	1	3	2	3
4	2	4	1	4	2	4

So all islands of height 2 are of the form $2 \times k$. In other words, the $2 \times n$ board must be cut vertically to islands of height 2. This idea can be used to show the following theorem.

THEOREM 6. *The number of islands in a $2 \times n$ board using heights $1, 2, \ldots, h$ only ($h \geq 3$) is*

$$\left[\frac{3n+1}{2}\right] + 1 - \left[\frac{n}{2^{h-2}}\right]^{+}$$

Proof. Let us consider the islands of height 2 in the $2 \times n$ grid. According to the note above this theorem, they are of the form $2 \times k_i$ ($i = 1, 2, \ldots p$) and they are separated by columns with cells of height 1. In these islands we can use heights $2, 3, \ldots, h$ to generate more sub-islands. If we forget for a minute that these islands are part of a larger board, then the maximum number of islands for these heights is the same as for the heights $1, 2, \ldots, h-1$ (we just need to add 1 for the height of each cell to get the required heights), and we can apply the note before this theorem again for these islands. Then in an island of size $2 \times k_i$ ($k_i \geq 2$) we must cut vertically by a column with cells of height 2, etc. This induction can be followed as long as the width of the island is at least 2, and naturally, in an island of size 2×1 we can have at most 2 sub-islands. This means that the $2 \times n$ board has two $1 \times n$ sub-boards in which (at most) every second cells's height can be different and hence the number of islands in the $2 \times n$ board is the number of islands in the $1 \times n$ board plus 1 island for each difference. But the construction also shows, that when the two $1 \times n$ boards differ, then they must contain the maximal heights in those cells, i.e. the heights h and $h-1$. By sorting these heights so that the first line of the $2 \times n$ board contains the height $h-1$ in these cells (this does not change the number of islands), we have there a $1 \times n$ board with heights $1, 2, \ldots, h-1$ and (of course) we want maximum number of islands. We already know how to do that and how many islands we obtain, then the $2 \times n$ board can have at most

$$\left(n + 1 - \left[\frac{n}{2^{h-2}}\right]^{+}\right) + \left[\frac{n+1}{2}\right] = \left[\frac{3n+1}{2}\right] + 1 - \left[\frac{n}{2^{h-2}}\right]^{+}$$

islands, which is exactly the number we need for the maximum height h. On the other hand this is also the construction: let us fill the first line of the $2 \times n$ board with height $1, 2, \dots, h-1$ so that we obtain the maximum number of islands for that $1 \times n$ board, repeat this height setting for the second line but change all heights $h-1$ to h (in exactly $[(n+1)/2]$ cells, increasing the number of islands by this number), and we get exactly the stated number islands. The following picture shows the result for a 2×11 grid and $h = 4$:

3	2	3	1	3	2	3	1	3	2	3
4	2	4	1	4	2	4	1	4	2	4

The $3 \times n$ case is a bit more difficult. Just as in the case of the $2 \times n$ board, we can prove here that the islands of height 2 cannot be of size $2 \times k$: we can repeat the line of the island neighboring the $1 \times k$ coast instead of the coast line and the number of islands does not change. Then this means that the islands of height 2 in this case are of size either $1 \times k$ or $3 \times k$. The construction for the case $2 \times n$ might give us an idea: let us fill the middle line in the $3 \times n$ board with heights $1, 2, \dots, h-1$ to get the maximum number of islands in that $1 \times n$ board, and repeat that line in the first and third lines while changing $h-1$ to h. This gives us all together

$$\left(n + 1 - \left[\frac{n}{2^{h-2}}\right]^{+}\right) + 2\left[\frac{n+1}{2}\right]$$

islands. But unfortunately, this is not optimal, we can do better for example for a 3×4 board and $h = 4$:

4	3	1	3
2	2	1	2
4	3	1	3

which has 9 islands, while the above construction gives 8 islands only. Looking at this example and the construction it is easy to see, that the construction always cuts vertically (just as in the case of a $2 \times n$ board), while in the example we cut horizontally too: for example in the 3×2 sub-board on the left-hand side. This is not surprising, we have just proved in the previous theorem that a 3×2 board must be cut horizontally.

To find out what is happening here, let us look at the construction of the maximum number of islands in a $1 \times n$ board with heights $1, 2, \dots, h-1$. One way the construction may go is to say that as long as $n \geq 2^{h-2}$ we always cut off an island of height 2 of width $2^{h-2} - 1$ by a cell of height 1. By repeating this procedure, we always cut off all together an even number of cells from the board. Then if n is even, we will always end up with a board of even width. Hence, after at the final vertical cut, this cut will be at the side of the board (which is clearly not optimal) or we will have a 3×2 sub-board, which must be cut horizontally, and we get one more island compared to the case if we would cut vertically again. Then the formula can be guessed and we have the following

Theorem 7. *The number of islands in a $3 \times n$ board using heights $1, 2, \dots, h$ only ($h \geq 3$) is*

$$2n + 2 - \left[\frac{n}{2^{h-2}}\right]^+$$

Proof. There are two cases, the $n < 2^{h-1}$ case is easier. Czédli's formula gives $[(m+1)(n+1)/2] - 1 = 2n + 1$ in this case, so the number of islands cannot be larger than this. On the other hand, let's cut the $3 \times n$ board into two $1 \times n$ boards by setting height 1 in the middle line, then the heights $2, 3, \dots, h$ are enough to get n islands in both $1 \times n$ boards, plus the whole board is an island too, and this construction gives exactly $2n + 1$ islands.

At first glance this construction seems to work for the $n \geq 2^{h-1}$ case too. Let's do the same, we cut the board into two $1 \times n$ boards and fill these board with the heights $1, 2, \dots, h$ optimally. This will give us

$$2\left(n + 1 - \left[\frac{n}{2^{h-1}}\right]\right) - 1 = 2n + 1 - 2\left[\frac{n}{2^{h-1}}\right]$$

islands, which is almost the number we needed. Let $n = k2^{h-1} + l$, where $0 \leq l < 2^{h-1}$ and we have two cases.

1. If $2^{h-2} \leq l < 2^{h-1}$ then

$$2\left[\frac{n}{2^{h-1}}\right] = 2k = \left[\frac{n}{2^{h-2}}\right] - 1$$

and we get the desired number of islands.

2. If $0 \leq l < 2^{h-2}$, then

$$2\left[\frac{n}{2^{h-1}}\right] = 2k = \left[\frac{n}{2^{h-2}}\right]$$

then we change the above construction a little. Let us first cut off a sub-board of size $3 \times \max\{1, l\}$ vertically, apply the above construction to the rest and we get there

$$2(n - 1 - \max\{1, l\}) + 2 - \left[\frac{n - 1 - \max\{1, l\}}{2^{h-2}}\right] = 2(n - \max\{1, l\}) + 1 - \left[\frac{n}{2^{h-2}}\right]$$

islands. On the $3 \times \max\{1, l\}$ board we cut it horizontally at the middle line by height 2, and the remaining two $1 \times \max\{1, l\}$ boards we can fill optimally with the heights $3, \dots, h$ to get $\max\{1, l\}$ islands each. With the $3 \times \max\{1, l\}$ board as one more island all together we obtain $2\max\{1, l\} + 1$ islands there. This construction provides us with

$$\left(2(n - \max\{1, l\}) - \left[\frac{n}{2^{h-2}}\right] + 1\right) + (2\max\{1, l\} + 1) = 2n + 2 - \left[\frac{n}{2^{h-2}}\right]$$

islands for the $3 \times n$ board. The following picture demonstrates this procedure for a 3×18 board and $h = 4$, where then $k = 2$ and $l = 2$:

4	3	1	4	3	4	2	4	3	4	1	4	3	4	2	4	3	4
2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	3	1	4	3	4	2	4	3	4	1	4	3	4	2	4	3	4

This proves the lower estimate for the maximum number of islands.

For the upper estimate we use induction by h and inside that induction another by n. To start we note that the $h = 2$ case is solved for all n. Let's suppose for induction that all cases, where the maximum height is less than h are proved, and also all cases where the maximum height is h and the width of the board is less than n (the $n = 1$ case for all $h \geq 3$ are trivial: there are 3 islands). Consider the islands of height 2 in the $3 \times n$-sized board, they are of size $3\times$(something) or $1\times$(something). There are two cases.

1. If there is a $3 \times k$-sized island of height 2, then we apply the induction assumptions for the $3\times k$ sub-board with heights $2, 3, \dots, h$ (i.e. heights $1, 2, \dots, h-1$) and for the rest of the board of size $3 \times (n-k-1)$ with heights $1, 2, \dots, h$. Then the number of islands is at most

$$2k+2-\left[\frac{k}{2^{h-3}}\right]^{+}+2(n-k-1)+2-\left[\frac{n-k-1}{2^{h-2}}\right]^{+} = 2n+2-\left[\frac{k}{2^{h-3}}\right]^{+}-\left[\frac{n-k-1}{2^{h-2}}\right]^{+}$$

To prove that this is at most what is stated in the theorem, we have three cases:

a; If $k < 2^{h-3}$ then we have

$$2n+1-\left[\frac{n-k-1}{2^{h-2}}\right]^{+}$$

islands in the construction and we want to prove that it is at most

$$2n+2-\left[\frac{n}{2^{h-2}}\right]^{+}$$

Since n and $n-k-1$ may differ by at most 2^{h-3}, the difference between the two numbers in the greatest integer functions is less than one, and this case is proved.

b; If $k \geq 2^{h-3}$ and $n-k-1 < 2^{h-2}$ then we need to prove that

$$2n+1-\left[\frac{k}{2^{h-3}}\right] \leq 2n+2-\left[\frac{n}{2^{h-2}}\right]^{+}$$

If $n < 2^{h-2}$ then we have $2n+1$ on the right hand side, and the estimation is OK. Otherwise we need that

$$\left[\frac{n}{2^{h-2}}\right] \leq \left[\frac{k}{2^{h-3}}\right]+1 = \left[\frac{2k+2^{h-2}}{2^{h-2}}\right]$$

Since $n = (n-k)+k \leq k+2^{h-2}$, this estimate also holds.

c; The only case left is when $k \geq 2^{h-3}$, $n-k-1 \geq 2^{h-2}$ and hence $n \geq 2^{h-2}$. Then we need

$$2n+2-\left[\frac{k}{2^{h-3}}\right]-\left[\frac{n-k-1}{2^{h-2}}\right] \leq 2n+2-\left[\frac{n}{2^{h-2}}\right]$$

Let $n = c2^{h-2}+d$ $(0 \leq d < 2^{h-2})$ then

$$c \leq c+\left[\frac{d-k-1}{2^{h-2}}\right]+\left[\frac{2k}{2^{h-2}}\right]$$

In the worst case we have $d = 0$: it is easy to check that for $2^{h-3} \leq k \leq 2^{h-2}$ the inequality holds. If $k > 2^{h-2}$ then $2k \geq k + 1 + 2^{h-2}$ and hence

$$\left[\frac{-k-1}{2^{h-2}}\right] \geq -\left[\frac{k+1+2^{h-2}}{2^{h-2}}\right] \geq -\left[\frac{2k}{2^{h-2}}\right]$$

and the proof of this case is complete.

2. If all islands of height 2 are of size $1 \times k$, then clearly the $3 \times n$ board is cut horizontally by putting height 1 in the middle line. In this case the number of islands can be computed using the proven formula for $1 \times n$-sized boards. There are two cases to check now:

a; If $n < 2^{h-1}$ then

$$2\left(n + 1 - \left[\frac{n}{2^{h-1}}\right]^{+}\right) + 1 = 2n + 1 = 2n + 2 - \left[\frac{n}{2^{h-2}}\right]^{+}$$

and this case is fine.

b; If $n \geq 2^{h-1}$ then

$$2\left(n + 1 - \left[\frac{n}{2^{h-1}}\right]\right) - 1 = 2n + 1 - 2\left[\frac{n}{2^{h-1}}\right] \leq 2n + 2 - \left[\frac{n}{2^{h-2}}\right]$$

and this completes the proof of the theorem. ■

If we would want to go even further, then the next smallest case would be a board of size 4×4 and $h = 3$. A little experimenting shows, that this is not an easy case, the best height setting is

2	1	2	3
3	1	1	1
1	1	1	2
2	3	1	3

This gives 9 islands but its construction does not bear any resemblance to the previous ones. The main difference is that the islands are not constructed by cutting though the board. For boards of size $m \times n$ $(m, n \geq 4)$ one needs to put optimally sized rectangles on the board optimally, whatever that means. To reconstruct a rectangle from rectangles is an NP-hard problem, very hard to study even using computer.

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