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A CONTRIBUTION TO THE DEVELOPMENT OF FUNCTIONAL THINKING RELATED TO CONVEXITY AND ONE-DIMENSIONAL MOTION

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Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday

Abstract. Mathematical concepts are defined precisely using the language of the branch of mathematics to which they belong. But their meaning can be enriched through different interpretations and those of them belonging to the real world situations, we call "vivid" mathematics. In contacts with Professor M. Marjanović, we investigated a case of "vivid" mathematics in some earlier papers and we continue to do so in this paper.

Suppose that a liquid (water) flow has a constant inflow rate and that a vessel has the form of a surface of revolution, and suppose that this process begins at moment $t = 0$ and ends at moment $t = T$. We study the dependence of the height $h(t)$ of the liquid level at the time t, which will be called the *height filling function*. It is convex or concave depending on the way how the level of the liquid changes. This vivid interpretation holds in general, namely we prove that given a strictly increasing convex (concave) continuous function on $[0, T]$ satisfying certain conditions, there exists a vessel such that its height filling function is equal to the given function. This is a fact that seems to be new and we continue paying attention to it.

In this way, we hope that we are providing a matter that can serve as a motivation and an illustration for a deeper understanding of basic concepts and ideas of the differential and integral calculus. It can also serve for a further development of functional thinking in teaching mathematics.

We also consider a more general concept of one-dimensional motion, including changes in direction of motion and the difference between velocity and acceleration defined by the position and the path as functions of time. We indicate how one can apply this for studying the height filling function of a liquid flow, which can be considered now as a one-dimensional motion of a liquid along the axis of rotation of the vessel.

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1. Characterization of monotony and convexity by pouring liquid into a vessel

We will first consider the notions of speed and acceleration for one-dimensional motion, as well as their relation with monotonous and convex functions, in the example of pouring liquids into a vessel.

Let $H > 0$ and let a function $r : [0, H] \to \mathbf{R}$ with the following properties be given:

(r1) r is continuous on $[0, H]$ and

(r2) $r(x) > 0$ for $x \neq 0$.

Rotating the curve

$$c_r = \{(0, y) : y \in [0, r(0)]\} \cup \{(x, r(x)) : x \in [0, H]\}$$

about x-axis we get a surface σ_r. The surface we get in this way is called an *elementary surface of revolution*, see Fig. 1.

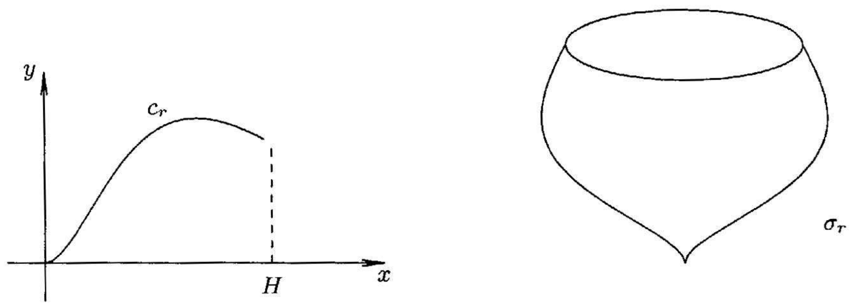

Fig. 1

It is convenient to identify the inner "wall" of the vessel (and the whole vessel) with σ_r. In this sense we will consider *the vessel* to be the part of the space bounded by σ_r and the plane $x = H$, while *the axis of rotation of the vessel* will be the rotation axis of surface σ_r. Function r (respectively the graph of r) will be called the *generatrix* of surface σ_r. The vessels we get in this way are called *elementary rotation vessels*.

Throughout this paper we assume that an elementary rotation vessel is placed so that its rotation axis is in vertical position and liquid is pouring evenly through the upper part so that the speed of change of the liquid volume is constant. Suppose that this procedure begins at moment $t = 0$ and ends at a moment $t = T$.

Denote by $h(t)$ the height of the liquid level at moment t. It can be shown that the function h has the following properties (see [10, 11]):

(h1) h is strictly increasing and continuous on $[0, T]$,

(h2) $h(0) = 0$,

(h3a) h is continuously differentiable on $[0, T]$ and for every $t \in [0, T]$, $h'(t) > 0$ (if $r(0) \neq 0$),

or

(h3b) h is continuously differentiable on $(0, T]$, $\lim_{t\to 0+} h'(t) = +\infty$ and for every $t \in (0, T]$, $h'(t) > 0$ (if $r(0) = 0$).

Of course, at the ends of intervals we consider one side derivatives.

The class of all functions satisfying conditions (h1), (h2) and (h3a) or (h3b) we call *the class of height filling functions* and denote it by $\mathcal{H}[0, T]$.

If we specify that a liquid (water) flow has a constant inflow rate v_0 into a vessel (whose generatrix is r), we can express the first and second derivatives of the corresponding function h in terms of r and v_0:

$$h'(t) = \frac{v_0}{\pi r^2(h(t))} \tag{1}$$

for $t \in (0, T]$ ($t \in [0, T]$, if $r(0) \neq 0$). Moreover, if we assume that the function r is differentiable on $(0, H)$, then we get

$$h''(t) = -\frac{2v_0^2}{\pi^2} \frac{r'(h(t))}{r^5(h(t))},$$

for $t \in (0, T)$.

Using the last formula we can prove the following:

If r is increasing then $r' \geq 0$ and $h'' \leq 0$, that is h is concave on $(0, T)$.

If r is decreasing then $r' \leq 0$ and $h'' \geq 0$, that is h is convex on $(0, T)$.

If we do not specify otherwise, we consider functions which are time dependent. Recall that it is convenient to consider liquid flow as a one-dimensional motion of a liquid along the axis of rotation of the vessel and apply it for studying the height filling function h. Denote by v instantaneous speed and by a instantaneous acceleration of this motion. Note that v is the rate of change of liquid level h in the vessel as a function of time (more precisely $v(t) = h'(t)$) and a is the rate of change of v (more precisely $a(t) = v'(t) = h''(t)$). It is clear that v is always positive.

If the generatrix of a vessel is a monotone increasing (respectively decreasing) function, one can conclude that the increment of liquid level is decreasing (respectively increasing) in the same interval of time Δt. We leave to the interested reader to give visual interpretation of this fact and to give detailed proof of the two following propositions.

Proposition 1. *If the generatrix of a vessel is a monotone increasing (respectively decreasing) function then v is a monotone decreasing (respectively increasing) function.*

Proposition 2. *The acceleration function $a = h''$ is positive (that is h is convex) in the case that the generatrix of a vessel is a monotone decreasing function, and the acceleration function $a = h''$ is negative (that is h is concave) in the case that the generatrix of a vessel is a monotone increasing function.*

Further in this section, when talking about the height filling function for a vessel σ_r, we assume that a liquid (water) flow has a constant inflow rate v_0 into the vessel.

For a given function r, we denote by $\mathcal{S}r$ the corresponding height filling function for the vessel σ_r. In the previous text we determined that if function r has properties (r1) and (r2), then the corresponding height filling function $h = \mathcal{S}r$ has the properties (h1), (h2) and (h3a) or (h3b). Now, let us see if the opposite is true, i.e., if h has the properties (h1), (h2) and (h3a) or (h3b), is there a function r with the properties (r1) and (r2) such that $h = \mathcal{S}r$?

Note that the formula (1) has an important role in our investigation.

Let a function $h : [0, T] \to \mathbf{R}$, with properties (h1), (h2) and (h3a) or (h3b) be given. Then there is a corresponding elementary surface of revolution, i.e., there is a function r with the properties (r1) and (r2) such that the dependence of the liquid level in the vessel σ_r is described exactly by the function h. The function r is specified by the formula

$$r(x) = \sqrt{\frac{v_0}{\pi h'(h^{-1}(x))}}, \quad x \in [0, h(T)].$$

Instead of functions from class $\mathcal{H}[0,T]$, we can consider function from some other classes, and for these new classes we try to determine if there is a corresponding class of surfaces of revolution. In this sense Professor M. Marjanović has suggested an interesting class, which we describe now.

Let $h : [0,T] \to \mathbf{R}$ be a function with the properties:
(ch1) h is continuous, convex and strictly increasing on $[0,T]$,
(ch2) $h(0) = 0$,
(ch3) $h'_+(0) > 0$ and $h'_-(T) < +\infty$.
The class of all functions satisfying the conditions (ch1), (ch2) and (ch3) we denote by $\mathcal{C}on[0,T]$.

Denote $H = h(T)$. Since h is strictly increasing and continuous on $[0,T]$ we get that h maps $[0,T]$ onto $[0,H]$, bijectively. We will examine if for a given $h \in \mathcal{C}on[0,T]$ there is a function $r : [0,H] \to \mathbf{R}$ such that h is the height filling function for vessel σ_r.

In the previous discussion we have indicated that there is an injective correspondence between the class of generatrices of elementary surfaces of revolution and the class $\mathcal{H}[0,T]$ of height filling functions. Since there is a function that satisfies the conditions (ch1), (ch2) and (ch3) which does not belong to the class of height filling functions, we conclude that for such function there is no corresponding generatrix r such that σ_r is an elementary surface of revolution. However, for a given function $h \in \mathcal{C}on[0,T]$, roughly speaking we show below that we can use function r given by the formula

$$r(h(t)) = \sqrt{\frac{v_0}{\pi h'(t)}}.$$

Since convex function h has the derivative everywhere except on at most countable set D, r is defined everywhere except on the set $h(D)$.

But in order to reach the surface of revolution which corresponds to function h we first introduce the notion of extended graph for monotone function.

Let us assume that we have an interval I and a decreasing function $f : I \to \mathbf{R}$. Then for all $x < y$, $f(x) \geq f(y)$ holds. Therefore, f can have only one type of simple discontinuity, where the right and left limit do not equal to each other. More precisely, if f has a discontinuity at point $p \in I$ then $f(p-) > f(p+)$, where $f(p-)$ indicates the limit from the left and $f(p+)$ indicates the limit from the right. Therefore, to discontinuity at point p we can assign the interval $I_p = [f(p+), f(p-)]$.

Let D be the set of points in I at which f is discontinuous. Notice that the set D is most countable. Define $J_p = \{(p,y) : y \in I_p\}$, $D_f = \bigcup_{p \in D} J_p$ and $\Gamma^*_f = \Gamma_f \cup D_f$, where Γ_f is the graph of f; we call Γ^*_f the *extended graph of* f.

The function r is determined by the formula

$$r(h(t)) = \sqrt{\frac{v_0}{\pi h'(t)}}.$$

More precisely, we define function $r : [0, H]\backslash h(D) \to \mathbf{R}$, where D is the set of points $t \in [0, T]$ such that $h'(t)$ does not exist (for simplicity the reader can assume that the set D is finite). As the function h is convex we get that function r is decreasing and the extended graph Γ_r^* of r is defined. Rotating the curve

$$\{(0, y) : y \in [0, r(0)]\} \cup \Gamma_r^*$$

about x-axis we get the surface $\Sigma_{\Gamma_r^*}$. The surfaces which are obtained in this way are called *extended elementary surfaces of revolution.* Similarly as we have defined a vessel (using the corresponding elementary surface of revolution) we define an *extended vessel* (using the corresponding extended elementary surface of revolution) and identify an extended elementary surface of revolution and the extended vessel.

The interested reader can check that the given function h is the corresponding height filling function for the extended vessel $\Sigma_{\Gamma_r^*}$ obtained in this way.

Therefore, we have proved the following:

THEOREM 1. *If $h \in \mathcal{C}on[0, T]$ then there exists an extended vessel such that h is its corresponding height filling function.*

EXAMPLE 1. If the inflow rate v_0 is equal to π and if the function $h : [0, 4\ln 2] \to \mathbf{R}$ is defined by the formula

$$h(t) = \begin{cases} e^{t/3} - 1, & \text{if } t \in [0, 3\ln 2] \\ 2t - 6\ln 2 + 1, & \text{if } t \in [3\ln 2, 4\ln 2], \end{cases}$$

then h satisfies the conditions (ch1), (ch2) and (ch3), and we get

$$r(x) = \begin{cases} \frac{1}{\sqrt{\frac{1}{3}(x+1)}}, & \text{if } x \in [0, 1) \\ \frac{1}{\sqrt{2}}, & \text{if } x \in (1, 2\ln 2 + 1]. \end{cases}$$

See Figs. 2 and 3.

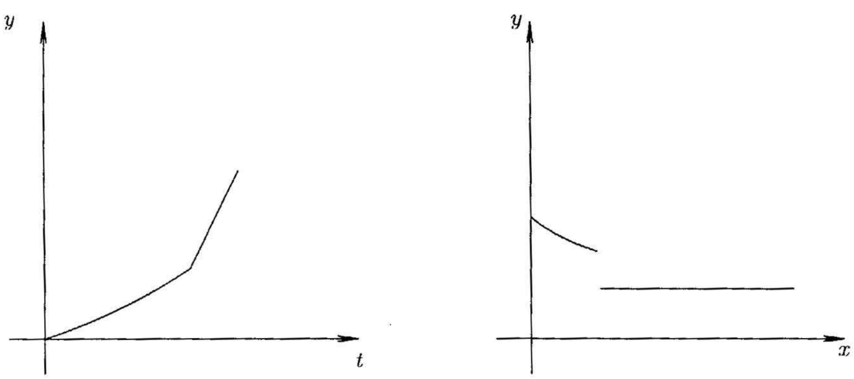

Fig. 2

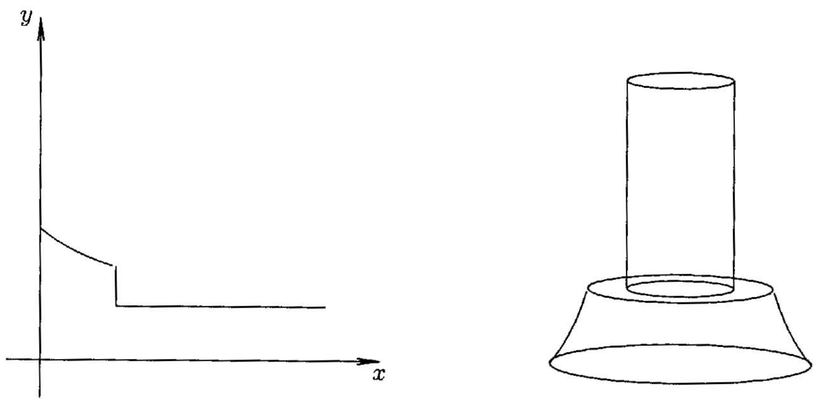

Fig. 3

2. Motion in one dimension

Motion of an object is the change of position of this object relative to another (reference) object. Motion takes place over time and we will observe the motions that take place during the time interval $[0, T]$.

Motion in one dimension is a motion along a straight line. The position of an object along a straight line can be uniquely determined by its distance (up to the sign, i.e., up to direction) from a (user chosen) origin. Note: the position is fully specified by one coordinate (that is why this a one-dimensional problem).

For a given problem, the origin can be chosen at whatever point is convenient. For example, the position of the object at the moment $t = 0$ is often chosen to be the origin. We will also follow this formalism (if not stated otherwise). The position of the object will in general be a function of time, and this function, $x : [0, T] \to \mathbf{R}$, is called the *position function.*

From mathematical point of view it is natural to assume that position function x is continuous and of bounded variation on interval $[0, T]$. The total variation of the position function over interval $[0, t]$ is the length of path $s(t)$ travelled by the object from the moment 0 until the moment t. The function $s : [0, T] \to \mathbf{R}$ is called a *path function.*

From physical point of view it is natural to assume that position function x is continuous and piecewise monotone on interval $[0, T]$ (see, for example, Example 3).

We say that a function x is *piecewise monotone* if there are finitely many moments $0 = t_0 < t_1 < \cdots < t_n = T$, and the restriction of x to each interval $[t_{j-1}, t_j]$ for $j \in \{1, \ldots, n\}$ is a monotone function. If $j_0 \in \{0, 1, \ldots, n\}$ is the largest index j such that $t_j \le t$ then we define

$$s(t) = \sum_{j=1}^{j_0} |x(t_j) - x(t_{j-1})| + |x(t) - x(t_{j_0})|$$

and $s(t)$ is the length of path travelled by the object from the moment 0 until the moment t. The function $s : [0, T] \to \mathbf{R}$ is called a *path function*.

We leave to the interested reader to check that a piecewise monotone function is of bounded variation and that the two definitions of path are consistent.

Further in the text, we assume that a position function is continuous and piecewise monotone.

The *average velocity* of an object over a time interval $[t_1, t_2]$ is defined as

$$v_{avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

The *average speed* of an object over a time interval $[t_1, t_2]$ is defined as

$$u_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Suppose now that function x is differentiable on interval $(0, T)$, and that there exists $x'_+(0)$ and $x'_-(T)$. It follows that the function s is also differentiable on interval $(0, T)$, and that $s'_+(0)$ and $s'_-(T)$ exist.

The *instantaneous velocity* at the moment t_1 is defined as an average velocity over time interval $[t_1, t_2]$, when t_2 is “infinitely close” to t_1. More precisely,

$$v(t_1) = \lim_{t_2 \to t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} = x'(t_1).$$

The *instantaneous speed* at the moment t_1 is defined as an average speed over time interval $[t_1, t_2]$, when t_2 is “infinitely close” to t_1. More precisely,

$$u(t_1) = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1} = s'(t_1).$$

Note that in the literature simpler terms “velocity” and “speed” are often used instead of “instantaneous velocity” and “instantaneous speed”.

Instantaneous velocity and instantaneous speed are also functions $v\colon [0, T] \to \mathbf{R}$ and $u\colon [0, T] \to \mathbf{R}$.

PROPOSITION 3. *For all $t \in [0, T]$ it is true that $u(t) = |v(t)|$.*

This proposition is a corollary of the fact that the function x is differentiable and piecewise monotone and we leave it to the reader to verify this.

If the object is moving in positive direction, then position function is not decreasing and $v \geq 0$; if the object turns around at t_0, then $v(t_0) = 0$; and if the object is moving in negative direction, then position function is not increasing and $v \leq 0$.

The velocity of an object is defined in terms of the change of position of that object over time. A quantity used to describe the change of the velocity of an object over time is the vector acceleration.

The *average vector acceleration* over a time interval $[t_1, t_2]$ is defined as

$$a_{v\text{-}avg} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}.$$

The *average path acceleration* over a time interval $[t_1, t_2]$ is defined as

$$a_{u\text{-}avg} = \frac{u(t_2) - u(t_1)}{t_2 - t_1}.$$

Note the similarity between the definition of the average velocity (average speed) and the definition of the average vector acceleration (average path acceleration).

We continue assuming further that function v is differentiable on $(0, T)$ and that there exist $v'_+(0)$ and $v'_-(T)$. In that case the following holds:

PROPOSITION 4. *The function u is differentiable on $(0, T)$ except perhaps in points in which function x has extreme values. In these points u'_+ and u'_- exist. Also, there exists $u'_+(0)$ and $u'_-(T)$.*

We leave to interested reader to prove this (see also Example 3 below).

We call $a_v(t) = v'(t)$ and $a_u(t) = u'(t)$ (when they exist) *instantaneous vector acceleration* and *instantaneous path acceleration*, respectively. It is convenient to say, shortly, "vector acceleration" and "path acceleration", respectively.

Note that in physical literature usually only notion of acceleration is used, which is in our terminology vector acceleration.

Notice that under assumptions listed above $a_v(t)$ exists for all $t \in [0, T]$, as far as $a_u(t)$ not necessarily exists for those moments t in which the position function x has extreme values. Also, both $a_v(t)$ and $a_u(t)$ can be positive as well as negative.

A positive vector acceleration is in general interpreted to mean an increase in speed. However, this is not correct. This is obviously true if the velocity is positive, and the velocity is increasing with time. However, it is also true for negative velocities if the velocity becomes less negative over time.

EXAMPLE 2. (Simple harmonic motion) Example of a motion where we can notice the difference between path and position function, as well as the difference between speed and velocity, and difference between path acceleration and vector acceleration, is the simple harmonic motion. Simple harmonic motion is any motion for whose position function x the following holds:

$$x''(t) = -k^2 x(t),$$

where k is a positive constant. We may assume that $k = 1$ and $t \in [0, 2\pi]$ as well as $x(0) = 0$ and $x'(0) = 1$. Under these conditions position function is uniquely determined, i.e., $x(t) = \sin t$.

EXAMPLE 3. If position function is defined as

$$x(t) = \sin t, \quad t \in [0, \pi]$$

then the path function is specified as

$$s(t) = \begin{cases} \sin t, & t \in [0, \pi/2] \\ 2 - \sin t, & t \in [\pi/2, \pi]. \end{cases}$$

For instantaneous velocity at the moment $t \in [0, \pi]$ it is true that

$$v(t) = x'(t) = \cos t$$

and for instantaneous speed at the moment $t \in [0,\pi]$ it is true that

$$u(t) = s'(t) = |\cos t|.$$

Finally, for instantaneous vector acceleration at the moment $t \in [0,\pi]$ we have that

$$a_v(t) = -\sin t$$

and for instantaneous path acceleration at the moment $t \in [0,\pi]$ we have

$$a_u(t) = \begin{cases} -\sin t, & t \in [0,\pi/2) \\ \sin t, & t \in (\pi/2,\pi] \end{cases}$$

where $a_u(\pi/2)$ does not exist. See Fig. 4.

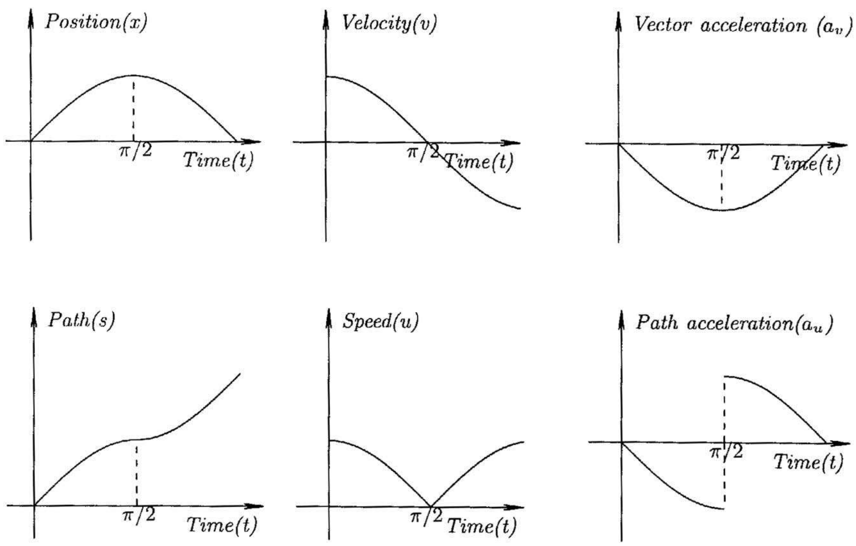

Fig. 4

EXAMPLE 4. (Constant acceleration) Objects falling under the influence of gravity are examples of objects moving with constant vector acceleration. A constant vector acceleration means that the vector acceleration does not depend on time:

$$a = v'(t).$$

Integrating this equation, the velocity of the object can be obtained:

$$v(t) = v_0 + at,$$

where v_0 is the velocity of object at the moment $t = 0$. From the velocity, the position of object as function of time can be calculated:

$$x(t) = x_0 + v_0 t + at^2/2,$$

where x_0 is the position of object at the moment $t = 0$.

A special case of constant vector acceleration is the free fall (falling in vacuum). In problems of free fall, the direction of free fall is defined along the y-axis, and the positive position along the y-axis corresponds to upward motion. The acceleration due to gravity is equal to constant g (along the negative y-axis). The equations of motion for free fall are very similar to those discussed previously for constant vector acceleration:

$$a(t) = v'(t) = -g,$$
$$v(t) = v_0 - gt,$$
$$y(t) = y_0 + v_0 t - gt^2/2,$$

where y_0 and v_0 are the position and the velocity of the object at the moment $t = 0$. Note that in this example we have not chosen the position of the object at time $t = 0$ as the origin. The origin is the position of the object at the moment when motion is completed.

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