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ON THE EXISTENCE OF $\lim\limits_{x\to x_0} f(g(x))$

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Abstract. In this article we discuss limits of composite functions in the general setting of topological spaces. We include here some of its technical applications.

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0. Introduction

Given $x_0 \in \mathbb{R}$, and functions f and g such that $\lim_{x\to x_0} g(x) = y_0$ and $\lim_{y\to y_0} f(y)$ exists, is it true that $\lim_{x\to x_0} f(g(x))$ always exists and equals $\lim_{y\to y_0} f(y)$? We can immediately check that neither $\lim_{x\to 0} f(x\sin(1/x)) = 1$ nor $\lim_{x\to 0} f(g(x)) = 1$ if

$$f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

and $g(x) := 0$. Thus, the answer to the question above is negative since, as shown by these examples, we have a case in which either $\lim_{x\to x_0} f(g(x))$ does not exist or it exists but does not equal $\lim_{y\to y_0} f(y)$.

However, it is correct to infer $\lim_{x\to 0} f(x\sin(1/x)) = 1$ if

$$f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$$

since we know, by the continuity of f at 0,

$$\lim_{x\to 0} f(x\sin(1/x)) = f(\lim_{x\to 0} x\sin(1/x)) = f(0) = 1.$$

Apart from that, interestingly, we may correctly deduce from $\lim_{y\to 0}(\sin y)/y = 1$ that

$$\lim_{x\to 0} \frac{\sin(x\sin(1/x))}{x\sin(1/x)} = 1,$$

and

$$\lim_{x\to 0} f(g(x)) = 1$$

if

$$f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

and $g(x) := x \sin x$, but what valid argument allows us to use this substitution method? Does putting the condition that f is not defined at y_0 (in addition to the conditions given) implies that the limit of the composition exists? Instead of the well-known continuity condition of f, is there any other additional condition, that implies the existence of limit?

1. The proposition and examples

The proposition below simply provides answers to the questions above. In fact, it is a particular version of Theorem 1 in [3]. We shall discuss and prove its general version in the next section. We will find in the sequel that the following notions are beneficial. We say that a function $f\colon D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}$, is *eventually distinct from* y_0 *towards* x_0 *on* $A \subseteq D_f$ if there exists a neighborhood N of x_0 such that for all $x \in N \cap A \setminus \{x_0\}$, $f(x) \neq y_0$; otherwise we say that f *frequently touches* y_0 *towards* x_0, i.e., there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $x_n \to x_0$ and $f(x_n) = y_0$ for all n. The function $f(x) := x \sin x$, for instance, is eventually distinct from 0 towards 0, while the function $g(x) := x \sin(1/x)$ frequently touches 0 towards 0. In fact, $f(x) \neq 0$ for all $x \in (-\pi, \pi)$, and if $x_n := \frac{1}{n\pi}$, then $x_n \to 0$ and $g(x_n) = 0$ for all n.

Proposition 1. *Let f and g be real-valued functions defined respectively on subsets D_f and D_g of $\mathbb{R}$, x_0 be a limit point of the domain $D_{f\circ g}$ of $f \circ g$, and y_0 is a limit point of D_f. Suppose that $\lim_{x\to x_0} g(x) = y_0$ and $\lim_{y\to y_0} f(y)$ exist. Then $\lim_{x\to x_0} f(g(x))$ exists and equals $\lim_{y\to y_0} f(y)$ if and only if, either one of the following conditions is satisfied:*

(i) f is continuous at y_0;

(ii) $y_0 \notin D_f$;

(iii) g is eventually distinct from y_0 towards x_0 on $D_{f\circ g}$.

As an illustration, let us consider each of the following pairs of f and g:

(a) $f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$ and $g(x) := x \sin(1/x)$'

(b) $f(y) := \frac{\sin y}{y}$ and $g(x) := x \sin(1/x)$;

(c) $f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$ and $g(x) := x \sin x$;

(d) $f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$ and $g(x) := x \sin(1/x)$;

(e) $f(y) := \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$ and $g(x) := 0$.

Notice that for each pair of f and g, both $\lim_{x\to 0} g(x)$ and $\lim_{y\to 0} f(y)$ exist. In particular, in (a), f is continuous at $0 = \lim_{x\to 0} x \sin(1/x)$; in (b), f is not defined at 0; and in (c), g is eventually distinct from 0 toward 0. Thus, in view of

Proposition 1, for each pair of f and g in (a), (b), and (c), $\lim_{x\to 0} f(g(x))$ exists and equals $\lim_{y\to 0} f(y)$; while, since both in (d) and (e), f is defined at 0, f is not continuous at 0, and g frequently touches 0 towards 0, it follows that either $\lim_{x\to 0} f(g(x))$ does not exist (case (d)) or $\lim_{x\to 0} f(g(x))$ exists but does not equal $\lim_{y\to 0} f(y)$ (case (e)).

2. A generalization and a proof

Here we generalize Proposition 1 in the general setting of topological spaces. This generalized version, consequently, has wider and more general forms of applications than both Proposition 1 and Theorem 1 of [3] do, particularly in parts where functions on metric spaces and topological rings are involved. We begin with the definition of *a limit of a function f at a point x_0* which, as usual, we write $\lim_{x\to x_0} f(x)$.

Definition 2. *Let X and Y be topological spaces, where Y is Hausdorff. Let $A \subseteq X$, x_0 be a limit point of A, $y_0 \in Y$, and $f : A \to Y$ be a function. We write $\lim_{x\to x_0} f(x) = y_0$ if, for every neighborhood W of y_0, there exists a neighborhood V of x_0 such that $f(V \cap A \setminus \{x_0\}) \subseteq W$.*

Notice that the condition Y to be Hausdorff guarantee that $\lim_{x\to x_0} f(x)$ has at most one value, as we are aware that in a non-Hausdorff space two points might share the same neighborhood system. Recall that f is continuous at $x_0 \in D_f$ if, for every neighborhood W of $f(x_0)$, there exists a neighborhood V of x_0 such that $f(V \cap D_f) \subseteq W$ or equivalently, by Definition 2, $\lim_{x\to x_0} f(x) = f(x_0)$. The notion of "eventually distinct" and "frequently touching" are defined in the same way as those in the first section, except that here we use the concept of a net instead of a sequence. These notions also apply to nets in the obvious way.

Proposition 3. *Let X, Y, and Z be topological spaces, with Y and Z being Hausdorff. Let f be a Z-valued function on $D_f \subseteq Y$, and g be a Y-valued function on $D_g \subseteq X$. Assume that $R_g \cap D_f \neq \emptyset$. Let x_0 be a limit point of $D_{f\circ g}$, and $y_0 \in Y$ is a limit point of D_f. Suppose that $\lim_{x\to x_0} g(x) = y_0$ and $\lim_{y\to y_0} f(y)$ exists. Then $\lim_{x\to x_0} f(g(x))$ exists and equals $\lim_{y\to y_0} f(y)$ if and only if, either one of the following conditions is satisfied:*

(i) f is continuous at y_0;

(ii) $y_0 \notin D_f$;

(iii) g is eventually distinct from y_0 towards x_0 on $D_{f\circ g}$.

Proof. Without loss of generality we can assume that $R_g \subseteq D_f$ and $D_g = X$, and so $D_{f\circ g} = D_g = X$. Here we shall prove it using Definition 2. For part ($\Longrightarrow$), assume that neither (i) nor (ii) hold, that is $y_0 \in D_f$ and f is not continuous at y_0. We wish to show that (iii) holds, i.e., g is eventually distinct from y_0 towards x_0. Suppose that g frequently touches y_0 towards x_0. Let $L := \lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y)$. Since f is not continuous at y_0, it follows that $L \neq f(y_0)$. As Z is Hausdorff, we can choose some neighborhoods W_1 and W_2 of L and $f(y_0)$

respectively such that $W_1 \cap W_2 = \emptyset$. Then, there exists a neighborhood U_1 of x_0 such that $(f \circ g)(U_1 \setminus \{x_0\}) \subseteq W_1$. From the assumption that f frequently touches y_0, there exists $x_1 \in U_1 \setminus \{x_0\}$ such that $g(x_1) = y_0$, and so $(f \circ g)(x_1) = f(g(x_1)) = f(y_0) \subseteq W_2$. Therefore $(f \circ g)(x_1) \in W_1 \cap W_2$, which contradicts the fact that $W_1 \cap W_2 = \emptyset$. Hence (iii) holds.

For part ($\Longleftarrow$), note first that (ii) implies (iii), since if $y_0 \notin D_f$, then for every $x \in D_{f\circ g}$, $g(x) \neq y_0$. Then, we only need to consider (i) and (iii). Suppose that (i) holds, that is f is continuous at y_0. Let W be a neighborhood of $f(y_0)$. Then there exists a neighborhood V of y_0 such that $f(V \cap D_f) \subseteq W$. Since $\lim_{x\to x_0} g(x) = y_0$, there exists a neighborhood U of x_0 such that $g(U \setminus \{x_0\}) \subseteq V \cap D_f$, and so $f(g(U \setminus \{x_0\})) \subseteq f(V \cap D_f) \subseteq W$. Therefore $\lim_{x\to x_0} f(g(x))$ exists and equals $f(y_0)$.

Suppose now that (iii) holds. Let W be any neighborhood of $L := \lim_{y\to y_0} f(y)$. Then there exists a neighborhood V of y_0 such that

$$f(V \cap D_f \setminus \{y_0\}) \subseteq W. \tag{1}$$

As g is eventually distinct from y_0, choose a neighborhood U_1 of x_0 such that for all $x \in U_1 \setminus \{x_0\}$, $g(x) \neq y_0$, that is

$$y_0 \notin g(U_1 \setminus \{x_0\}). \tag{2}$$

Since $\lim_{x\to x_0} g(x) = y_0$, there exists a neighborhood U_2 of x_0 such that

$$g(U_2 \setminus \{x_0\}) \subseteq V \cap D_f. \tag{3}$$

Now set a neighborhood $U := U_1 \cap U_2$. Then, it follows from (2) that $g(U \setminus \{x_0\})$ does not contain y_0, and so by (3), $g(U \setminus \{x_0\}) \subseteq V \cap D_f \setminus \{y_0\}$, and hence $f(g(U\setminus\{x_0\})) \subseteq f(V\cap D_f\setminus\{y_0\})$. It then follows from (1) that $f(g(U\setminus\{x_0\})) \subseteq W$. Thus $\lim_{x\to x_0} f(g(x))$ exists with the limit L. ■

We will need the convenient fact below as it enables us to prove statements about limits in the language of nets (e.g., using this fact, we may want to exhibit a proof of Proposition 3 in the language of nets, rather than in that of neighborhoods as above). If $\{U_\alpha \mid \alpha \in \Delta_{x_0}\}$ is the neighborhood system at a point x_0, then "$\geq$" directs the index set Δ_{x_0} where, for α, $\beta \in \Delta_{x_0}$, $\alpha \geq \beta$ if and only if $U_\alpha \subseteq U_\beta$. We write $(x_\alpha)_{\alpha\in\Delta_{x_0}}$ to indicate a net whose directed set is Δ_{x_0}, and write (x_α) to indicate a net with a non-specified directed set. If a net (x_α) converges to a point x_0, we write $x_\alpha \to x_0$.

Fact 4. *Let X and Y be topological spaces, where Y is Hausdorff. Let $A \subseteq X$, x_0 be a limit point of A, $y_0 \in Y$, and $f\colon A \to Y$ be a function. Then the following statements are equivalent:*

(i) $\lim_{x\to x_0} f(x) = y_0$;

(ii) For every net (x_α) in $A \setminus \{x_0\}$ such that $x_\alpha \to x_0$, there exists a subnet (x_{α_β}) such that $f(x_{\alpha_\beta}) \to y_0$;

(iii) For every net (x_α) in $A \setminus \{x_0\}$ such that $x_\alpha \to x_0$, $f(x_\alpha) \to y_0$.

Proof. For part ((i) $\Longrightarrow$ (ii)), let (x_α) be any net in $A\setminus\{x_0\}$ such that $x_\alpha \to x_0$. Let W be any neighborhood W of y_0. By (i), there exists a neighborhood V of x_0 such that $f(V \cap A \setminus \{x_0\}) \subseteq W$. Since $x_\alpha \to x_0$, it follows that x_α is eventually in $V \cap A \setminus \{x_0\}$, and so $f(x_\alpha)$ is eventually in $f(V \cap A \setminus \{x_0\})$ and hence in W. Therefore $f(x_\alpha) \to y_0$, where $(f(x_\alpha))$ is a subnet of itself.

For part ((ii) $\Longrightarrow$ (iii)), we shall prove its contrapositive. Suppose that (iii) does not hold. Then, there exists a net (x_α) in $A \setminus \{x_0\}$ and a neighborhood W of y_0 such that $x_\alpha \to x_0$ and $(f(x_\alpha))$ is not eventually in W. Axiom of choice, therefore, allows us to have a subnet (x_{β_α}) such that $f(x_{\beta_\alpha}) \notin W$ for all β_α. Thus, we have a net (x_{β_α}) that converges to x_0 and has no subnet whose image under f converges to y_0, that is (ii) does not hold.

For part ((iii) $\Longrightarrow$ (i)), as above, we shall prove the contrapositive. Suppose (i) does not hold. Then there exists a neighborhood W of y_0 such that for any neighborhood V of x_0, $f(V\cap A\setminus\{x_0\}) \nsubseteq W$. By Axiom of choice, there exists a net $(x_\alpha)_{\alpha\in\Delta_{x_0}}$ in $A\setminus\{x_0\}$ such that for every neighborhood V_α of x_0, $x_\alpha \in V_\alpha \cap A\setminus\{x_0\}$ and $f(x_\alpha) \notin W$. Since $(x_\alpha)_{\alpha\in\Delta_{x_0}}$ is eventually in any neighborhood of x_0, it follows that $x_\alpha \to x_0$, while $f(x_\alpha) \nrightarrow y_0$ as $f(x_\alpha) \notin W$ for all α, so that (iii) does not hold. ■

3. Applications

Below we give three examples of the applications. One application will be an example supplementing the illustration in Section 1, with which we show the existence of the limit of a function, and at the same time compute it, using a thorough substitution method based on Proposition 1. The other two are concerned with parts of the proof arguments of the inverse function theorem and the derivative of composite function theorem; each of which works for two general underlying spaces: one using Proposition 3, while the other using Proposition 3 and a proof-method by cases in terms of "eventually distinct" and "frequently touching" concepts.

Substitution method

Knowing that $\lim_{x\to 0}(\sin x)/x = 1$, we shall show the existence of, and at the same time, compute

$$\lim_{x\to 1} \frac{\arcsin(x-1)}{x-1}.$$

Set $f(x) := x/\sin x$ and $g(x) := \arcsin(x-1)$. Then $D_{f\circ g} = [0,2] \setminus \{1\}$, and 1 is a limit point of $D_{f\circ g}$. Since sin function is one-to-one on $(-\pi/2, \pi/2)$ and is continuous at $0 = \arcsin 0$, it follows that its inverse arcsin is also continuous at 0, and therefore $\lim_{x\to 1} g(x) = \arcsin(\lim_{x\to 1}(x-1)) = \arcsin 0 = 0$. We observe that $\lim_{x\to 0} f(x) = 1/(\lim_{x\to 0}(\sin x)/x) = 1/1 = 1$. Since f is not defined at $0 = \lim_{x\to 1} g(x)$, it follows from the Proposition 1 that $\lim_{x\to 1} f(g(x))$ exists, and is equal to $\lim_{x\to 0} f(x) = 1$. Thus

$$\lim_{x\to 1} \frac{\arcsin(x-1)}{x-1} = \lim_{x\to 1} \frac{\arcsin(x-1)}{\sin(\arcsin(x-1))} = \lim_{x\to 1} f(g(x)) = 1.$$

In the next applications we will need the following lemma whose proof is also a direct application of Proposition 3.

Lemma 5. *Let X be a topological space and Y is a Hausdorff topological group. Let f and g be Y-valued functions on $A \subseteq X$. Let x_0 be a limit point of A.*

(i) If both $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist, then $\lim_{x\to x_0}(f(x)\cdot g(x))$ exists, and

$$\lim_{x\to x_0}(f(x)\cdot g(x)) = \lim_{x\to x_0} f(x)\cdot \lim_{x\to x_0} g(x).$$

(ii) If $\lim_{x\to x_0} f(x)$ exists, then $\lim_{x\to x_0}(f(x))^{-1}$ exists, and

$$\lim_{x\to x_0}(f(x))^{-1} = (\lim_{x\to x_0} f(x))^{-1}.$$

Proof. For (i), consider the functions $\pi : x \mapsto (f(x), g(x))$, $x \in A$,

$$\mu : (y_1, y_2) \mapsto y_1 \cdot y_2,\ (y_1, y_2) \in Y \times Y$$

and

$$f \cdot g : x \mapsto f(x) \cdot g(x)$$

from which we have $\mu\circ\pi = f\cdot g$. Let $L := \lim_{x\to x_0} f(x)$ and $M := \lim_{x\to x_0} g(x)$. We shall show that $\lim_{x\to x_0}(\mu\circ\pi)(x)$ exists and equals $L\cdot M$. Recall that convergence in the product topology is pointwise convergence, so that

$$\lim_{x\to x_0} \pi(x) = (L, M). \tag{4}$$

Since μ is continuous,

$$\lim_{(y_1,y_2)\to(L,M)} \mu((y_1, y_2)) = \mu((L, M)) = L\cdot M. \tag{5}$$

By noting (4), (5) and the continuity of μ, it then follows from Proposition 3 that $\lim_{x\to x_0} \mu(\pi(x))$ exists and equals $\lim_{(y_1,y_2)\to(L,M)} \mu((y_1, y_2)) = L\cdot M$.

For (ii), consider the functions $\iota : y \mapsto y^{-1}$, $y \in Y$, and

$$(f)^{-1} : x \mapsto (f(x))^{-1},\ x \in A$$

(we distinguish $(f)^{-1}$ from f^{-1}) from which we have $\iota\circ f = (f)^{-1}$. Let

$$K := \lim_{x\to x_0} f(x). \tag{6}$$

We shall show that $\lim_{x\to x_0}(\iota\circ f)(x) = K^{-1}$. But

$$\lim_{y\to K} \iota(y) = \iota(K) = K^{-1} \tag{7}$$

since ι is continuous, and hence by noting (6), (7) and the continuity of ι, it follows from Proposition 3 that $\lim_{x\to x_0} \iota(f(x))$ exists and equals $\lim_{y\to K} \iota(y) = K^{-1}$. ∎

Inverse function theorem

Here we shall prove parts of the proof arguments of the theorem (where Proposition 3 is used) for each underlying Banach space and topological field. We write $(DG)_a$ for the (Frechet) derivative of G at a.

FACT 6. *Suppose that X and Y are Banach spaces, W is an open set of X, $x_0 \in W$, and $F : W \to X$ is a function that is differentiable at x_0 with $(DF)_{x_0}$ is invertible. If F is one-to-one, then F^{-1} is differentiable at y_0 with $(DF^{-1})_{y_0} = (DF)^{-1}_{x_0}$.*

Proof. Write, for convenience, $D := (DF)_{x_0}$. Let

$$\varphi(y) := \frac{\|F^{-1}(y) - F^{-1}(y_0) - D^{-1}(y - y_0)\|}{\|y - y_0\|}, \quad y \in F(W) \setminus \{y_0\}$$

and

$$\omega(x) = \frac{\|x - x_0 - D^{-1}(F(x) - F(x_0))\|}{\|F(x) - F(x_0)\|}, \quad x \in W \setminus \{x_0\}.$$

Then $\omega \circ F^{-1} = \varphi$. We shall show that $\lim_{y\to y_0}(\omega \circ F^{-1})(y) = 0$. First, write

$$\omega(x) := \frac{\|D^{-1}(D(x - x_0) - (F(x) - F(x_0)))\|}{\|x - x_0\|} \cdot \frac{\|x - x_0\|}{\|F(x) - F(x_0)\|}. \tag{8}$$

Since F is differentiable at x_0, there exists a neighborhood N of x_0 such that for all $x \in N \setminus \{x_0\}$,

$$\frac{\|F(x) - F(x_0) - D(x - x_0)\|}{\|x - x_0\|} < \frac{1}{2\|D^{-1}\|}$$

which implies that

$$\frac{\|F(x) - F(x_0)\|}{\|x - x_0\|} > \frac{\|D(x - x_0)\|}{\|x - x_0\|} - \frac{1}{2\|D^{-1}\|}$$

where $\|D(x-x_0)\|/\|x-x_0\| = \|D(x)-D(x_0)\|/\|D^{-1}(D(x)-D(x_0))\| > 1/\|D^{-1}\|$, so that $\|x - x_0\|/\|F(x) - F(x_0)\|$ is bounded on $N \setminus \{x_0\}$. This implies that F^{-1} is continuous at y_0, that is

$$\lim_{y\to y_0} F^{-1}(y) = F^{-1}(y_0) = x_0. \tag{9}$$

By (8) and the fact that D is the derivative of F at x_0, this also implies that

$$\lim_{x\to x_0} \omega(x) = 0. \tag{10}$$

Noting (10), (9), and the fact that F^{-1} is eventually distinct from $F^{-1}(y_0)$, it follows from Proposition 3 that $\lim_{y\to y_0} \omega(F^{-1}(y)) = \lim_{x\to x_0} \omega(x) = 0$. ∎

FACT 7. *Let X be a Hausdorff topological field, $W \subseteq X$ is open, and $x_0 \in W$ is a limit point. Let $f : W \to X$ be a function with*

$$f'(x_0) := \lim_{x\to x_0} (x - x_0)^{-1}(f(x) - f(x_0)) \tag{11}$$

exists and is not 0. If f is one-to-one and $f^{-1}\colon f(W) \to W$ is continuous at $y_0 := f(x_0)$, then

$$f'(y_0) := \lim_{y\to y_0} (y-y_0)^{-1}(f^{-1}(y) - f^{-1}(y_0))$$

exists and equals $(f'(x_0))^{-1}$.

Proof. Let

$$F(x) := (x-x_0)^{-1}(f(x) - f(x_0)), \quad x \in W \setminus \{x_0\}. \tag{12}$$

Since for $y \in f(W) \setminus \{y_0\}$,

$$F(f^{-1}(y)) = (f^{-1}(y) - f^{-1}(y_0))^{-1}(y-y_0)$$

we have

$$(F(f^{-1}(y)))^{-1} = (y-y_0)^{-1}(f^{-1}(y) - f^{-1}(y_0)). \tag{13}$$

We shall prove that $\lim_{y\to y_0}(F(f^{-1}(y)))^{-1}$ exists and equals $(\lim_{x\to x_0} F(x))^{-1}$. Since f^{-1} is continuous at $y_0 = f(x_0)$, $\lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$. Since $\lim_{x\to} F(x)$ exists, and f^{-1} is eventually distinct from $f^{-1}(y_0)$, it follows from Proposition 3 that $\lim_{y\to y_0} F(f^{-1}(y))$ exists and

$$\lim_{y\to y_0} F(f^{-1}(y)) = \lim_{x\to x_0} F(x). \tag{14}$$

Since $\lim_{x\to} F(x) \neq 0$, it follows from Lemma 5 that $\lim_{y\to y_0}(F(f^{-1}(y)))^{-1}$ exists and equals $(\lim_{x\to x_0} F(x))^{-1}$. ■

The derivative of a composite function

Here we shall prove the existence and the formula of the derivative of a composite function, both on Banach spaces and topological fields.

FACT 8. *Suppose that X, Y and Z are Banach spaces. Let $G\colon X \to Y$ and $F : Y \to Z$ be functions such that G is differentiable at $x_0 \in X$ and F differentiable at $G(x_0)$. Then $F \circ G$ is differentiable at x_0 with $(D(F\circ G))_{x_0} = (DF)_{G(x_0)} \circ (DG)_{x_0}$.*

Proof. Let for $x \in X \setminus \{x_0\}$,

$$\psi(x) := \frac{(F\circ G)(x) - (F\circ G)(x_0) - ((DF)_{G(x_0)} \circ (DG)_{x_0})(x-x_0)}{\|x-x_0\|}.$$

We shall prove that $\lim_{x\to x_0} \|\psi(x)\| = 0$ in view of Fact 4, where we use sequences instead of general nets, since every Banach space is a first-countable topological space. For $G(x) \neq G(x_0)$, write

$$\psi_1(x) := \frac{(F\circ G)(x) - (F\circ G)(x_0) - (DF)_{G(x_0)}(G(x) - G(x_0))}{\|G(x) - G(x_0)\|}$$

and for $x \in X \setminus \{x_0\}$,

$$\psi_2(x) := \frac{G(x) - G(x_0) - (DG)_{x_0}(x-x_0)}{\|x-x_0\|}.$$

Then, it can immediately be checked that either

$$\psi(x) = \|G(x) - G(x_0)\|\psi_1(x) + (DF)_{G(x_0)}(\psi_2(x)) \tag{15}$$

if $G(x) \neq G(x_0)$, or

$$\psi(x) = (DF)_{G(x_0)}(\psi_2(x)) \tag{16}$$

if $G(x) = G(x_0)$. Let x_n be any sequence in $X \setminus \{x_0\}$ such that $x \to x_0$. Then, either $(G(x_n))$ is eventually distinct from $G(x_0)$ or it frequently touches $G(x_0)$. In the former case, since F is differentiable at $G(x_0)$ and G is differentiable at x_0, it follows that both $(\|\psi_1(x_n)\|)$ and $(\|\psi_2(x_n)\|)$ converge to 0, and hence (15) implies that $\|\psi(x_n)\| \to 0$. In the latter case, we have a subsequence (x_{n_k}) such that $G(x_{n_k}) = G(x_0)$ for all k, so that by (16), $\|\psi(x_{n_k})\| \to 0$. This completes the proof. ■

In the following fact we have the notion of the derivative of a function f at x_0 as formulated in (11), and as usual, when such a value is defined (or the limit exists) at x_0, we say that f is *differentiable* at x_0.

FACT 9. *Let X be a Hausdorff topological field, and $x_0 \in X$. If f and g are X-valued functions on X such that g is differentiable at x_0 and f is differentiable at $g(x_0)$, then $f \circ g$ is differentiable at x_0 and*

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0). \tag{17}$$

Proof. Let

$$H(x) := (x - x_0)^{-1}((f \circ g)(x) - (f \circ g)(x_0)), \quad x \in X \setminus \{x_0\}$$

and

$$F(y) := (y - g(x_0))^{-1}(f(y) - f(g(x_0)), \quad y \in X \setminus \{g(x_0)\}.$$

Then

$$H(x) = F(g(x)) \cdot (x - x_0)^{-1} \cdot (g(x) - g(x_0)) \tag{18}$$

for all $x \in X \setminus \{x_0\}$, $g(x) \neq g(x_0)$. First, consider the case when g is eventually distinct from $g(x_0)$. By Proposition 3, $\lim_{x\to x_0} F(g(x))$ exists and equals $\lim_{y\to y_0} F(y) = f'(g(x_0))$ as f is differentiable at $g(x_0)$, and hence by Lemma 5, $\lim_{x\to x_0} H(x)$ exists and equals $f'(g(x_0)) \cdot g'(x_0)$ as g is differentiable at x_0, so that (17) holds.

Now consider the case when g frequently touches $g(x_0)$, where we assert that (17) holds by showing $g'(x_0) = 0$ and $(f \circ g)'(x_0) = \lim_{x\to x_0} H(x) = 0$. But in this case $g'(x_0)$ must be 0, since we have a net (x_α) in $X \setminus \{x_0\}$ such that $x_\alpha \to x_0$ and $g(x_\alpha) = g(x_0)$ for all α, and that $(x_\alpha - x_0)^{-1}(g(x_\alpha) - g(x_0)) \to g'(x_0)$ as g is differentiable at x_0; so it remains to show that $\lim_{x\to x_0} H(x) = 0$. Let (x'_α) be any net in $X \setminus \{x_0\}$ such that $x'_\alpha \to x_0$. Then, either $(g(x'_\alpha))$ is eventually distinct from $g(x_0)$ or it frequently touches $g(x_0)$. The former case, as the argument above, implies that $H(x'_\alpha) \to f'(g(x_0)) \cdot g'(x_0) = 0$ as $g'(x_0) = 0$. In the latter case, we have a subsequence (x'_{α_β}) such that $g(x'_{\alpha_\beta}) = g(x_0)$ for all α_β, which implies that $H(x'_{\alpha_\beta}) \to 0$, and this completes the proof. ■

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