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AN EXPLORATION OF STUDENTS' CONCEPTUAL KNOWLEDGE BUILT IN A FIRST ORDINARY DIFFERENTIAL EQUATIONS COURSE (PART II)

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Abstract. This is the second part of a study published in a previous issue of this Journal whose main goal was to analyze and document the types of knowledge that university students exhibit to deal with fundamental issues that they had studied in a first ordinary differential equation course. Here, we focus on analyzing and discussing the following research questions: (ii) How do students make sense of, interpret and deal with the concept of solution to an ODE? (iii) What systems of representation do they use to represent and explore the information embedded in those questions in order to answer them? And, to what extent do the students privilege the use of certain type of representation?

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Introduction

This is the second part of a study published in a previous issue of this Journal whose main goal was to analyze and document the types of knowledge that university students exhibit to deal with fundamental issues that they had studied in a first ordinary differential equation course. In the first part, we presented the conceptual framework and methodological features associated with the research. Here, we focus on analyzing and discussing the following research questions:

(ii) How do students make sense of, interpret and deal with the concept of solution to an ODE?

In this question, we document the ways in which students think of and operate ideas around the concept of solution to an ODE. In addition, we examine the types of difficulties students experience during the process of verifying whether a function fulfils the necessary conditions to be the solution to a given ODE.

(iii) What systems of representation do they use to represent and explore the information embedded in those questions in order to answer them? And, to what extent do the students privilege the use of certain type of representation?

In this question, we analyze the extent to which students show consistent tendencies or preferences to select and use a set of resources to deal with problems or questions associated with ODEs. To answer and discuss these questions we relied on data gathered from task-based interviews. The interviews used in this part involved problems of types 2 and 3 which were described in the first part of this study (see methodology) (see also appendix 1 at the end of this part).

Data Analysis and Presentation of Results

The analysis of the information gathered in our research is divided into two parts. First, we focus on analyzing the answers given by the twenty-one participating students to the problems set in the questionnaire. To this end, each student's answer was analyzed in terms of identifying salient mathematical features that were summarized in Appendix 2. These tables were used as a heuristic aid to identify initial global patterns in students work related to their level of proficiency to deal with basic concepts associated with the study of ODEs. This global behavior is complemented with data from interviews with the students. That is, we follow up students' answers to the questionnaire by considering their ideas expressed during the interview. In addition, we also show examples of the students' work to illustrate their answers.

On the concept of solution of an ODE

To analyze the students' ways of dealing with the concept of *solution of an ODE*, we focused on what students answered to questions that were grouped as Type 2 questions (see Appendix 1). Knowledge needed to deal with these questions includes understanding the concept of solution to an ODE, recognizing explicit and implicit functions and ways of differentiating those functions; and the meaning of both a particular and general solution of an ODE.

Based on the work shown by students to answer questions Q3, Q4 and Q11 we can infer that for some students the meaning of solution of an ODE is restricted to the result of the process of solving the equation. This is the case of Mary and Gaby who choose to solve the equations in order to find the solutions of the three equations (Tables 1, 2 and 4, Appendix 2). Other students think of the concept of the solution to a differential equation as the set of functions which substituted into the equation, together with their derivatives, transform this into an identity. Six students (Angie, Jordan, Edna, Laure, Stella and Betty) decided to differentiate the possible function solution in order to verify whether it was a solution to the given ODE. However, it is important to mention that when these students encountered difficulties while differentiating such expressions, they were not able to think of another alternative like solving directly the equation.

Jordan is one of those students who chose differentiation to check that a function was a solution in Q3(a), but failed to answer Section Q3(b). In the interview we could see that the difficulty this student encounters when answering these questions is related to the process of differentiation. Jordan recognizes the implicit expression of the function given in Q3(b) but does not differentiate correctly in any of the subsections of this question. In the first section he recognizes that he does not know how to differentiate the function and he expresses doubts about the use of the chain rule to differentiate the implicit function, a differentiation that he also performs incorrectly.

J (Jordan): [...] Well, if we differentiate here (the implicit expression) with respect to x, if I have this OK,

$$0 + \frac{3dy}{dx} - 3y^2 \cdot \frac{dy}{dx} = 0$$

[...] we'll get this

$$3\frac{dy}{dx}(1-y^2) = 0 \Leftrightarrow \frac{dy}{dx}(1-y^2) = 0$$

and [...] substituting here ... we'll get $x^2 = 0$. I think that this is wrong. [...]

R: You said before "if I have this OK". Have you any doubts that you have it OK?

J: No, this happens, the chain rule thing. I think I have it OK.

Jordan only sets about solving the differential equation when we insisted on him doing so during the interview.

R: And do you have to calculate the derivative of that function necessarily to answer?

J: It's probably not so easy, but ... you have to do it.

R: Can you think of any other way of going about the problem or isn't there any?

J: Calculate the solutions of the equation (the ODE).

R: Will that help?

J: If we get all the solutions ... we can know if this is one.

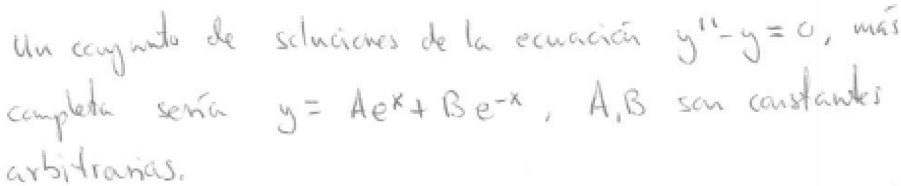
Stella uses differentiation as the procedure for checking that a function is the solution to an ODE, but, like Jordan, she fails to differentiate the function given implicitly, which leads us to suppose that this concept might be an obstacle to answer this part (Q3(b)). In the interview Stella answers these questions by solving the equations. For her, as well as for other twelve students, the concept of solution of a differential equation has several meanings. However, based on the answer she gives in the interview when asked to solve Q3(b) differentiating the function, we could see that the implicit expression of a function has no meaning for Stella.

S (Stella): Section b is strange ... So, $f(x)$, what is it?, $f(x)$ might be anything, any function of x, but obviously, as they tell you that it holds this [$-x^3 + 3y - y^3 = C$]. If it satisfies this [$-x^3 + 3y - y^3 = C$], does that mean that $f(x)$ is the same as that?

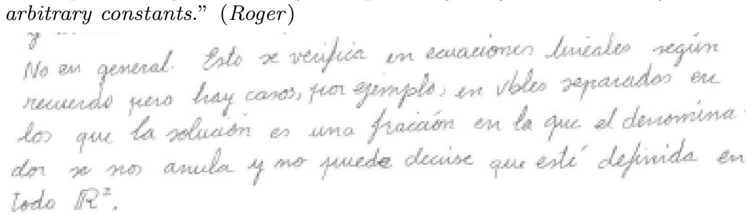
When analyzing Wanda's answers we observe that her understanding of the concept of implicit function can also be an obstacle when undertaking Q3(b). In this case it is not because of any lack of meaning of the expression, but rather because of her idea of this concept. Wanda believes that the implicit expression of a function is of the form $F(x, y)$, and reflects this idea when she differentiates to answer the questionnaire, the derivative of the expression $F(x, y) = C$ with respect to x being correct. As she does not know what to do with the expression obtained from differentiation, this student chooses to solve the equation and classifies it as exact. Her answer is wrong as she makes a mistake when applying the algorithm for solving this type of equation. During the interview she classifies the ODE as exact, basing this idea on the fact that she remembers that this was the expression of the solutions to these equations, but she cannot remember how to check if this equation is exact and how to calculate the expression of the exponential function. This is an example of the meaninglessness use of algorithms in students' actions.

W (Wanda): How can you know if something is the solution of something? Well, we should substitute the y from here [she indicates the implicit expression] into this [she indicates the equation] and see if this verifies the differential equation. What happens is that finding from here isn't easy because you have y^3 and y [...] When you see something like this [the ODE] and you see an answer of this type [the implicit function] ... To me they look like equations, I think they're called exact equations but I'm not sure, and the solutions are going to be an exponential function, the same as a constant. I can't remember precisely why, or how ...

Roger, Wanda and Angie seem to have a more robust understanding of the concept of solution of an ODE than the rest of the students. They rely on the use of heuristic strategies to solve Type 2 problems. Roger uses the relationship between the number of integration constants that appears in the algebraic expression of the general solution of an ODE with the order of this equation in order to solve Q4. Wanda and Angie give counterexamples to demonstrate that the statement made in Q5 is false (Figure 1).



"*A more complete set of solutions of the equation* $y'' - y = 0$ *would be* $y = Ae^x + be^{-x}$, *A and B are arbitrary constants.*" (*Roger*)



"*Generally not. As far as I can remember, this is verified in linear equations, but there are cases, for example, in separate variables, where the solution is a fraction where the denominator is cancelled and you can't say that it is delimited to all* $\mathbb{R}^2$." (*Wanda*)

(*Angie*)

Figure 1. Procedure followed by Roger in Q4 and Wanda and Angie in Q5

Comment. Raychaudhuri (2008) cited a definition found in a classic textbook (Boyce & DiPrima [1]) "any differentiable function $y = \phi(t)$ that satisfies the DE

(Differential Equation) $y' = f(t, y)$ for all t in some interval is called a solution" to highlight that the concept of solution of an ODE requires students to understand "what the solution represents as an entity in the context of differential equations (a function) and what mathematical process makes that function a solution—that it satisfies the DE for all x in a certain interval". He distinguishes two processes related to the concept of solution of an ODE: the generating process, solutions are the result of solving the equation, and the defining process, solutions are functions that satisfy the ODE for all values of the independent variable. In this context, there is evidence that in general, students do not consider the properties associated with the function that represents the solution in the context of the differential equation involved. In addition, when students direct their attention to differentiability or integration processes, they often face serious difficulties to accomplish successfully the task of verifying the solution. In terms of the strands that characterize proficiency in comprehending the concept of solution on an ODE, it is clear that the students privilege the use of derivation or integration rules (procedural fluency), isolated from the others strands, to answer the questions. However, they also face difficulties to perform efficiently even those rules or procedures (see Appendix 2).

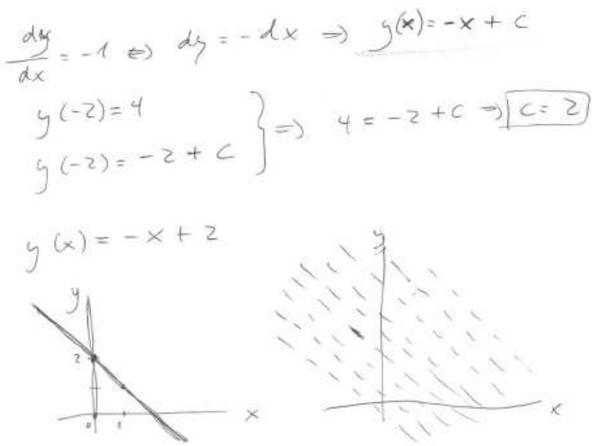
On the concept of "direction field"

To what extent do students recognize the importance of representing a slope field or direction field associated with a first-order differential equation in order to visualize its solution? Do they rely on using their knowledge of functions and derivatives to construct and interpret a direction field? The process shown by the participants to answer problems of Type 3 helps us document the ideas and resources they use, and difficulties they encounter to construct and interpret the slope field associated with the given differential equations.

We observe that half the students experience difficulties with questions that demand the interpretation or construction of a direction field (Type 3 problems). Many students left their answers blank, even though they had studied the elements needed to go about the problems using both graphical as well as algebraic systems of representation (Tables 5, 6 and 7).

Twelve students appeared not to have the slightest knowledge of the processes of representation and interpretation of a direction field. Thus, they failed to relate the geometric meaning of the derivative, the slope of the tangent line to a function at one point, to the direction field. In an interview one of these students, Jordan, showed that he could interpret this concept but not represent it. This student's behavior might be due to the fact that the he answered the questionnaire during his study of the subject, while the interview was conducted after he had finished the course and had thus developed certain resources.

Several students, among them Jordan, give the concept of differential equation a single role, that of something that has to be solved. This is what this student does in all the Type 3 questions in the interview: he always tries to solve the equation immediately. At no time does he refer to the monotony of the function, which might be the reason why he fails to represent any direction field correctly. He only attempts to represent the direction field in question Q15 and he does this based on the solution to the problem that he had previously represented after solving the ODE (Figure 2).



J: Draw the direction field . . . I'm not going to do that!

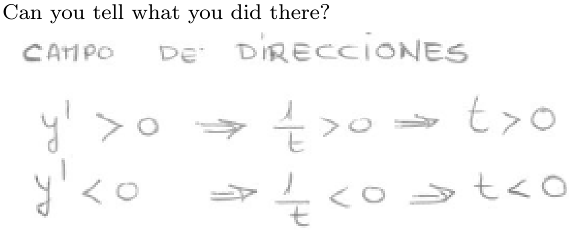
E: Why? . . . Are you tired?

J: . . . No, I'm not tired, it's just . . . I couldn't do it in other questions . . . Well, I'll try . . . Let's see . . . I will graph the solution of this [points to the PVI and represents the solutions] [. . .] That's the particular problem; the general problem would be plus C. As the lines indicate the derivative the solution has at that point . . . [Jordan's represention of the direction field]

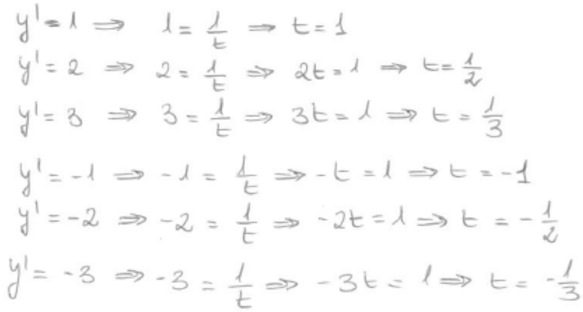
Figure 2. Jordan's answer to Q15

Answering Q6 in the questionnaire, Stella uses the differential equation both to obtain the algebraic expression of the function solution as well as to analyze the monotony of this, but she does not represent the direction field. In the interview Stella merely solves the equation, leading us to think that in the questionnaire she was attempting to apply an algorithm that she could not altogether remember. This algorithm acquires meaning for her during the interview, giving rise to the geometric meaning of the derivative and permitting correct representation of the direction field.

R: What I've just given you is what you did. Here there is a series of calculations where it says direction field. Can you tell what you did there?



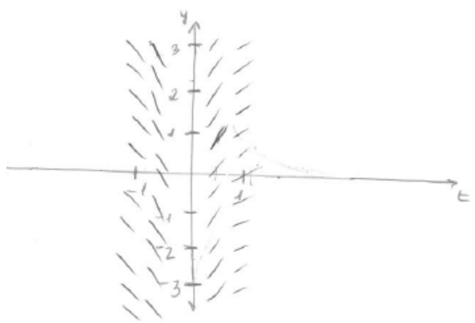
S: Here I analyze the derivative, whether it's positive or negative [. . .]



and here I begin to calculate the derivative ... I calculate how much the parameter t is in accordance with the values of the derivative.

R: And how would you use these calculations to draw the direction field?

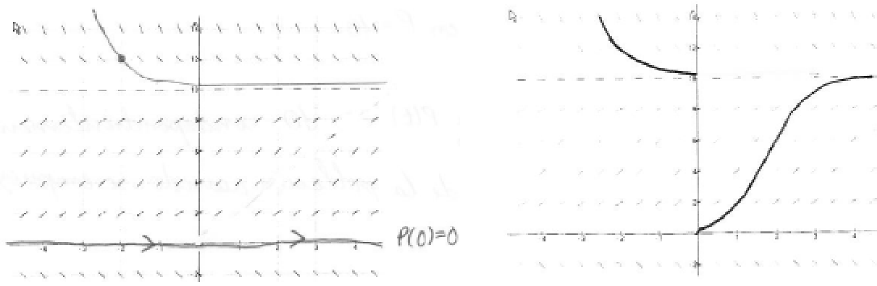
S: It'd be like that, at $t = 1$, the derivative is supposed to be the tangent line to the curve at that point, so at $t = 1$, the curve has slope 1, something like that, at $t = 1$ it has slope 1, so that it'd be like this ... At $t = 1/2$ it has slope 2, sloping further and at $t = 1/2$ ... Well, to tell you the truth, I knew that this was the slope of the curve and I don't know what, but I didn't know what it was ... And here it's the same with the opposing slope.



It is interesting to observe that during the interview, Stella was able to relate the symbolic representation of the derivative to its geometric meaning.

Regarding the information that can be obtained from a direction field associated with an equation we have observed that the source of students' greatest difficulties is the representation of solutions to the differential equation that holds a certain condition; only four students tried to represent the solutions by considering initial conditions $P(00$ and $P(-2) = 12$ in Q8 (Table 6) and only Edna and Wanda did it correctly while Jeremy and Angie made a mistake with the solution corresponding to $P(0) = 0$ (Figure 3).

In the questionnaire, neither Jordan nor Stella manage to extract any sort of information from the direction field given in activity Q8; however, in the interview, they managed to answer the questions in this problem related to the monotony of the function and its behavior at infinity, making use only of the direction field. Stella could not represent the specific solutions asked for in the problem; she appears to confuse the differential equation with its solutions.



Wanda's representation in Q8 Jeremy's representation in Q8

Figure 3

S: What does it mean that $P(0)$ is 0? That it is annulled when the equation is zero, but ...

R: What does $P(-2) = 12$ mean?

S: That when you substitute into the equation you put -2 at t, and this gives twelve.

R: What gives you twelve?

S: The equation, the result of ... I don't know ... This is as if we had, for example, a polynomial, then when instead of x you put -2; if it is, for example, x squared plus one, then if you put this it gives you a value. Well, the same but with P, I suppose.

R: Could you draw that solution?

S: No.

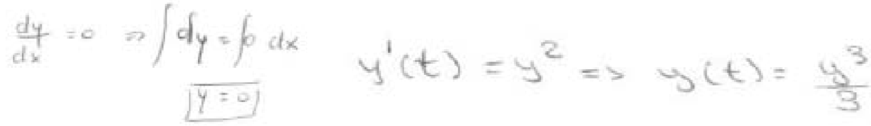
Comment. There is evidence that students experience serious difficulties in identifying and accessing a series of concepts and ideas previously studied to make sense of problem statements that are not asked for explicitly. It is also interesting to observe that when some students were asked to reflect on those concepts (during the interview), then they were able to recall and use them to draw the direction field.

A focus on the students' use of systems of representation

The way students represent concepts and problems play a relevant role in the identification and exploration of mathematical relations. To analyze the types of representation that students used to answer the questionnaire, we focus on three cognitive approaches that Duval [5] deems fundamental in the learning of mathematical concepts: the types of representations used, the treatment or operations carried out within and between the representations, and the transit, conversion, and connections utilized by the students among the different representations.

Generally, students recognize the algebraic expression of a differential equation and carry out the necessary operations within this register to solve the problem. Some of the difficulties that students found when handling the algebraic system of representation are related to the unsuitable choice of the method of solution to the equation (Figure 5 from Camacho-Machn et al. [4]), or an incorrect process of integration (Figure 4).

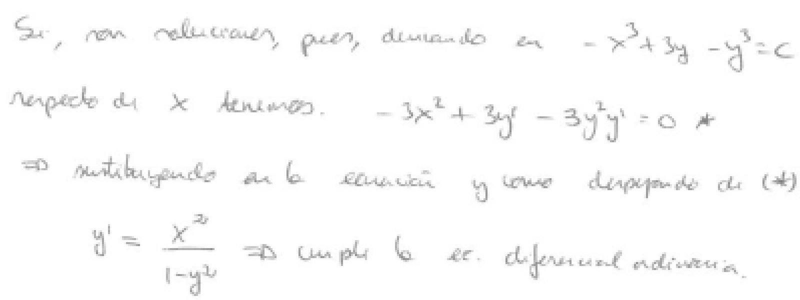
In general, all students recognize the algebraic representation of the solution to an ODE as an explicit function, except one who fails to answer any of the problems



Mary Laure

Figure 4. Mistakes in the integration of functions

included in Type 2. Representation of the solution as an implicit function is very different, as this is recognized by only four of the twenty-one students answering the questionnaire (Figure 5).



"Yes, they're solutions, because differentiating in the equation with respect to x we get $-3x^2 + 3y^2 - 3y^2y' = 0$. Substituting into the equation and as if finding from ($*$) the differential equation is satisfied."

Figure 5. Carina recognizes and transforms the implicit function

With regard to the graphical system of representation, the students in Type 1 problems (see Appendix 1 from Camacho-Machn et al. [4]) are faced with the representation of basic functions such as linear, exponential, trigonometric and hyperbolic functions. We can see that some students are not rigorous enough when making the graphical representations referred to, and merely give sketches of the functions without any detail (Figure 6).

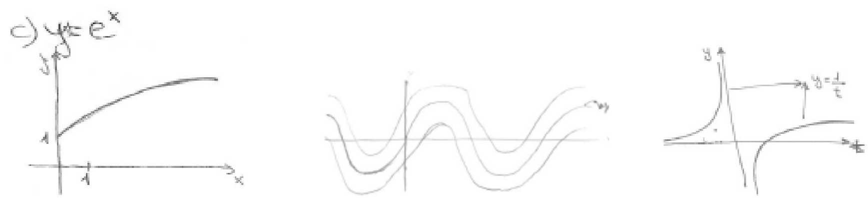


Figure 6. Some students' graphical representations

Other students make mistakes when representing the functions, even with the most basic functions, as is the case with Stella who incorrectly represents the function $\sin x$. This student also finds it difficult to transit from the algebraic to the graphical register and makes mistakes when graphically representing the function $y = ax + b$ (Figure 7).

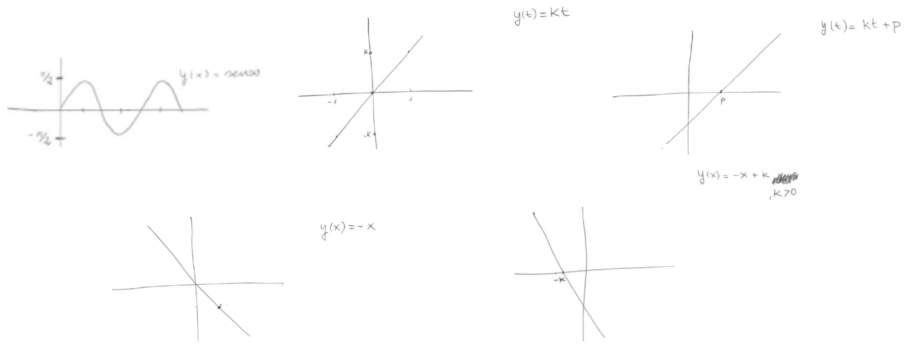


Figure 7. Errors in handling the graphical register (Stella)

Another piece of data that might tend to show us a lack of coordination between registers is the fact that Betty, in order to solve the equation $\frac{dy}{dx} = \cos x$, merely analyzes the sign of the derivative and does not establish any connection with the corresponding graph (Figure 3 from Camacho-Machn et al. [4]).

In question Q6, included in both the questionnaire and the interview with the aim of creating contradictions for those students who incorrectly solved the integral of the logarithmic function, Jordan provides us with an example of the importance of the activity of conversion between different registers of representation. Jordan solves the ODE in this problem by separating the variables but, when integrating the logarithmic function, he fails to take into account the absolute value of the argument, so that this solution would only be defined for positive values of the independent variable, in this case t. The student then shows that he does not know how to represent the direction field associated with this equation. Although he represents a function corresponding to $t = -1$, he does not accept this as a solution to the ODE.

R: You said before that for $t = -1$ the logarithm isn't defined, so the solution couldn't be drawn. And now, with the direction field, you've drawn it. What does this mean to you?

J: [. . .] If the differential equation had a solution, it'd be like this (he indicates the drawing) but then if the problem is that the solution, lest's say, wasn't defined, then we wouldn't have to draw it.

R: So, what you drew for $t = -1$, is it a solution or isn't it a solution to the differential equation?

J: I think it isn't.

R: Why?

J: Because if doesn't verify this solution [$y = \ln t + C$].

Another example of a student's lack of coordination between registers, and that leads him to a flaw interpretation, is shown by Jordan when solving question Q8 in the interview. This student graphically represents the solutions satisfying $P(0) = 0$ and $P(-2) = 12$, following the trace of the direction field (Figure 8).

This leads Jordan into the error of considering that the function that initially has value 0 is increasing, when the differential equation tells us that it is an equilibrium solution.

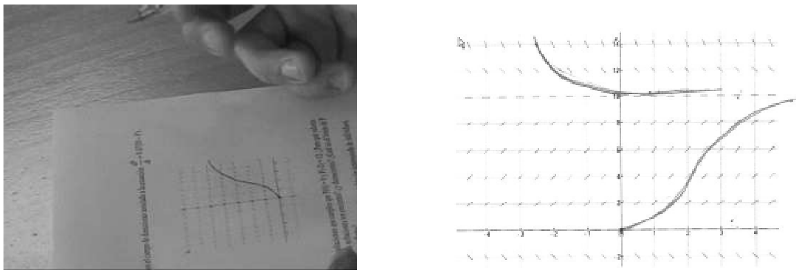


Figure 8. Jordan interprets the direction field

However, Jordan also provides us with an example of how the use of different systems of representation helps develop heuristic strategies to solve these problems. He attempted to solve problem Q8, where an ODE and its direction field are presented simultaneously, by solving the equation. The solution leads to the integration of a rational function, the algorithm of which he cannot remember, making him think again about the situation and he now takes into account the direction field represented. It can be seen that the graphical system of representation allows him to solve the activity correctly, something he would not have managed using only the algebraic register which requires the subject to remember at least two algorithms—the algorithm for approaching the ODE and the algorithm for solving integrals of rational functions.

J: For what positive values of P are the solutions increasing? Up here (for $P > 10$) they're decreasing and down here (for $0 < P < 10$) increasing.

J: What is the limit of P when t tends to infinity? The two of them would tend to 10.

Comment. The way students make sense of problem statements or situations is essential for them to engage in meaningful thinking or reasoning activities. In the conceptual framework, we argue that the students' development of mathematical proficiency involves not only understanding concepts and ways of operating with them in different contexts; but also developing mental habits consistent with mathematical practices. A crucial habit for students to develop such proficiency is to examine or explore problems or situations through the use of different systems of representation. In this context, it is evident that the participants mainly focus on the use of one system and fail to transit, in terms of meaning, from one system to another.

Final remarks

A central theme in this paper is to characterize the extent to which university students have developed mathematical proficiency to deal with basic ideas studied in a first-year course in ordinary differential equations. To this end, we rely on a framework (Kilpatrick et al. [8]) that identifies five interdependent or interconnected strands to discuss the development of mathematical proficiency. Thus, to document features of students' proficiency we use a set of mathematical tasks where the students had the opportunity to exhibit their ways of thinking to approach the tasks. The process shown by the students to deal with the tasks provides us with useful information in order to characterize their ways of thinking about and dealing with the concept of solution to an ordinary differential equation, the use derivative knowledge to make sense of differential equations and ways of constructing and interpreting the concept of slope or direction fields associated with some differential equations. In this context, we should consider the patterns of behaviors shown throughout this study.

Regarding the concept of solution, there is evidence that the students experience difficulties in identifying the different meanings of the concept of solution to a differential equation. For example, in general, they failed to report general solutions. It was also observed that to verify whether a function was a solution of an equation, they either chose to solve the equation and compare the obtained result with the given solution or they obtained the derivative of the function to check whether it satisfied the equation. In this process, they encountered difficulties in identifying and accessing proper knowledge and strategies to operate that knowledge. For example, some students could not recall how to apply the chain rule to get the derivative of an implicit function, and as a consequence they could not verify whether an implicit function was the possible solution to a differential equation.

In relation to the students' construction and interpretation of slope fields associated with some ODEs, it was observed that, in general, students also seem to rely initially on general sketches of elementary functions, which could not be used to relate their graphic representation to the algebraic expressions. That is, they could not transit in terms of meaning from geometric to algebraic representations or vice versa. As we can see in other research works (Gonzlez-Martn & Camacho [4]; Camacho, Depool & Santos-Trigo [2]), the graphical system of representation produces a certain degree of rejection among students, especially when it is related to a concept that is relatively new to them, as is the direction field associated with a differential equation. The student does not feel comfortable in this system when representing and exploring mathematical concepts. In particular, they had difficulties when relating the monotonicity of the function with its derivative and graphic behaviors. Graphical representation of the direction field consists of converting (in terms of meaning) from the algebraic to the graphical system. It cannot be limited to a codification of a process as this might lead to mistakes when solving certain problems. Thus, students need to focus their attention to the global behavior of the derivative function and its relation to the original function. That is, they should move back and forth from the graphic representation of the derivative function and the behavior of the function.

It seems that most students possess the conceptual resources needed to answer the questionnaire (differentiation, integration, graphical representation of functions, properties of the functions and their derivatives, algorithms to solve differential equations, etc.), but they cannot exploit these resources efficiently. We have also noted that students' learning of algorithms is not durable over time. Also, students lack, or at least do not rely on, monitoring strategies to focus on ways to recall procedures and forms of reasoning related to the expression involved. For example, students experienced difficulties to derive implicit functions. In this context, students need to become aware of the resources they possess and that they can use these resources in approaching different situations or problems. Rasmussen & Kwon [10, p. 190] suggest that an inquiring approach could guide the students construction and development of mathematical concepts learn new mathematics through inquiry by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems. We also argue that the way in which the mathematical contents are structured and presented to students plays an important role in the students' development of conceptual learning (Camacho, Perdomo & Santos [3]).

In this perspective, the results in this study provide important information to restructure and connect a first calculus course with an introductory differential equation course. For example, to review, extend, and articulate the meanings associated with the concept of derivative studied in a first calculus course, students could also relate the interpretation and/or meaning of the derivative to finding the solution of type of equations that involves $y'(t) = k$ or $\frac{dy}{dx} = \cos x$. Similarly, the geometric interpretation of the derivative could also help students represent the direction fields associated with a particular ODE or to analyze certain solution to an ODE without expressing them algebraically. That is, in a calculus course students can discuss certain types of differential equations in terms of the meaning of the concept of derivative without focusing yet on the use of particular algorithm to solve it. In this context, the use of the derivative to solve this type of problems could be the bridge for students to connect the concept of derivative with the initial concepts that appear in an ordinary differential equation course. Indeed, we argue that discussing this type of ODEs adds another meaning associated to the concept of derivative to the list of meaning proposed by Thurston [13].

Suitable representation of mathematical objects leads to greater perception of the intrinsic properties of these objects (Santos-Trigo & Barrera-Mora [11]). This does not mean that it is merely enough for students to use different systems of representation so that they can discover these properties. The students who took part in our research work recognized that their previous experiences of dealing with mathematics problems involve the use of graphical and algebraic systems of representation, though principally the latter system. However, very few students tackled those questions related to the direction field associated with a differential equation. Instruction based on the use of different systems of representation does not guarantee success in the solution of non-traditional activities, as noted by Rasmussen [9]. Rasmussen observed that, in this context, learning activities that in traditional instruction were performed through mechanical algebraic manipulations now get to be carried out as mechanical symbolic manipulations.

In view of the results obtained in this research work we believe that algebraic treatment of differential equations should not be limited to classification and later solution using algorithms, but that the equations should be analyzed from other points of view, relating them to the concept of derivative of a function and extracting from them as much information as possible about the solutions. In the case that an equation can be solved, the algebraic expression of the solution as well as the properties observed in the equation itself should be contrasted, leading to the development of heuristic strategies that the students appear to have forgotten. Similarly, students need to discuss and use the geometric meaning of the derivative concepts and not only to focus on rules or procedures to construct the direction field associated with an equation.

Finally, we argue that in order for students to develop mathematical proficiency, they need to develop a way of thinking that includes not only different ways of representing, connecting, using, or extending mathematical concepts; but also ways of overcoming difficulties that may arise during the problem solving approaches (Camacho, Perdomo & Santos [3]). We also recognize that proficiency is not all or nothing in terms of presence or absence of the strands that characterize it. It involves a process where students' behaviors associated with the strands need to be explicitly integrated. The results in this study show that the participants' initial attempts to make sense of problem statements is guided by the methods that they believe match those methods studied in the specific course and often fail to relate them to other concepts or ideas. Speer, Smith III & Horvath [12] underline the need to examine systematically collegiate mathematical practices in order to inform on solid knowledge or research grounds what is involved in teaching college mathematics. They state that:

> Studies that have targeted the classrooms practices of K-12 teachers have been productive in understanding the choices and acts of teaching, the factors that shape them, and the practices of teacher education. We expect that similar research at the collegiate level holds equal promise for understanding teachers' choices (and their rationales for them) and for aiding beginners by informing the design of professional development. . . . we also acknowledge that there are important differences between college and pre-college teachers and teaching. Collegiate teachers, for example, are less likely to face limits in their content knowledge. On the other hand, they also have less time with students, making experimenting with new content and activities potentially harder . . . In the space that is not constrained, collegiate teachers make judgments, and decisions, before, during, and after teaching, based on their sense of content, what their students do and do not understand, and what is possible in the time remaining in their courses. This is the space of teaching practice that we consider worthy of examination and analysis (pp. 100–101).

Thus, a research inquiry into collegiate teaching practices will help us understand and explain why in general students exhibit a fragmented or isolated knowledge in their attempts to solve problems and experience difficulties to overcome obstacles that they might encounter during the solution process.

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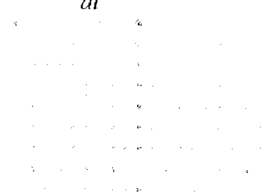
Appendix 1: List of activities and aims thereof

Problems of Type 2	***Setting***	***Aims***
Q3. Say whether the following statements are true or false and give reasons for your answer: a) The function $y = e^{\int e^{t^2} dt}$ is a solution for the differential equation $\frac{dy}{dt} = 4e^{t^2} y$. b) The functions $y = f(x)$ which satisfy $-x^3 + 3y - y^3 = C$ are solutions for the differential equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.	Questionnaire Interview	To determine the extent to which they take into account different forms of algebraic expression of the solutions to an ODE. To analyze: ▪ The solution strategy they use. ▪ If they take into account alternative solution strategies in case they encounter difficulties. ▪ If they differentiate correctly functions given both implicitly and explicitly. ▪ If they know the algorithms to solve first-order ODEs and separable variable equations and if they apply them correctly.
Q4. Say, giving your reasons, if the following statement is true or false: "The solutions of the ODE $y''-y=0$ are of the form $y(x) = Ae^{x} + e^{-x}$, where A is an arbitrary constant." **Q5.** Say whether the following statement is true or false and give reasons for your answer: "Take the first order differential equation $y'(x) = f(x,y)$. If the function $f(x,y)$ is defined as $\mathbf{R}^2$, solutions for the differential equation will also be defined as $\mathbf{R}^2$".	Questionnaire	To document whether the students recognize general properties of the solution to the ODEs and if they use counterexamples to justify their arguments. To analyze: ▪ The solution strategies the students use. ▪ If they distinguish a particular solution from a general solution of an ODE. ▪ If they know the algorithms for solving second-order, linear and constant coefficients ODEs and it they use them correctly.
Q11. If the function $Q = Ce^{kt}$ satisfies the differential equation $\frac{dQ}{dt} = -0.03Q$, what properties satisfy the values of C and k?	Questionnaire	To analyze the solution strategies they use and if they take into account alternative strategies in case of difficulties. To determine if the students correctly use the algebraic meaning of the concept of solution of an ODE.

Problems of Type 3	*Setting*	*Aims*
Q6. Solve the differential equation $y'(t)=\frac{1}{t}$. Draw the direction field associated with this equation and a solution for a t = -1.	Questionnaire Interview	▪ To create a contradiction for those students who use the integral of the logarithmic function incorrectly. ▪ To determine which systems of representation the students give greater validity to. ▪ To analyze: - The method of algebraic solution they use, if in fact they achieve this. - The graphical representation of the direction field and of the solution asked for. - The coordination between the graphical and algebraic systems of representation.
Q8. Let's consider the direction field associated with the equation $\frac{dP}{dt}=0.1P(10-P)$.  Represent the solutions that satisfy that P(0) = 0 and P(-2) = 12. For what positive values of P are the solutions increasing? For what values are they decreasing? What is the limit of P when t tends to infinity?	Questionnaire Interview	To analyze: ▪ The method chosen to solve the activity, as well as the system of representation. ▪ The meaning they give to the representation of the direction field associated with an ODE. To determine if the students transfer knowledge about the relationship of the derivative of a function with the monotony of this derivative.

Q10. Draw the direction field for the differential equations $\frac{dy}{dx}=1$ and, based on this, solve the following initial value problem $\begin{cases} \frac{dy}{dx}=1 \\ y(-2)=4 \end{cases}$.	Questionnaire	To identify the method chosen to solve the activity, as well as the system of representation.
Q14. Explain what a direction field associated with an ODE consists of.	Interview	To examine the mental image students have of the concept of direction field.
Q15. Solve the following initial value problem. Draw the direction field for the differential equations and represent graphically the solution of the IVP. $\begin{cases} \frac{dy}{dx}=-1 \\ y(-2)=4 \end{cases}$	Interview	To identify the method chosen to solve the activity as well the system of representation.

Appendix 2: Procedures used by students in the questionnaire

Type 1 activities

Q3(a)	Q3(b)	Students
10 students decided to differentiate the expression of the function given explicitly. Two of them did not differentiate the expression correctly. All of them concluded that the function was not a solution to the given ODE.	3 students differentiated the implicit expression of the function and concluded that the functions holding that property were solution of the ODE.	Carena, Angie, Melvin
	2 students decided to solve the equation. One used the separation of variable method to argue that the functions were solutions of the ODE. Another student differentiated the function but failed to use that information to answer the question. Later, she thought of the ODE as exact type; but could not solve it. She concluded that functions were not solutions to the ODE.	Wanda, Sam
	5 students did not answer this part.	Jordan, Stella, Edna, Betty, Laure
4 students chose to solve the equation in both cases by using the method of separation of variables. Two of them found the correct solutions. And two students made several mistakes while integrating the expressions involved.		Roger, Mary, Gaby, Silvana
1 student used an incorrect statement for the first part; but he was able to differentiate the implicit function (second part) to argue that the functions were solutions of the ODE.		Jeremy
1 student did not answer the first part; but she solved the equation involved in the second part to argue correctly that the functions were a solution of the ODE.		Berenice
5 students did not answer this question.		Jason, Eddy, Franklin, Helen, Rosy.

Table 1: How did students deal with question Q3?

Q4	Students
9 students solved the ODE using the characteristic equation. One student did not solve correctly; another one did not express the general solution correctly. One student also added a general argument related with the number of constants the general solution has depending on the order of the ODE.	Mary, Gaby, Jason, Eddy, Franklin, Carena, Melvin, Sam, Jeremy
1 student makes an argument in function of general properties of the solutions of the ODEs.	Roger
7 students check if the algebraic expression is the solution of the ODE but were not concerned about the general solution. All, except one who used a wrong equation, concluded that the statement was right.	Jordan, Wanda, Angie, Edna, Laure, Silvana, Berenice
4 students did not answer this question.	Stella, Betty, Helen, , Rosy

Table 2: Students' procedures in Q4

Q5	Students
2 students showed a counterexample for the statement.	Stella, Wendy
Another 4 students considered the statement to be false. The argument of one of them was "y'(x) depends only on x". Another student assumed a nonexistent hypothesis. A third one showed conceptual mistakes and the last one used an argument that is valid only for linear equations.	Sam, Roger, Edna, Franklin
3 students considered the statement was true. Two students stated that $y = \int f(x,y)dx$ was defined in R^2. Other students expressed that a constant was defined in R^2.	Melvin, Laure, Mary
4 students said the statement is true or false without arguments.	Silvana, Carena, Jeremy, Rosy
8 students did not answer this question.	Jordan, Stella, Berenice, Betty, Jason, Helen, Eddy, Gaby

Table 3: Students' responses in Q5

Q11		Students
4 students did not answer this question.		Jason, Jeremy, Helen, Eddy
4 students solved the ODE and compared the solution with the expression of the function in the statement.	2 students solved the ODE correctly. One of them made a mistake with the properties of the exponential function.	Mary, Gaby,
	2 students made mistakes while using the method of separation of variables.	Berenice, Rosy
13 students differentiated the expression of the function Q and made it equal to the second member of the ODE for obtaining the values of k and C.	7 students answered correctly.	Edna, Betty, Melvin, Jordan, Roger, Wendy, Carena
	3 students carried out the mathematical procedures correctly but failed in the answer. Two students said nothing about C and the other student did not interpret correctly the result she obtained.	Stella, Angie, Sam
	One student made a mistake simplifying an algebraic expression.	Laure
	One student made a mistake while differentiating the exponential function.	Franklin
Silvana firstly differentiated the function Q and made it equal to the second member of the ODE, getting k=-0.03. Then she replaced this value in the ODE and solved the equation using the separation of variable method. She made a mistake while integrating. Finally she made the function in the statement equal to the function obtained integrating, concluding that C=0.		

Table 4: Students' responses in Q11

Type 3 activities

Q6	Students
8 students solved the ODE, represented the direction field (DF) and a function. Six of the students used the method of separating variables. Five students did not assign the absolute value to the argument of the logarithmic function and one of them also failed to consider the constant of integration. One student was coherent with her algebraic solution and she did not represent the DF for negative values of the independent variable. Three students represented the correct DF but only two of them drew the specific solution for t=-1.	Roger, Wanda, Angie, Jeremy, Gaby, Franklin, Melvin, Laure.
7 students only solved the ODE. Five students used the method of separating variables; only one of them assigned the absolute value to the argument of the logarithmic function and three of them also failed to consider the constant of integration. One student, after solving the ODE, made some calculations for representing the DF but she did not know how to use them. One student tried to use the method for solving linear equations with constant coefficients.	Jordan, Berenice, Betty, Mary, Rosy, Sam, Stella.
4 students solved the ODE and represented a solution function but they did not represent the DF. Two students used the method of separating variables but they did not consider either the absolute value in the argument of the logarithmic function or the constant of integration. One student made a mistake while transforming the ODE. One student raised a problem of initial values and represented two particular solutions, one for a positive value of t and another one for negatives values of t.	Edna, Carena, Silvana, Jason.
2 students did not answer this question.	Helen, Eddy.

Table 5: Students' responses in Q6

<table>
<tr><th colspan="2">Q8</th><th>Students</th></tr>
<tr><td colspan="2">9 students did not answer this question.</td><td>Carena, Jordan, Roger, Berenice, Betty, Helen, Eddy, Franklin, Sam.</td></tr>
<tr><td rowspan="3">5 students solved the problem using only the DF.</td><td>3 students did not represent the solutions asked for. Two of them only answered the monotony of the function.</td><td>Stella, Melvin, Laure.</td></tr>
<tr><td>One student represented an increasing function for the condition P(0)=0.</td><td>Jeremy</td></tr>
<tr><td>One student answered this question correctly.</td><td>Edna</td></tr>
<tr><td colspan="2">5 students used only the ODE for solving this question. None of them answered correctly. Three of them used the method of separating variables but only one integrated correctly. She did not consider the constant of integration so she did not know what to do with the conditions P(0)=0 and P(-2)=12. Two students made mistakes while solving the ODE.</td><td>Silvana, Jason, Mary, Gaby, Rosy.</td></tr>
<tr><td colspan="2">2 students used both the ODE and the DF for solving the problem. Two students analyzed the monotony of the solutions using the ODE and the geometric meaning of the derivative. Only one of them represent correctly the two particular solutions corresponding to P(0)=0 y P(-2)=12.</td><td>Wanda, Angie.</td></tr>
</table>

Table 6: Students' responses in Q8

Q10	Students
9 students did not answer this question.	Carena, Jordan, Berenice, Betty, Jason, Eddy, Franklin, Rosy, Sam.
2 students did not use the ODE. One student represented the DF and the solution of the initial value problem without using algebraic expresions. Other students obtained the straight line equation using the point-slope form.	Melvin, Roger.
2 students did not use the graphic system. They did not represent the DF or the solution. One student used the method of separating variables but she did not consider the constant of integration. The other student only re-wrote the ODE.	Mary, Stella.
4 students solved the ODE separating variables. Two students did not consider the constant of integration; one of them stated that the problem did not have a solution and the other one represented the DF.	Wanda, Silvana, Angie, Jeremy.
4 students directly wrote the expression of the solution of the ODE. Three students did it correctly and represented the correct DF too. One student stated that the solution was y=x; she represented this function, the DF and the initial conditions.	Edna, Gaby, Helen, Laure.

Table 7: Students' responses in Q10