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# HOW TO UNDERSTAND GRASSMANNIANS?

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*Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday*

**Abstract.** Grassmannians or Grassmann manifolds are very important manifolds in modern mathematics. They naturally appear in algebraic topology, differential geometry, analysis, combinatorics, mathematical physics, etc. Grassmannians have very rich geometrical, combinatorial and topological structure, so understanding them has been one of the central research themes in mathematics. They occur in many important constructions such as universal bundles, flag manifolds and others, hence studying their properties and finding their topological and geometrical invariants is still a very attractive question.

In this article we offer a quick introduction into the geometry of Grassmannians suitable for readers without any previous exposure to these concepts.

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## 1. Introduction

Before we start with formalization of abstract ideas and objects, we are going to do several elementary exercises involving matrices, equivalence relations and other objects familiar to the reader. Understanding "invisible" mathematical spaces is nothing but deep understanding of mathematics that we believe to know well. Here we demonstrate how some of the standard facts about $(2 \times 4)$-matrices facilitate understanding of Grassmannians $G_2^+(\mathbb{R}^4)$.

Definition 1.1. Let $M_2(2\times 4)$ be the set of all $2\times 4$ matrices of rank 2. Two matrices $B$, $C \in M_2(2\times 4)$ are called equivalent $B \sim C$ if and only if there exists a $2\times 2$ matrix $A$ such that $C = A\cdot B$ and $\det A > 0$.

Exercise 1. Show that the relation $\sim$ is indeed an equivalence relation on the set $M_2(2\times 4)$ of all $(2\times 4)$-matrices.

For a $2\times 4$ matrix $B$ let $B_{ij}$ be the associated $[ij]$-minor, i.e. the determinant of the $2\times 2$ matrix whose columns are $i$-th and $j$-th column of the matrix $B$.

Exercise 2. Show that the following relation holds for each matrix $B \in M_2(2\times 4)$,

$$B_{12}\cdot B_{34} - B_{13}\cdot B_{24} + B_{14}\cdot B_{23} = 0.$$

Indeed let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$ be a $2\times 4$ matrix. Then

$B_{12}\cdot B_{34}-B_{13}\cdot B_{24}+B_{14}\cdot B_{23}=(b_{11}b_{22}-b_{12}b_{21})(b_{13}b_{24}-b_{14}b_{23})-$
$(b_{11}b_{23}-b_{13}b_{21})(b_{12}b_{24}-b_{14}b_{22})+(b_{11}b_{24}-b_{14}b_{21})(b_{12}b_{23}-b_{13}b_{22})=$
$b_{11}b_{13}b_{22}b_{24}-b_{12}b_{13}b_{21}b_{24}-b_{11}b_{14}b_{22}b_{23}+b_{12}b_{14}b_{21}b_{23}-$
$b_{11}b_{12}b_{23}b_{24}+b_{12}b_{13}b_{21}b_{24}+b_{11}b_{14}b_{22}b_{23}-b_{13}b_{14}b_{21}b_{22}+$
$b_{11}b_{12}b_{23}b_{24}-b_{12}b_{14}b_{21}b_{23}-b_{11}b_{13}b_{22}b_{24}+b_{13}b_{14}b_{21}b_{22}=$
$=0.$

The following exercise is also a consequence of elementary linear algebra.

EXERCISE 3. Let $B$, $C\in M_2(2\times 4)$ be two matrices such $B\sim C$ and let $C=A\cdot B$. Then

$$C_{ij}=\det(A)\cdot B_{ij}.$$

Consider now the subset $\Omega\subset\mathbb{R}^6\setminus\{0\}$ such that $(x,y,z,t,u,v)\in\Omega$ iff $xy-zt+uv=0$. Define the relation $\equiv$ as $(x_1,y_1,z_1,t_1,u_1,v_1)\equiv(x_2,y_2,z_2,t_2,u_2,v_2)$ iff $x_2=\lambda x_1$, $y_2=\lambda y_1$, $z_2=\lambda z_1$, $t_2=\lambda t_1$, $u_2=\lambda u_1$ and $v_2=\lambda v_1$ for some positive real number $\lambda$. Next exercise is obvious.

EXERCISE 4. Relation $\equiv$ is an equivalence relation on set $\Omega$.

Our aim is to find connections between sets of equivalence classes of $\sim$ and $\equiv$, $M_2(2\times 4)/\sim$ and $\Omega/\equiv$.

Now, we define the function $f: M_2(2\times 4)\to\Omega$ by

$$f(B)=(B_{12},B_{34},B_{13},B_{24},B_{14},B_{23}).$$

In fact we are interested in function induced with $f$ which we will also denote with $f$, $f: M_2(2\times 4)/\sim\to\Omega/\equiv$ defined on classes with

$$f([B])=[(B_{12},B_{34},B_{13},B_{24},B_{14},B_{23})].$$

Using previous exercises we get:

EXERCISE 5. Function $f: M_2(2\times 4)/\sim\to\Omega/\equiv$ is well defined.

We shall prove that $f$ is a bijection.

EXERCISE 6. Function $f: M_2(2\times 4)/\sim\to\Omega/\equiv$ is injective.

If $f([B])=f([C])$ then $B_{ij}=\lambda C_{ij}$, for some $\lambda>0$. There exist $B_{ij}\neq 0$. Let $\overline{B}_{ij}$ be the $2\times 2$ matrix whose columns are $i$-th and $j$-th column of matrix $B$. Obviously, because $\overline{B}_{ij}$ is nonsingular, so is $\overline{C}_{ij}$, and there is a matrix $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$ such that $\overline{B}_{ij}=A\cdot\overline{C}_{ij}$. We directly get $detA=\lambda$. Also $b_{1i}=a_{11}c_{1i}+a_{12}c_{2i}$, $b_{1j}=a_{11}c_{1j}+a_{12}c_{2j}$, $b_{2i}=a_{21}c_{1i}+a_{22}c_{2i}$ and $b_{2j}=a_{21}c_{1j}+a_{22}c_{2j}$. Let $k\neq i$, $j$. From equations $B_{ik}=\lambda C_{ik}$ and $B_{jk}=\lambda C_{jk}$ we get

$$(b_{1i}b_{2j}-b_{1j}b_{2i})b_{1k}=\lambda((b_{1j}c_{1i}-b_{1i}c_{1j})c_{2k}+(b_{1i}c_{2j}-b_{1j}c_{2i})c_{1k}).$$

After substitutions and cancellations we get more convenient form

$$\det A\cdot(c_{1i}c_{2j}-c_{1j}c_{2i})b_{1k}=\det A\cdot(c_{1i}c_{2j}-c_{1j}c_{2i})(a_{11}c_{1k}+a_{12}c_{2k}).$$

Thus $b_{1k} = a_{11}c_{1k} + a_{12}c_{2k}$. Continuing in this fashion we prove $B = A \cdot C$ which proves injectivity of $f$.

Exercise 7. Function $f : M_2(2 \times 4)/\sim \longrightarrow \Omega/\equiv$ is surjective.

Suppose $[(x, y, z, t, u, v)]$, $(x, y, z, t, u, v) \in \Omega$ is given. Without loss of generality we suppose $x \neq 0$ and $x = 1$. Then it is easy to check that (following from the definition of $\Omega$) $f$ maps $\begin{bmatrix} 1 & 0 & -v & -t \\ 0 & 1 & z & u \end{bmatrix}$ to $[(1, y, z, t, u, v)]$ and it is surjective.

So far, we have concluded that we can identify sets $M_2(2 \times 4)/\sim$ and $\Omega/\equiv$. One could object: "This is interesting, but we cannot visualize neither one of the objects." Luckily this is not true. For $[(x, y, z, t, u, v)] \in \Omega/\equiv$ we can always take $(x, y, z, t, u, v) \in \Omega$ such that $x^2 + y^2 + z^2 + t^2 + u^2 + v^2 = 1$. There exists real numbers $p$, $q$, $r$, $s$, $m$ and $n$ such that

$$x = p + q,\ y = p - q,\ z = r + s,\ t = r - s,\ u = m + n \text{ and } v = m - n.$$

Set this into $x^2 + y^2 + z^2 + t^2 + u^2 + v^2 = 1$ and $xy - zt + uv = 0$, and get

$$p^2 + q^2 + r^2 + s^2 + m^2 + n^2 = \tfrac{1}{2} \text{ and } p^2 + s^2 + m^2 = q^2 + r^2 + n^2.$$

Thus

$$p^2 + s^2 + m^2 = q^2 + r^2 + n^2 = \frac{1}{4}.$$

Obviously if we take $p$, $q$, $r$, $s$, $m$ and $n$ such that $p^2 + s^2 + m^2 = q^2 + r^2 + n^2 = \frac{1}{4}$ by reverse process we get $[(x, y, z, t, u, v)] \in \Omega/\equiv$ and we can identify this two sets. But, the set $\{(p, q, r, s, m, n) \in \mathbb{R}^6 \mid p^2 + s^2 + m^2 = q^2 + r^2 + n^2 = \frac{1}{4}\}$ is nothing but $S^2 \times S^2$.

Besides getting this nice result using elementary methods, we proved that the set of all $2 \times 4$ matrices of rank 2 modulo $\sim$ is in fact $S^2 \times S^2$. But the set of all $2 \times 4$ matrices of rank 2 modulo $\sim$ is nothing but the set of oriented 2-planes in $\mathbb{R}^4$, that is, rows of a matrix of rank 2 are two linearly independent vectors in $\mathbb{R}^4$ and the multiplication with a $2 \times 2$ matrix with positive determinant is an orientation preserving change of base for 2-plane in $\mathbb{R}^4$. This set is called the oriented Grassmannian $G_2^+(\mathbb{R}^4)$.

Now we proceed with more formal treatment of Grassmannians. We will try to illuminate them from the viewpoint of various branches of mathematics.

## 2. Topological manifolds and coordinate charts

Since Grassmannians are examples of manifolds let us provide a brief introduction to manifolds in general.

Suppose $M$ is a topological space. We say that $M$ is a *topological manifold of dimension $n$* or a *topological $n$-manifold* if it is *locally Euclidean of dimension $n$*. That means that every point $p \in M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Example 2.1. It follows directly from the definition that every open subset of $\mathbb{R}^n$ is a topological $n$-manifold.

Let $M$ be a topological $n$-manifold. A *coordinate chart* (or just a *chart* on $M$) is a pair $(U, \varphi)$, where $U$ is open subset of $M$ and $\varphi : U \to \tilde{U}$ is homeomorphism from $U$ to open subset $\tilde{U} \subset \mathbb{R}^n$. The definition of topological manifold implies that each point $p \in M$ is contained in the domain of some coordinate chart $(U, \varphi)$.

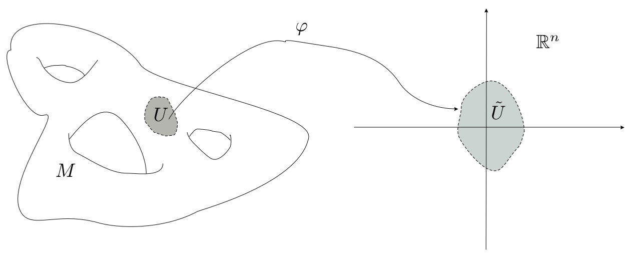


Fig. 1

Given a chart $(U, \varphi)$, we call the set $U$ a *coordinate domain*, or a *coordinate neighborhood* of each of its points. We say that chart $(U, \varphi)$ contains $p$. The map $\varphi$ is called a *local coordinate map*, and the component coordinate functions of $\varphi$ are called *local coordinates* on $U$.

Example 2.2. (*Spheres*) Let $\mathbb{S}^n$ denote the unit $n$-sphere, the set of unit-length vectors in $\mathbb{R}^{n+1}$:

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Let $U_i^+$ denote the subset of $\mathbb{S}^n$ where the $i$-th coordinate is positive:

$$U_i^+ = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_i > 0\}.$$

Similarly, $U_i^-$ is the subset where $x_i < 0$, Fig. 2.

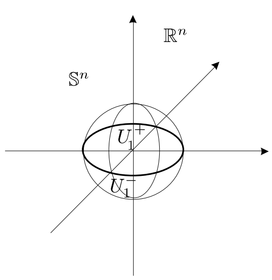


Fig. 2

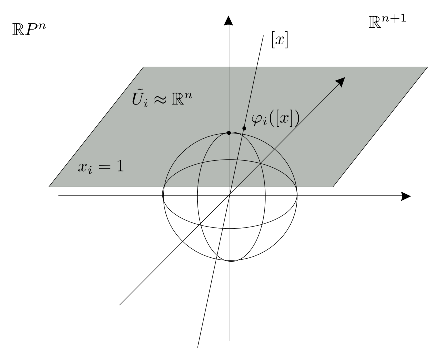


Fig. 3

For each $i$ define maps $\varphi_i^{\pm} : U_i^{\pm} \to \mathbb{R}^n$ by:

$$\varphi_i^{\pm}(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

where the hat over $x_i$ indicates that $x_i$ is omitted. Each $\varphi_i^{\pm}$ is evidently a continuous map. It is a homeomorphism onto its image, the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$, because it has a continuous inverse given by

$$(\varphi_i^{\pm})^{-1}(u_1, u_2, \ldots, u_n) = (u_1, u_2, \ldots, u_{i-1}, \pm\sqrt{1 - \|u\|}, u_{i+1}, \ldots, x_{n+1}).$$

Thus, every point of $\mathbb{S}^n$ is contained at least in one of these $2n+2$ charts, hence $\mathbb{S}^n$ is topological $n$-manifold.

EXAMPLE 2.3. (*Projective spaces*) The $n$-dimensional real projective space, denoted by $\mathbb{R}P^n$, is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$. We give it the quotient topology determined by the natural projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ sending each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the line through $x$ and 0. For any point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ let $[x] = \pi(x)$ denote the equivalence class of $x$.

For each $i = 1, 2, \ldots, n+1$, let $\tilde{U_i} \subset \mathbb{R}^{n+1} \setminus \{0\}$ be the set where $x_i \neq 0$, and let $U_i = \pi(\tilde{U_i}) \subset \mathbb{R}P^n$. Since natural projection $\pi$ is a quotient map, it is continuous and an open map, so $U_i$ is open in $\mathbb{R}P^n$ and $\pi : \tilde{U_i} \to U_i$ is quotient map. Define a map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x]) = \varphi_i([x_1, x_2, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

This map is well defined because its value is unchanged when multiplying $x$ by nonzero constant. Since composition $\varphi_i \circ \pi$ is continuous and $\pi$ is quotient map then $\varphi_i$ is also continuous. We easily see that $\varphi_i$ is homeomorphism because it has inverse

$$(\varphi_i)^{-1}(u) = (\varphi_i)^{-1}(u_1, u_2, \ldots, u_n) = [u_1, \ldots, u_{i-1}, 1, u_i, \ldots, u_n].$$

Geometrically, if we identify $\mathbb{R}^n$ in the obvious way with $x_i = 1$ then $\varphi_i([x])$ can be interpreted as the point where the line $[x]$ intersect this subspace, Fig. 3. Because every point of $\mathbb{R}P^n$ lies in some chart $U_i$ (thus has neighborhood homeomorphic to $\mathbb{R}^n$) we proved that $\mathbb{R}P^n$ is topological $n$-manifold.

## 3. Smooth maps and smooth manifolds

Let $U$ be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ with coordinate functions $f_i : U \to \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$.

DEFINITION 3.1. Function $f : U \to \mathbb{R}^m$ is *smooth* if all of its partial derivatives $\frac{\partial^k f_i}{\partial x_{i_1} \ldots \partial x_{i_k}}$ exist and are continuous on $U$. If $f$ is bijection and $f^{-1}$ is smooth than $f$ is *diffeomorphism*.

Let $M$ be a topological manifold; an *atlas* for $M$ is a collection of charts $(U, \varphi)$ whose domain covers $M$.

DEFINITION 3.2. A topological $n$-manifold $M$ is a *smooth manifold* if for every two charts $(U, \varphi)$ and $(V, \phi)$ such that $U \cap V \neq \emptyset$, the composite map

$$\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V),$$

called *transition map*, is a diffeomorphism, Fig. 4.

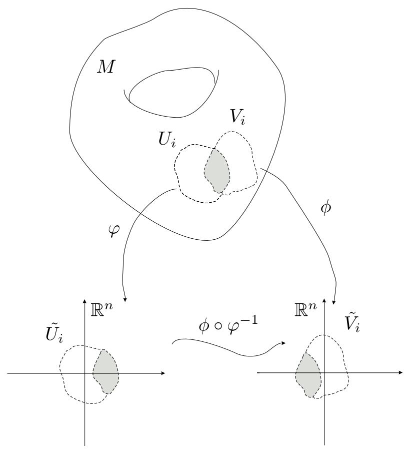


Fig. 4

PROPOSITION 3.1. *A unit sphere $\mathbb{S}^n$ with charts from example 2.2 is a smooth manifold.*

PROPOSITION 3.2. *Real projective space $\mathbb{R}P^n$ with charts from example 2.3 is a smooth manifold.*

## 4. Grassmannians

Let $V$ be an $n$-dimensional real vector space. For any integer $0 \leq k \leq n$, we let $G_k(V)$ denote the set of all $k$-dimensional linear subspaces of $V$. We will show that $G_k(V)$ is a smooth manifold of dimension $k(n-k)$. First we need to construct charts which cover $G_k(V)$ and then prove that transition maps are diffeomorphisms.

Let $P$ and $Q$ be any complementary subspaces of $V$ of dimensions $k$ and $(n-k)$, respectively, so that $V$ decomposes as a direct sum: $V = P \oplus Q$. The graph of any linear map $A : P \to Q$ is a $k$-dimensional linear subspace $\Gamma(A) \subset V$ , defined by

$$\Gamma(A) = \{x + Ax \mid x \in P\}.$$

Notice that for all $A$ we have $\Gamma(A) \cap Q = 0$ because $x + Ax \in \Gamma(A) \cap Q$ implies $x + Ax \in Q$, and since $Ax \in Q$, we got $x \in Q$. But $P \cap Q = 0$ so $x = 0$. Now we shall prove that any $k$-dimensional linear subspace $T$ with property $T \cap Q = \{0\}$ is the graph of a unique linear map $A : P \to Q$. We conclude this from a fact that every $t \in T$ has unique decomposition $t = p + q$, $p \in P$, $q \in Q$. Set $Ap = q$. At first $A$ is well defined because $t = p + q$ and $t_1 = p + q_1$ implies $q - q_1 = t - t_1 \in T$, and since $q - q_1 \in Q$ we get $q - q_1 = 0$. Obviously, $A$ is linear, Fig. 5.

Let $L(P, Q)$ denote the vector space of linear maps from $P$ to $Q$, and let $U_Q$ denote the subset of $G_k(V)$ consisting of $k$-dimensional subspaces whose intersection with $Q$ is trivial. Define a map $\psi : L(P, Q) \to U_Q$ by

$$\psi(A) = \Gamma(A).$$

The discussion above shows that $\psi$ is a bijection. Let $\varphi = \psi^{-1} : U_Q \to L(P, Q)$. By choosing bases for $P$ and $Q$, we can clearly identify $L(P, Q)$ with $M((n-k) \times k, \mathbb{R})$ (space of all $(n-k) \times k$ matrices), thus with $R^{(n-k)k}$, so we can think about $(U_Q, \varphi)$ as a coordinate chart.

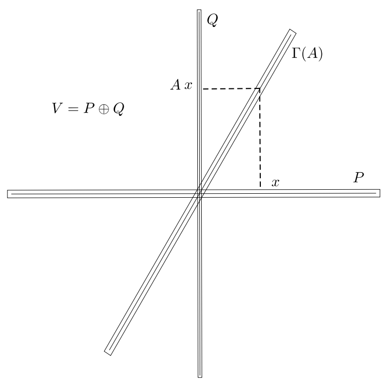


Fig. 5

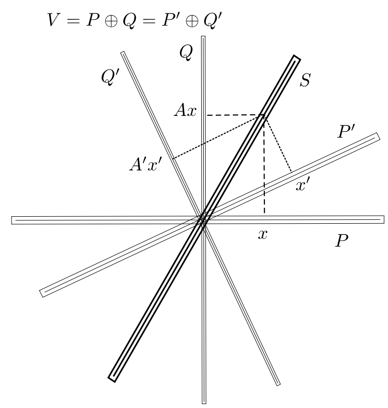


Fig. 6

We need to prove that all transition maps are smooth. Let $(P, Q)$ and $(P', Q')$ be a pair of subspaces such that $V = P \oplus Q = P' \oplus Q'$ and $\dim P = \dim P' = k$, $\dim Q = \dim Q' = n - k$. Let $(U_Q, \varphi)$ and $(U_{Q'}, \varphi')$ be corresponding charts, and $\psi = \varphi^{-1}$, $\psi' = \varphi'^{-1}$. The set $\varphi(U_Q \cap U_{Q'}) \subset L(P, Q)$ consist of all $A \in L(P, Q)$ whose graphs intersects both $Q$ and $Q'$ trivially. It is easily seen this set is open. The transition map is $\varphi' \circ \varphi^{-1} = \varphi' \circ \psi$.

Suppose $A \in \varphi(U_Q \cap U_{Q'}) \subset L(P, Q)$ is arbitrary, and let $S$ denote the set $\psi(A) = \Gamma(A) \subset V$, Fig. 6. If we put $A' = \varphi' \circ \psi(A)$, then $A'$ is the unique linear map from $P'$ to $Q'$ whose graph is equal to $S$. To identify this map, let $x' \in P'$ be arbitrary, and note that $A'x'$ is the unique element of $Q'$ such that $x' + A'x' \in S$, which is to say that

$$x' + A'x' = x + Ax \text{ for some } x \in P.$$

In fact, such $x$ is unique and has the property

$$x + Ax - x' \in Q'.$$

If we let $I_A : P \to V$ denote the map $I_A(x) = x + Ax$ and let $\pi_{P'} : V \to P'$ be the projection onto $P'$ with kernel $Q'$, then $x$ satisfies

$$0 = \pi_{P'}(x + Ax - x') = \pi_{P'} \circ I_A(x) - x',$$

because $x' \in P'$, $P' \cap Q' = 0$. As long as $A$ stays in the open subset of maps whose graphs intersect both $Q$ and $Q'$ trivially, $\pi_{P'} \circ I_A : P \to P'$ is invertible since projection of $\Gamma(A)$ on both $P$ and $P'$ is bijection, and thus we can solve this last equation for $x$ to obtain $x = (\pi_{P'} \circ I_A)^{-1}(x')$. Therefore, $A'$ is given in terms of $A$ by

$$A'x' = I_A x - x' = I_A \circ (\pi_{P'} \circ I_A)^{-1}(x') - x'. \tag{1}$$

Let $(e'_i)$ be basis for $P'$ and $(f'_i)$ basis for $Q'$. The columns of matrix representation of $A'$ are the components of $A'v_i$ vector. By (1) this could be written

$$A'e'_i = I_A \circ (\pi_{P'} \circ I_A)^{-1}(e'_i) - e'_i.$$

The matrix entries of $I_A$ clearly depend smoothly on those of A, and so do those of $\pi_{P'} \circ I_A$. By Cramer's rule, the components of the inverse of a matrix are rational functions of the matrix entries, so the expression above shows that the components of $A'e'_i$ depend smoothly on the components of $A$. Since $A$ is linear map, this proves that $\varphi' \circ \varphi^{-1}$ is a smooth map.

The smooth manifold $G_k(V)$ is called the Grassmann manifold of $k$-planes in $V$, or simply a Grassmannian. In the special case $V = \mathbb{R}^n$, the Grassmannian $G_k(\mathbb{R}^n)$ is often denoted by some simpler notation such as $G_{k,n}$ or $G(k,n)$. Note that $G_1(\mathbb{R}^{n+1})$ is exactly the $n$-dimensional projective space $\mathbb{R}P^n$.

## 5. Other interpretations of Grassmannians

Let $L$ be a $k$-dimensional plane through the origin in $\mathbb{R}^n$. There is a unique operator of orthogonal projection $P$ (equivalently its matrix) onto $L$ with respect to the scalar product. We identify each such $k$-dimensional plane with corresponding operator of orthogonal projection.

Proposition 5.1. *The operator of orthogonal projection $P$ onto $k$-dimensional plane through the origin in $\mathbb{R}^n$ is idempotent, in other words it satisfies $P^2 = P$.*

Let $L$ be a $k$-dimensional plane in $\mathbb{R}^n$ and $P$ corresponding operator of orthogonal projection. Then for every $u \in L$ we have $Pu = u$. Since $Pv \in L$ for arbitrary $v \in \mathbb{R}^n$, then $P^2 v = P(Pv) = Pv$ and proposition is proved.

Proposition 5.2. *The operator of orthogonal projection $P$ onto $k$-dimensional plane in $\mathbb{R}^n$ through the origin is symmetric, in other words, its matrix satisfies $P^t = P$.*

To prove this we shall use the following fact from linear algebra:

Proposition 5.3. *The operator $P$ is symmetric if and only if for every $u$, $v \in \mathbb{R}^n$,*

$$\langle Pu, v\rangle = \langle u, Pv\rangle,$$

*where $\langle\ ,\ \rangle$ is scalar product in $\mathbb{R}^n$.*

Decompose $\mathbb{R}^n = L \oplus L^\perp$ and let $e_1, e_2, \cdots, e_k \in L$, $e_{k+1}, e_{k+2}, \cdots, e_n \in L^\perp$ be orthogonal basis for $R^{n+1}$. For every $u \in L^\perp$ we have $P_u = 0$. According to Proposition 5.3., since $P$ is linear operator we need to check that $\langle Pe_i, e_j\rangle = \langle e_i, Pe_j\rangle$ for every $e_i$, $e_j$. If both $e_i$, $e_j \in L$ then $\langle Pe_i, e_j\rangle = \langle e_i, e_j\rangle = \langle e_i, Pe_j\rangle$. If both $e_i$, $e_j \in L^\perp$ then $\langle Pe_i, e_j\rangle = \langle 0, e_j\rangle = 0 = \langle e_i, 0\rangle = \langle e_i, Pe_j\rangle$. If $e_i \in L$ and $e_j \in L^\perp$ then (using fact $e_i \perp e_j$) $\langle Pe_i, e_j\rangle = \langle e_i, e_j\rangle = 0 = \langle e_i, 0\rangle = \langle e_i, Pe_j\rangle$, and case $e_i \in L^\perp$ and $e_j \in L$ is analogous. Thus we proved proposition 5.2 .

PROPOSITION 5.4. *The operator $P$ of orthogonal projection onto a $k$-dimensional plane in $\mathbb{R}^n$ through the origin satisfies*

$$\operatorname{tr} P = k.$$

Let $L$ be a $k$-dimensional plane in $\mathbb{R}^n$ and $P$ corresponding operator of orthogonal projection. Since $P_u = 0$ for $u \in L^\perp$ and $Pv = v$ for $v \in L$ then 0 and 1 are only eigenvalues for $P$. Since $\dim L = k$ multiplicity of eigenvalue 1 is greater or equal $k$, and $\dim L^\perp = n-k$ multiplicity of eigenvalue 0 is greater or equal $n-k$. So equality must hold and multiplicity of 1 is $k$. Then $\operatorname{tr} P$ is equal to the sum of its eigenvalues, so $\operatorname{tr} P = k$.

PROPOSITION 5.5. *Every operator $P$ (matrix $n\times n$) such that $P^2 = P$, $P^t = P$ and $\operatorname{tr} P = k$ is the operator of orthogonal projection onto some $k$-dimensional plane in $\mathbb{R}^n$ throw origin.*

We first prove that $P$ does not have eigenvalues other then 0 and 1. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $P$ and let $v \in \mathbb{R}^n$ be a corresponding eigenvector. Because $P$ is symmetric we have

$$\lambda^2\|v\| = \langle \lambda v, \lambda v\rangle = \langle Pv, Pv\rangle = \langle P(Pv), v\rangle = \langle Pv, v\rangle = \langle \lambda v, v\rangle = \lambda\|v\|.$$

Now, $\lambda^2 - \lambda = 0$, and $\lambda \in \{0, 1\}$.

Let $L = \{z \mid Pz = z\}$. For $v \in L^\perp$ we have

$$0 = \langle z, v\rangle = \langle Pz, v\rangle = \langle z, Pv\rangle,$$

so $Pv \in L^\perp$. On the other hand $P(Pv) = Pv$ and $Pv \in L$. Thus $Pv \in L \cap L^\perp$ so $Pv = 0$. Since $\operatorname{tr} P = k$ then $\dim L \le k$ and $\dim L^\perp \le n-k$. But $\dim L + \dim L^\perp = n$ and $\dim L = k$ so $P$ is orthogonal projection onto $L$.

Let $H(n)$ be the space of symmetric $n \times n$ matrices which itself is Euclidean space of dimension $\frac{n(n+1)}{2}$ . Then. from previous discussion we get that identification $\Phi : G_k(\mathbb{R}^n) \to H(n)$, $L \to P_L$ is a homeomorphism onto its image

$$\Phi(G_k(\mathbb{R}^n)) = \left\{P \in H(n) \mid P^2 = P, \operatorname{tr} P = k\right\}.$$

From this interpretation, the compactness of $G_k(\mathbb{R}^n)$ follows directly.

## 6. Group actions

Let $G$ be a group and $X$ a topological space. A left action of $G$ on $X$ is a continuous map

$$\varrho : G \times X \to X$$

such that

(i) $\varrho(g, \varrho(h, x)) = \varrho(gh, x)$ for $g, h \in G, x \in X$,

(ii) $\varrho(e, x) = x$ for $x \in X, e \in G$ unit.

A left $G$-space (also, a transformation group) is a pair $(X, \varrho)$ consisting of a space $X$ together with a left action $\varrho$ of $G$ on $X$. It is convenient to denote $\varrho(g, x)$ by $gx$. Then rules $(i)$ and $(ii)$ take the familiar form $g(hx) = (gh)x$ and $ex = x$.

A right action is a map $X \times G \to X$, $(x, g) \to xg$ satisfying $(xh)g = x(hg)$ and $xe = x$. If $(x, g) \to xg$ is right action, then $(g, x) = xg^{-1}$ is a left action.

The left translation $L_g : X \to X$, $x \to gx$ by $g$ is a homeomorphism of $X$ with the inverse $L_{g^{-1}}$. This follows from the rules $L_g L_h = L_{gh}$, $L_e = id(X)$, which are just reformulations of definition of group action. Thus the map $g \to L_g$ is homomorphism of $G$ into the group of homeomorphism of $X$.

Let $X$ be a $G$-space. Then $R = \{(x, gx) \mid x \in X, g \in G\}$ is an equivalence relation on $X$. The set of equivalence classes $X \pmod R$ is denoted $X/G$. The quotient map $q : X \to X/G$ is used to provide $X/G$ with the quotient topology. This space is called the orbit space of the $G$-space $X$. The equivalence class of $x \in X$ is called the orbit $Gx$ through $x$. A more systematic notation would be $G \setminus X$ for the orbit space of a left action and $X/G$ for the orbit space of a right action. An action of $G$ on $X$ is called transitive if $X$ consists of single orbit.

Example 6.1. Let $H$ be a subgroup of topological group $G$. The group multiplication $H \times G \to G$, $(h, g) \to hg$ is a left action. There is similar right action. A group also acts on itself by conjugation $G \times G \to G$, $(g, h) \to ghg^{-1}$.

A subset $F \subset X$ of a $G$-space $X$ is called a fundamental domain of this $G$-space if $F \subset X \to X/G$ is bijective. A fundamental domain contains exactly one point from each orbit. Usually, there are many different fundamental domains, and the problem then is to choose one with particularly nice geometric properties. For each $x \in X$, the set $G_x = \{g \in G \mid gx = x\}$ is a subgroup of $G$. This subgroup is called the isotropy group of the $G$-space $X$ at $x$.

Proposition 6.1. *Let $X$ be a $G$-space and $x \in X$. The map $G \to X$, $g \to gx$ is constant on cosets $gG_x$ and induces an injective map $q_x : G/G_x \to X$ whose image is the orbit through $x$.*

Let $g_1$, $g_2 \in G$ be such that $g_1 G_x = g_2 G_x$ that is $g_1^{-1} g_2 \in G_x$. Then we have $g_1^{-1} g_2 x = x$ and thus $g_1 x = g_2 x$ that proves $q_x$ is well defined. Now let $g$, $h \in G$ be such that $q_x(gG_x) = q_x(hG_x)$. Then $gx = hx$ and $g^{-1}hx = x$. Thus $g^{-1}h \in G_x$ and $gG_x = hG_x$ and $q_x$ is injective map. Image of $q_x$ is obviously the orbit through $x$.

Let $G_k(\mathbb{R}^n)$ be the set of $k$-dimensional linear subspaces of $\mathbb{R}^n$. The standard action of $O(n)$ (group of all orthogonal matrices $AA^t = I$) on $\mathbb{R}^n$ maps $k$-spaces to $k$-spaces and thus induces action of $O(n)$ on $G_k(\mathbb{R}^n)$. This action is transitive since any $k$-space can be transformed into any other $k$-space by some $A \in O(n)$. Let $x \in G_k(\mathbb{R}^n)$ be a $k$-space spanned by $e_1, e_2, \dots, e_k$, where $e_i$, $i = 1, 2, \dots, n$ is standard basis for $\mathbb{R}^n$. Then, the corresponding isotropy group $G_x$ is the subgroup

$O(k) \times O(n-k)$ of $O(n)$ consisting of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A \in O(k),\ B \in O(n-k).$$

Thus from the proposition 6.1, since the action is transitive we get

$$G_k(\mathbb{R}^n) \cong O(n)/(O(k) \times O(n-k)).$$

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