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A UNIVERSAL SEQUENCE OF CONTINUOUS FUNCTIONS

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Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday

Abstract. We show that for each positive integer k there is a sequence $F_n\colon \mathbb{R}^k \to \mathbb{R}$ of *continuous* functions which represents via point-wise limits *arbitrary* functions $G\colon X^k \to \mathbb{R}$ defined on domains $X \subseteq \mathbb{R}$ of sizes not exceeding a standard cardinal characteristic of the continuum.

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1. Introduction

More than sixty years ago Sierpiński posed a general problem[1] asking for which index-sets X and for which families $\mathcal{G}$ of real functions defined on X can we find a single sequence f_n of real functions defined on X with the property that every $g \in \mathcal{G}$ is a point-wise limit of a subsequence of f_n. In [2], Rothberger showed that this is the case when both the family $\mathcal{G}$ and the index-set X have cardinalities at most $\aleph_1$. In Theorem 6.4 of [3], we have extended Rothberger's results to families and index-sets of size at most $\mathfrak{p}$ (a characteristic of the continuum defined below) that appears to be a result of optimal generality. The purpose of this note is to reinterpret this idea and prove the following result as well as its extensions to all other finite dimensions.

1.1. Theorem. *There is a sequence*

$$F_n : \mathbb{R}^2 \to \mathbb{R}$$

of continuous functions such that for every set X of reals of size at most $\mathfrak{p}$ and every function

$$G : X^2 \to \mathbb{R}$$

there is a one-to-one map $h : X \to \mathbb{R}$ such that for all $(x, y) \in X^2$,

$$G(x, y) = \lim_{n\to\infty} F_n(h(x), h(y)).$$

[1]See Fund. Math., vol. 27 (1936) p. 293, *problème de M. Sierpiński.*

Recall that $\mathfrak{p}$ is the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that $\bigcap \mathcal{F}_0$ is infinite for all finite $\mathcal{F}_0 \subseteq \mathcal{F}$ but there is no infinite subset b of $\mathbb{N}$ such that $b \subseteq^* a$ for all $a \in \mathcal{F}$, where, as customary, $b \subseteq^* a$ denotes the fact that $b \setminus a$ is a finite set, the fact that the set a *almost includes* the set b. This is a well studied cardinal characteristic which while not provably equal to the continuum it has this maximal value under many standard assumptions such us, for example, the Continuum Hypothesis. One of its most useful formulation of this cardinal characteristic of the continuum is that it is exactly equal to the Baire-category number for the class of compact separable spaces. More precisely, $\mathfrak{p}$ is the minimal cardinality of a family of dense open subsets of some separable compact Hausdorff space with empty intersection (see [1]). We shall use below the dual form of this formulation of $\mathfrak{p}$.

2. Two variables

Note that when proving Theorem 1.1, without loosing generality, we can replace the reals with the Cantor set $2^{\mathbb{N}}$. Thus, if we identify $2^{\mathbb{N}}$ with the power-set of $\mathbb{N}$ in the natural way, the irrationals correspond to the collection of infinite subsets of $\mathbb{N}$. Let us first show that there is a sequence $F_n : 2^{\mathbb{N}} \to 2$ of continuous $\{0,1\}$-valued functions universal in this way for functions $G : X^2 \to 2$ defined on sets of irrationals of size at most $\mathfrak{p}$. In this case it will also be convenient to identify $2^{\mathbb{N}}$ with its cube $2^{\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ in some natural way so that a given $x \in 2^{\mathbb{N}}$ gets its three coordinates $(x)_0$, $(x)_1$ and $(x)_2$. For a given integer n, we define $F_n : (2^{\mathbb{N}})^2 \to 2$ by setting

$$F_n(x,y) = \begin{cases} 1, & \text{if } \max((x)_1 \cap \{0,1,\dots,n\}) \in (y)_2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, this defines a continuous function from $(2^{\mathbb{N}})^2$ into $2 = \{0,1\}$. Let us show that the sequence F_n is universal for mappings $G : X^2 \to 2$ with domains $X \subseteq 2^{\mathbb{N}} \setminus \mathbb{Q}$ of cardinality at most $\mathfrak{p}$.[2] Given such a mapping $G : X^2 \to 2$, we apply Theorem 6.4 of [3] and find a sequence (x_a, y_a) $(a \in X)$ of pairs of infinite and co-infinite subsets of $\mathbb{N}$ such that

(a) $G(a,b) = 1$ implies $x_a \subseteq^* y_b$, and

(b) $G(a,b) = 0$ implies $x_a \subseteq^* \mathbb{N} \setminus y_b$.

Then it is readily seen that $h : X \to 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ defined by $h(a) = (a, x_a, y_a)$ is the required map satisfying the conclusion

$$G(a,b) = \lim_{n\to\infty} F_n(h(a), h(b))$$

for all $(a,b) \in X^2$.

Now we treat the general case of finding a sequence $F_n : \mathbb{R}^2 \to \mathbb{R}$ of continuous functions that codes an arbitrary mapping $G : X^2 \to \mathbb{R}$ defined on a set of reals of size at most $\mathfrak{p}$. Again we work with the Cantor set $2^{\mathbb{N}}$ instead of the set of reals.

[2] In this context, by $2^{\mathbb{N}} \setminus \mathbb{Q}$, we denote the set of all $x \in 2^{\mathbb{N}}$ that are not eventually constant.

However we shall now identify $2^{\mathbb{N}}$ with its infinite power $(2^{\mathbb{N}})^{\mathbb{N}}$ and so every $x \in 2^{\mathbb{N}}$ decomposes naturally as a sequence $(x)_n$ of its coordinates. We shall again need to identify $2^{\mathbb{N}}$ with the power-set of $\mathbb{N}$ and in order to simplify the notation and avoid the confusion, for an infinite subset x of $\mathbb{N}$ and an integer n, we let

$$x[n] = \max(x \cap \{0, 1, \ldots, n\}),$$

where we set $\max(\emptyset) = 0$. This way we make the distinction with the notation $x(n)$ which is the value of the characteristic function at n, i.e., $x(n) = 1$ iff $n \in x$. For an integer n define $F_n : (2^{\mathbb{N}})^2 \to 2^{\mathbb{N}}$ by setting

$$F_n(x, y)(k) = \begin{cases} 1, & \text{if } k \leq n \text{ and } (x)_{2k+1}[n] \in (y)_{2k+2}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this indeed defines a continuous function from $(2^{\mathbb{N}})^2$ into $2^{\mathbb{N}}$. We need to show that the sequence F_n of continuous functions is universal for all mappings $G : X^2 \to \mathbb{R}$ defined on sets X of reals of cardinality $\mathfrak{p}$. Clearly, we may assume that X is a subset of $2^{\mathbb{N}} \setminus \mathbb{Q}$ and that the range of G is $2^{\mathbb{N}}$ rather than $\mathbb{R}$. To this end we apply the above argument to each of the coordinate functions $G_k : X^2 \to 2$ defined by $G_k(x, y) = G(x, y)(k)$ getting the sequences (x_a^k, y_a^k) $(a \in X)$ $(k \in \mathbb{N})$ of pairs of infinite and co-infinite subsets of $\mathbb{N}$ such that for all $k \in \mathbb{N}$ and $(a, b) \in X^2$,

(c) $G_k(a, b) = 1$ implies $x_a^k \subseteq^* y_b^k$, and

(d) $G_k(a, b) = 0$ implies $x_a^k \subseteq^* \mathbb{N} \setminus y_b^k$.

Finally, let $h : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ be defined by

$$h(a) = (a, x_a^0, y_a^0, x_a^1, y_a^1, \ldots, x_a^k, y_a^k, \ldots).$$

We again leave to the reader the simple checking that

$$G(a, b) = \lim_{n \to \infty} F_n(h(a), h(b))$$

for all $(a, b) \in X^2$.

2.1. Remark. Note that in general we cannot say much about the nature of the mapping h since the set X might have more that continuum many maps of the form $G : X^2 \to \mathbb{R}$.

3. Higher dimensions

In this section we show how a higher dimensional version of the coding designed above gives us the following more general result.

3.1. Theorem. *For ever positive integer k there is a sequence*

$$F_n : \mathbb{R}^k \to \mathbb{R}$$

of continuous functions such that for every set X of reals of size at most $\mathfrak{p}$ and every function

$$G : X^k \to \mathbb{R}$$

there is a one-to-one map $h : \mathbb{R} \to \mathbb{R}$ *such that for all* $(x_1, \ldots, x_k) \in X^k$,

$$G(x_1, \ldots, x_k) = \lim_{n\to\infty} F_n(h(x_1), \ldots, h(x_k)).$$

We again concentrate first on the case of representing $\{0,1\}$-valued functions $G : X^k \to 2$ and we work with the Cantor space $2^{\mathbb{N}}$ instead of $\mathbb{R}$. We shall use the following variation of the coding of [3] which is proved along similar lines.

3.2. Lemma. *For every set* X *of size at most* $\mathfrak{p}$ *every positive integer* k *and every* $G : X^k \to 2$ *there exist a sequence* y_a $(a \in X)$ *of infinite subsets of* $\mathbb{N}$ *and a sequence* $(x_a^1, \ldots, x_a^k)$ $(a \in X)$ *of* k*-tuples of infinite subsets of* $\mathbb{N}$ *such that* $G(a_1, \ldots, a_k) = 1$ *if and only if*

$$(\exists 1 \le i \le k)(\forall^\infty n)|(\bigcap_{j=1}^{k} x_{a_j}^j) \cap \{0, \ldots, n\}| \ge |y_{a_i} \cap \{0, \ldots, n\}|. \tag{1}$$

Proof. For the convenience of reader we sketch the proof. In fact, we shall prove the symmetric version of the result, i.e., we shall first show how to code mappings of the form $G : [X]^k \to 2$ defined on the family $[X]^k$ of all k-element subsets of X rather than on the power X^k. We fix a well-ordering $<_w$ of X such that for all $b \in X$, the set $X(b) = \{a \in X : a <_w b\}$ has cardinality $< \mathfrak{p}$. We shall first select a sequence x_a $(a \in X)$ of infinite subsets of $\mathbb{N}$ such that for every $s \in [X]^k$ the set $x_s := \bigcap_{a\in s} x_a$ is infinite if and only if $G(s) = 1$. When this is done, for each $b \in X$, we choose an infinite subset y_b of $\mathbb{N}$ whose enumeration function grows much faster than the enumeration function of any infinite set of the form $\bigcap_{a\in t} x_a$ for t a finite subset of $X(b) \cup \{b\}$. In particular, we will have that for $s \in [X]^k$,

$$G(s) = 1 \text{ iff } (\forall^\infty n)\ |x_s \cap \{0, \ldots, n\}| \ge \min\{|y_a \cap \{0, \ldots, n\}| : a \in s\}.$$

The sequence x_a $(a \in X)$ is selected by recursion on the well-ordering $<_w$. The extra inductive hypothesis at a given stage $b \in X$ is that $x_p \setminus \bigcup_{a\in q} x_a$ is infinite for every pair p and q of disjoint finite subsets of $X(b)$ such that $|p| < k$. This extra inductive hypothesis guarantees (via the natural σ-centered poset $\mathcal{P}$ of finite approximations) the existence of an infinite subset x_b that is almost disjoint from every element of the family

$$\{x_s : \ s \in [X(b)]^{k-1}, G(s \cup \{b\}) = 0\}$$

and that will, if sufficiently generic, also have infinite intersection with every element of the family

$$\{x_s : \ s \in [X(b)]^{k-1}, G(s \cup \{b\}) = 1\}.$$

Moreover, if sufficiently generic, the set x_b will also have infinite intersection with every member of the family

$$\{x_p \setminus \bigcup_{a\in q} x_a : \ p \in [X(b)]^{<k-1}, q \in [X]^{<\omega}, p \cap q = \emptyset\}$$

and will not almost include any element of the family

$$\{x_p \setminus \bigcup_{a\in q} x_a : \ p \in [X(b)]^{<k}, q \in [X]^{<\omega}, p \cap q = \emptyset\}.$$

Since 'sufficiently generic' requires meeting only $< \mathfrak{p}$ dense open sets of the σ-centered poset $\mathcal{P}$, Bell's formulation of the number $\mathfrak{p}$ (see [1]), gives us a choice of x_b that will preserve all our inductive hypotheses.

To deduce the general case from the symmetric one, consider an arbitrary $G : X^k \to 2$. Let $\bar{X} = X \times \{1, \ldots, k\}$ and choose a mapping $\bar{G} : [\bar{X}]^k \to 2$ such that for all $(a_1, \ldots, a_k) \in X^k$,

$$\bar{G}(\{(a_1, 1), \ldots, (a_k, k)\}) = G(a_1, \ldots, a_k).$$

Obtain sequences $\bar{x}_a$ $(a \in \bar{X})$ and $\bar{y}_a$ $(a \in \bar{X})$ of infinite subsets of $\mathbb{N}$ such that for all $s \in [\bar{X}]^k$, we have that $\bar{G}(s) = 1$ if and only if

$$(\exists a \in s)(\forall^\infty n)\ |\bar{x}_s \cap \{0, \ldots, n\}| \geq |\bar{y}_a \cap \{0, \ldots, n\}|.$$

Then if for $a \in X$ and $1 \leq i \leq k$, we set $x_a^i = \bar{x}_{(a,i)}$, and if we set y_a to be any infinite subset of $\mathbb{N}$ whose enumeration function is faster than the enumeration functions of each of the sets $\bar{y}_{(a,i)}$ $(1 \leq i \leq k)$, we will obtain sequences satisfying the conclusion of the lemma. ∎

In order to define the corresponding sequence $F_n : (2^{\mathbb{N}})^k \to 2$ of continuous functions we identify $2^{\mathbb{N}}$ with $(2^{\mathbb{N}})^{k+1}$ in the natural way so that for a given $x \in 2^{\mathbb{N}}$ and $i \leq k$ we have well defined coordinate $(x)_i$. Thus, for a given integer n, we define $F_n : (2^{\mathbb{N}})^k \to 2$ by setting $F_n(x_1, \ldots, x_k) = 1$ if and only if there is some $j \in \{1, 2, \ldots, k\}$ such that the intersection $\bigcap_{i=1}^k (x_i)_i$ has at least as many points below n as the set $(x_j)_0$. Clearly, each F_n is a continuous function on $(2^{\mathbb{N}})^k$. Let us show that this sequence captures an arbitrary $G : X^k \to 2$ where X is a set of reals of size at most $\mathfrak{p}$. We apply the coding of Lemma 3.2 and get a sequence y_a $(a \in X)$ of infinite subsets of $\mathbb{N}$ and a sequence $(x_a^1, \ldots, x_a^k)$ $(a \in X)$ of k-tuples of infinite subsets of $\mathbb{N}$ such that $G(a_1, \ldots., a_k) = 1$ if and only if the inequality (1) is satisfied for all but finitely many n. We may assume $a \mapsto y_a$ is a one-to-one mapping on X. Recalling our identification of $2^{\mathbb{N}}$ and $(2^{\mathbb{N}})^{k+1}$ we define $h : X \to 2^{\mathbb{N}}$ as follows

$$(h(a))_0 = y_a \text{ and } (h(a))_i = x_a^i \text{ for } i = 1, \ldots, k.$$

It follows from the equation (1) and our definition of F_n that

$$G(a_1, \ldots, a_k) = \lim_{n\to\infty} F_n(h(a_1), \ldots, h(a_k))$$

for all $(a_1, \ldots, a_k) \in X^k$. To treat the general case we work again on the Cantor space and identify $2^{\mathbb{N}}$ with its infinite power $(2^{\mathbb{N}})^{\mathbb{N}}$ so that every $x \in 2^{\mathbb{N}}$ decomposes naturally as a sequence $(x)_n$ of its coordinates. Note that given an arbitrary $G : X^k \to 2^{\mathbb{N}}$ defined on a set of reals X of size at most $\mathfrak{p}$, we can apply the coding of Lemma 3.2 to each of the coordinate functions $G_m(a_1, \ldots, a_k) = G(a_1, \ldots, a_k)(m)$ and obtain a sequence y_a $(a \in X)$ of infinite subsets of $\mathbb{N}$ and a sequence $(x_a^\ell, \ldots, x_a^\ell)$

$(a \in X)$ $(\ell \in \mathbb{N}^+)$ such that for all $(a_1, \ldots, a_k) \in X^k$ and all $m \in \mathbb{N}$, we have that $G_m(a_1, \ldots, a_k) = 1$ if and only if

$$|(\bigcap_{i=1}^{k} x_{a_i}^{km+i}) \cap \{0, \ldots, n\}| \geq \min\{|y_{a_i} \cap \{0, \ldots, n\}| : 1 \leq i \leq k\} \tag{2}$$

for all but finitely many n. Again we assume that $a \mapsto y_a$ is a one-to-one mapping on X.

Having in mind the identification of $2^{\mathbb{N}}$ and $(2^{\mathbb{N}})^{\mathbb{N}}$, for each integer n we define $F_n : (2^{\mathbb{N}})^k \to 2^{\mathbb{N}}$ by letting $F_n(x_1, \ldots, x_k)(m) = 1$ if and only if there is some $j \in \{1, 2, \ldots, k\}$ such that the intersection $\bigcap_{i=1}^{k} (x_i)_{km+i}$ has at least as many points below n as the set $(x_j)_0$. Clearly, each F_n is a continuous function on $(2^{\mathbb{N}})^k$. To see that the sequence F_n is as required let $G : X^k \to 2^{\mathbb{N}}$ be an arbitrary function defined on a set of reals X of size at most $\mathfrak{p}$. Apply the above coding procedure and obtain y_a $(a \in X)$ and $(x_a^\ell, \ldots, x_a^\ell)$ $(a \in X)$ $(\ell \in \mathbb{N}^+)$ satisfying the inequality (2) for all m and all but finitely many n. Define $h : X \to 2^{\mathbb{N}}$ as follows

$$(h(a))_0 = y_a \text{ and } (h(a))_\ell = x_a^\ell \text{ for } \ell \in \mathbb{N}^+.$$

Then it readily follows that

$$G(a_1, \ldots, a_k) = \lim_{n \to \infty} F_n(h(a_1), \ldots, h(a_k))$$

for all $(a_1, \ldots, a_k) \in X^k$.

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