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AN EMPIRICAL STUDY IN CONVERGENCE VIA ITERATION AND ITS VISUALIZATION

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Abstract. Our aim is to provide an educative experience for High School students leading to a precise verbal description of the notion of convergence of a sequence of numbers generated by an iterative process triggered by the visualization of an unending geometric progression. Iteration as a guiding idea is chosen because it is easy to grasp, opens the door to further mathematical topics of the curriculum and encourages the use of technology. The experience is structured as an interview where a computer-generated tool, providing data generated by iteration and their visual dynamic representations, is available as an unavoidable aid to our goal of getting insight before formality. It all ends up to a very simple observation: the step-by-step implementation of a manual/visual routine together with reporting verbally what you see and what you don't see at every step is the clue to the understanding of the Weierstrassian definition of convergence, showing that its reputation of unintelligibility is hardly deserved. A detailed discussion of the experience concludes the exposition.

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Key words and phrases: Iteration; symbol; visualization; convergence.

1. Introduction

In [4] we conducted a study where functions appeared as models of certain relationships that were observed, trying to interest students in explaining changes verbally and transferring to their graphical (Cartesian) representations leading to the idea of rate of change and local rate of change, i.e. the derivative. In [3] we focused in the transfer from verbal to symbolic representation to justify the delicate equality $0.999\ldots = 1$. Both studies were structured as empirical interviews presenting data in a semi-fictitious Socratic narrative considering typical misconceptions and barriers students might encounter and how these might be overcome as well as emphasizing the role and need for a computer generated tool as a visual aid. Since the same presentation is followed here, we refer to our considerations there, where a more detailed and complementary justification for choosing this methodology was provided. In an amalgamation of procedure and concept-driven instruction, our purpose here is to trigger a verbal notion of convergence through visualization as the link between a non-terminating process (defined algebraically by iteration and visualized in a computer screen) and the realization of this process, which should ease the transition to the formal definition of convergence when the time comes and logical and algebraic maturity is at hand.

2. Our approach: interview combined with an applet as a learning experience

In the second half of the last century, a powerful educational model was designed for producing an elite cadre of highly trained and motivated students destined for sophisticated scientific and technical careers where the pedagogy used was formal, didactic, brilliant and severe. Since countries did not require vast numbers of mathematically trained professionals, there was sufficient mathematical talent and motivation to survive any pedagogy. By broadening the population which should be trained mathematically and in order to avoid generalized failure, the most pressing problem nowadays is how to motivate the desire to acquire conceptual knowledge. It is our impression that student population without intellectual curiosity in mathematics seems to be growing and the well detected declining overall performance of students in public university courses seems to confirm it. The pedagogical challenge is to figure out how to motivate, encourage and help students come to flexible and connected genuine mathematical insight since the usual classroom collective instruction methods, placing the teacher in front to the collective of students and conceiving teaching as a formal communication through a discourse, is found wanting to that effect.

According to the constructivist perspective which focuses in individual thinking and the continuing act of creating learning opportunities, we prefer to accomplish progress to insight via an appropriate discovery-oriented Socratic interview incorporating the use of visualization so that interviewees can encode relevant information in more linked ways than one. Two designs have to take place: one for the interview itself, which places the interviewer as the architect of the learning process, and hence due attention has to be paid to the existence of those cognitive obstacles which are bound to appear as described in the research literature, and another one for visualization as provider of images of abstracts. Since simplicity is always a bonus, an applet as a computer-generated tool (sometimes called mathlet) endowed with down-to-basics visual capabilities, has to be constructed with computational facilities added for versatility. The main concern is to create a learning environment which makes difficult the appearance of attitudes common in students who resign themselves to learning strategies in order to cope without understanding. In this spirit, both designs should emphasize exploration under guidance to encourage activities including conjecture and reflection which are supposed to engender deeper learning.

The applet provides the interviewee with a manipulative perspective to complement, expand and translate back and forth its already existent symbolic one. It offers the considerable advantages of information technology as multiplier of individual mathematical capacity, which can empower beginners to do what only more learned individuals once did. It is our experience that the applet, apart from easing the burden of calculations, inviting exploration and carrying out procedures which are not desired to be the focus of attention, promotes higher levels of engagement encouraging students' participation and provides them with numerical data and visual realizations which, paradoxically as it may sound, may facilitate abstraction by linking the processes of discovery, understanding and conceptualization in an increasingly cohesive network. Assuming the applet is up to the task of addressing the relevant information related to the intended goal, its mere presence during instruction is insufficient since students can avoid using the technology, use it unthinkingly, modify the intended task or draw incorrect conclusions (images or tables are not self-explanatory). Language acts as a mediator between the representation and the mental images the student is constructing and hence linguistically-based instruction can be reinforced by encouraging interviewees to talk about the pictures they have generated. We do so in the framework of a semi-structured interview, where concrete imagery needs to be coupled with thought processes at roughly the same time to be effectively used in any mathematical endeavor. Since most students have not had ample opportunities to hone visualization skills and some of them do not like thinking in terms of pictures when analytical thinking is at hand, it is important to encourage students to verbalize profusely their visual observations as well as write them down clearly and precisely.

Apart from making irrelevant any educational task that can be reduced to a routine, one wonders on the future impact of computers in instruction as programmers teach them to deal with tone and linguistic ambiguity. But the power of words is still under our control; we have to focalize in getting insight through explanation; what matters is the argument and how to engage students in this activity. The interview is by its very nature a tension and conflict filled process as new knowledge, skills, or attitudes are achieved through confrontation to surmount misconceptions as underlying beliefs which govern mistake or error. We shall proceed by breaking the interviewees' production of brain work into tiny slices leaving the more tedious tasks to the applet to allow them to focus in what they should be best at. We need to proceed slowly, promoting the view of mathematics as an evolutionary process and with deliberate silences implanted, since we believe that the path to insight through instruction requires time to learn, to make mistakes and rectify them; time to reflect, share and to think creatively and time to develop skills with construction and work with mental and pictorial images. Redundancy, usually avoided in mathematics instruction, can be used in small doses to be able to repeat experiences with the same information and to allow focusing on ideas to compensate for the lack of a long period of incubation time.

All things considered, an 'interview with applet' as described above is an effective way to promote reasoning by creating an environment in which students can engage in the production of arguments for explanation and verification. Its use as a learning experience allows us teachers to know our students and to be able, not only to explain things to them, but to understand their perspectives by listening to them closely and with understanding. Interviews have to be taped: it is essential to analyze the recordings of student's interviews focusing in mathematical and verbal explanations. It is expected that the diverse material for analysis gathered by the interview should enable us to get a detailed impression of the students' train of thoughts and strategies (or lack thereof), since one of our objectives is to understand the students' negotiations in order to constitute mathematical sense, the other being providing guidance to accomplish this task. What kind of mathematical material is suited to this strategy? Concepts, more than skills or results, are the best candidates: starting from a common ground, which allows an exchange of ideas, our selected concept should be simple enough to be endowed with a naïve understanding of it which is rooted in earlier experiences; it should allow the transition to higher forms of reasoning by placing it in different contexts (from verbal to symbolic to visual and backwards to symbolical-algebraic) which should ease the transition to the formal definition of the concept through abstraction.

3. Our selected mathematical concept

There are several options available for the introduction of the idea of convergence to High School students. We chose doing it through the use of iteration because we want our students to grasp at early stages of their education one of those ideas in mathematics that form unifying themes across different branches of mathematics, including functions, transformations, proof, and data. Iteration has important applications throughout the curriculum; understanding iteration, the way in which systematically repeated simple operations can build complex structures, can shed light on many important concepts in arithmetic, algebra, geometry, fractals, calculus, and mathematical modelling. Even in a traditional presentation of Calculus, as a fragmented collection of assorted topics and techniques often buried under a mountain of unnecessary formalism obscuring the intuitively clear notion of smooth change, it is important to emphasize the idea of iteration as those aforementioned topics and techniques are perceived by students as united only by the ease in which they can be reduced to step-by-step procedures: the finding of limits and derivatives, in which a value becomes smaller and smaller, is an iterative process; Riemann sums accumulate the value of a function while an interval is iteratively divided into smaller and smaller subintervals. Because iteration often involves large numbers of operations, which would be difficult and time-consuming to carry out by hand, it is particularly suited to the use of a computer generated tool which performs repetitive mental tasks much faster than human beings.

The concrete realization of the idea of iteration via geometrical series (i.e., a series where there is a constant ratio between successive terms) is chosen because they play an important role in the development of Calculus, their sum (if existent) is easily calculated by self-similarity arguments and they have important applications in physics, computer science, finance and economics. We shall pursue a visualization of this series to facilitate the search of a judicious conjecture for the sum of the series and a precise verbal explanation of what getting close to it should mean, which is accomplished by putting into words the action of another iteration, namely the process of repeating a zooming operation around the chosen conjecture. Visualization comes after training in verbal, symbolic and numerical reasoning, because excessive reliance on visualization at the expense of other representations can result in a lower mathematical performance.

Our experience was conducted on High School students in Seville and Valencia which (i) were proficient in algebraic manipulations, (ii) had familiarity with periodic decimal expressions (the result of division between integers) as numbers with place in the number line in spite of their unending representations (iii) had been subjected to detailed exposure to a notion of function (as a relationship between changing variables symbolized by $y = f(x)$, a single rule over a whole domain) and to its associated bi-dimensional graph in the Cartesian plane, but not to any limiting process aiming at the definition and manipulation of continuity, differentiation or integration of functions and (iv) had no specific training in developing abilities to construct formal arguments. Twenty individual interviews were carried out with the purpose of studying how they reacted to new ideas under instruction and how potential collisions with previous non-well digested knowledge could be negotiated and surmounted. Each successful one took roughly one and a half hour time. Students were selected on the basis of their willingness to participate and every volunteer was accepted. They agreed to the audio recording of the interviews and also to the use of their corresponding anonymous transcriptions in our analysis. Their previous upbringing in mathematics was the usual consistent (but sadly declining) one provided by the Baccalaureate in Spain.

4. The tool

The tool is an interactive screen and the user does not need to have previous knowledge on the program (MATLAB 6.1) used to design it. The screen shows two windows (Figure 1), a graphical (GW) and a computational one (CW): GW allows the plotting of two functions and CW shows three columns corresponding to order and iterates of the two entered functions and the number of digits present in CW can be chosen. Several buttons allow us to write the analytic expression of the functions to be considered, start the iterative process from abscissa 0 and choose the number of iterations to be performed, which will appear as a spiral or a staircase when GW is activated (pushbuttons below). We keep equal scales for both abscissa and ordinate.

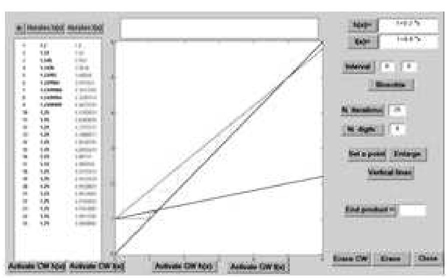


Figure 1

Zooming capability by a factor of one third is available at every point of the screen by enlarging a square: the mouse allows us to choose its center by pushing on "*Set a point*" and the square to be enlarged appears at the screen as well as the abscissa of its center (Figure 2). When the zoom is executed, the square occupies the whole graphic window GW and therefore only a portion of the spiral/staircase will be visible; a message above will appear reminding us of the number m of zooming operations performed and indicating how many iterates n are left outside the visual field (Figure 3). We may add more iterates to the process and/or perform new zooming operations. The pushbutton "*Vertical lines*" allow us to see iterates in abscissas as projections of the staircase/spiral steps on a number-marked parallel line to the x-axis (Figure 4) and "*End product*" allows us to write the conjectured limit of the sequence of iterates, an abscissa which marks its corresponding point at the bisectrix line, and picture automatically the following square to be enlarged and its projection on abscissas. The pushbutton "Erase" allows us to start again with original scales, select a new iterative function and change the number of digits to be seen in CW.

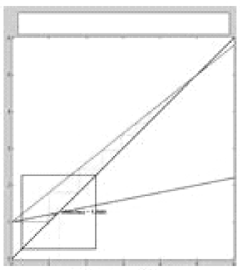


Figure 2

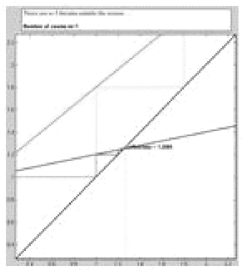


Figure 3

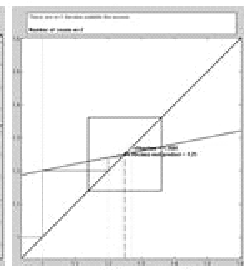


Figure 4

5. The interview

In what follows, we have assembled a single interview with a fictitious student whose answers come from a variety of students who were able to reach the interview's conclusion. Hoping that the experience might offer some guidance to teachers wanting to introduce the topic to their students, we have unified our transcriptions and edited heavily the final text in order to present a narrative in mature language avoiding clumsiness, false starts, hesitations and fragmented and incoherent arguments hopefully preserving what we were able to experience in terms of clarity of thought and adaptability of our most successful students, if not in their exact actual words. As a warming up exercise, we start by briefly revisiting the concept of function.

5.1. Functions and processes

Pr.: Do you remember what a function is?

St.: A formula relating two variables written as $y = f(x)$ where x and y are the variables. A variation in x is reflected in the y and that manipulation carried out on x produces y.

Pr.: By the term "variable" what do you mean?

St.: A magnitude whose value changes continuously, such as space or time.

Pr.: And what do you understand under the term "continuously"?

St.: As moving on a line segment; no holes or gaps.

Pr.: Is it a reasonable assumption? I mean, do we perceive in everyday life any boundary between one moment and the next?

St.: (pensive) No, we perceive time as flowing continuously. The same applies to space.

Pr.: What is the use of the function f?

St.: It establishes the relation between variables, for instance, displacement as a function of time as in physics. As time flows, displacement changes and we can picture the relationship in the Cartesian plane as the graph of the function where time is in abscissas and displacement in ordinates.

We mention generically the term procedure as the steps taken in an action and we fix the meaning of the term *process* as a sequence of linked mathematical procedures which converts inputs into outputs.

Pr.: Can you understand a function as a process consisting in a sequence of commands?

St.: I am not sure what you mean; if I understand f as a formula, say $f(x) = x^2 + 1$, I can see it as a process where commands are 'square the variable and then add one' which can be applied to any value of the variable x.

Pr.: Right. You are allowed to perform substitution and then evaluation by addition and multiplication.

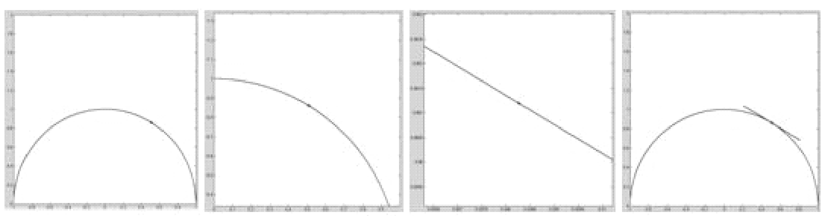
St.: And I obtain output $f(x)$ from input x. This is how it works.

Pr.: Moreover, we can understand a function with temporal variable as a process in continuous time. But you may imagine other processes in time which, however, can be broken into discrete steps.

St.: Discrete as opposed to continuous?

Pr.: You can think of discrete time as something like a sequence of snapshots, whereas continuous time is more like a movie. Folding a sheet of paper is a good example of this; each fold is more or less an instantaneous event and the time between folds serves as a boundary. Folding can, therefore, be modelled, approximately, in discrete time. Discrete time breaks a process up into the inputs and outputs at individual, separated moments in time (or space).

5.2. Iteration



Figures 5

As a first contact with the tool and in order to provide some training in its graphing and scaling capabilities, we activate GW and enter an analytical expression for a certain function. We place the cross on some point of its graph (scale 1 : 1) and we ask him to perform repeatedly a zooming operation, as if a magnifying glass of larger and larger resolution power was used, by the same amplifying factor. Asked about what he ends up seeing after several operations he will suggest that a straight line is appearing at some step (see Figures 5). Now is not the time to argue if a line is produced after a finite number of such operations or if one should carry on indefinitely, but assuming a straight line is what appears, we ask him what would be the effect of further zooming operations. He does not hesitate in saying that a straight line will remain no matter how many times the operation is executed and provides a rationale for his words in terms of the effects equal scale produces on figures. We fix the line in the screen, proceed to undo the zooming operations recuperating the curve and we ask about the relation between curve and line; he adventures that the line stays tangent to the curve. Thus, the process of repeatedly performing the same zooming operation seems to produce something of relevance, namely the tangent line to the curve at the chosen point.

We explain further where we are heading to. We are looking for a description of a process which takes at each moment the output of a given step to be used as the input for the next step in the process. Such a process is called iteration; complete a step by performing an action that generates a new value; use that new value as the starting point as repeat the same action; repeat this action for as long as you like. Iteration means the act of repeating an action usually with the aim of approaching a desired goal or target or result. Each repetition in the process is also called again iteration, and the result of one iteration (called iterate) is used as the starting point for the next iteration which produces the next iterate.

St.: Apart from folding paper and performing zooming operations, what type of process and operation are you suggesting?

Pr.: The operation may be a simple arithmetic operation of addition or multiplication.

St.: Such as ...

Pr.: For example, we assume that there is some abstract quantity called a "one" and that you can add it to numbers to get other numbers, which, to give them names, we might as well call "two" and "three" and so on. Natural numbers are defined as the collection of numbers that are finite sums where all terms are 1. For instance, $1 + 1 + 1 + 1 + 1$ is the natural number "five".

St.: You start with 1 and add one repeatedly.

Pr.: Is there any reason why this process should stop?

St.: (smiling) Boredom or physical impossibility, take your pick.

Pr.: Even if you can't do it, can you imagine yourself doing it?

St.: No matter how many times I have done it, I can also do it one more time. Theoretically, the process can go forever.

Pr.: What does this process guarantee?

St.: That every natural number has ... a successor, a larger one.

Pr.: Conclusion: there is no largest number, then.

Can the notion of function help in describing iteration?

Pr.: Try to describe the iteration process above by an algebraic operation, such as using a function in a variable x, such that when x is conveniently fed with an initial iterate you can go to the next one.

St.: Do you mean a function which can be used at every step, from 1 to 2, from 2 to 3 and so on?

Pr.: Yes, but the same function at every repeated action, not a different rule for each one; call it $h(x)$.

St.: I start with 1 and I use $h(x) = x + 1$ to produce $1 + 1 = 2$; then $h(2) = 2 + 1 = 3$ and so on.

Now we consider a word problem to be turned into an (algebraic) iteration process.

5.3. A word problem suggests a symbol and an iteration process

Pr.: Suppose now that I want you to have a certain amount of money, say A Euros, but you have to pay VAT, say 20% of it. How much money do I have to deliver so that you can end up with A EUR after taxes?

St.: A will not do; you need to deliver $A + 0.2\,A$ to compensate for VAT.

Pr.: It does not seem right because you have to pay $0.2\,A + 0.2^2 A$ back.

St.: (taking paper and pencil) I am left $0.2^2 A$ short. Then, give me $A + 0.2\,A + 0.2^2 A$.

Pr.: But then you have to pay tax on the extra amount $0.2^2 A$, that is, $0.2^3 A$.

St.: Hmm ... I see myself adding and adding without end.

Pr.: Can you provide a symbolic expression for this unending process?

St.: Hmm ... , the process could be written as $A + 0.2\,A + 0.2^2 A + 0.2^3 A + \cdots$.

We want to know which meaning is assigned to the ellipsis 'three dots'.

Pr.: Please, explain the three dots.

St.: I don't care if there are three or more dots; what I want to express is that the process never ends ... something similar to what we do when writing down decimal expressions, I guess.

Pr.: Such as 1.111 ... ?

St.: Right! Well, there is something more ... it also means that I know which digits should follow in the expression; if I write 1.111 ... the dots say that a '1' comes after another '1' but if I write 1.739 ... without further explanation, I don't know what the three dots stand for.

Pr.: The same meaning here?

St.: Well, yeah in the sense that the process is unending and that I know what follows if I want to write more terms, namely $0.2^4 A$ and so on. But ... here the three dots say that ... somehow ...

Pr.: The information in the terms preceding the dots should set the pattern ...

St.: Of what comes next, that's it!

Pr.: Does $A + 0.2\,A + 0.2^2 A + 0.2^3 A + \cdots$ stand for an ordinary addition?

St.: Ordinary is not, since this process never ends. Anyway, since I can't calculate it, what is the use?

Now we ask him to analyze if we are dealing with an iterative process.

Pr.: Can this process be described as iteration?

St.: I start with A and produce $A+0.2\,A$ ... well, by adding $0.2\,A$; now we obtain $A+0.2\,A+0.2^2 A$ by adding $0.2^2 A$ and so on. Each time I add a different quantity, not the same situation as the natural numbers. It doesn't look as if I am repeating the same operation over and over.

Pr.: Iteration is the process of performing some operation repeatedly, but that operation is not necessarily adding the same quantity. You have to change your perspective and think of a different operation.

St.: The operation transforms A in $A+0.2\,A$; now the same operation has to produce $A+0.2\,A+0.2^2 A$ from $A + 0.2\,A$. Is that what you mean?

Pr.: Yes, step-by-step input and outputs are seen; what you need is figure out the common mechanism of conversion from one to the other. Let us think in terms of functions. Can you think of this operation as a function which performs this duty step-by-step in terms of input and output?

St.: What you want me to do is to use a function and the same function for every step, I see.

Trial and error takes place. After several injudicious guesses, he focuses in going from A to $A + 0.2\,A$ and proposes $x \mapsto x + 0.2\,x$ to describe the rule. Once the next step is understood as feeding the function with $A + 0.2\,A$, he realizes that his choice doesn't work. He changes the rule to $x \mapsto A + 0.2\,x$ and then, output $A + 0.2\,A + 0.2^2 A$ is obtained from input $A + 0.2\,A$. Further checks confirm him of the suitability of his choice.

Pr.: Thus the process is described by the repeated action of the function $h(x) = A + 0.2\,x$ and our starting data A is there by choosing $x = 0$.

St.: Right ... I got it ... Iteration here doesn't mean adding always the same amount, but applying repeatedly the same function $h(x)$; first to 0, then to A, then to $A + 0.2\,A$ and so on.

Uneasiness can be felt since we haven't provide a solution to our original question, namely the amount of money to be delivered. The interviewee comments that it should be possible to calculate it and we are happy to oblige: if B denotes to amount to hand over, it should happen that B minus tax should equal A; that is, $B - 0.2\,B = A$; hence B equals A divided by $1 - 0.2$. The simplicity of the argument comes as an anticlimax as he feels cheated for all the previous work done above.

Pr.: Please, be patient and indulge me for a moment. What about our last simple argument in terms of $h(x)$? Can you think of B as the solution of some equation involving $h(x)$ such as $h(x)$ equals something?

St.: (hesitating) B being $A/(1 - 0.2)$ ... solving $h(x) = x$, I get $x = B$.

5.4. Making sense of an equality

He points out that, should we have started with this last argument, the whole 'process argumentation' seems quite irrelevant. We agree that this is the case if our interest is limited to knowing the solution, but we state that we are more interested in the problem itself as a mirror in where to look when confronted with many others problems where iteration plays a role and where a solution may not be so easily at hand.

Returning to our problem, we want him to keep in mind two facts: on one hand, a symbol $A + 0.2\,A + 0.2^2 A + 0.2^3 A + \cdots$ for the kind of process he should follow to end up with A EUR after taxes has been developed and, on the other, we have been able to know what this amount is by a simple algebraic argumentation related to the statement of the word problem. We emphasize (i) that symbol and algebraic argumentation have something in common, namely the use of the same function $h(x)$ and (ii) that we are in the presence of a process of iteration in which we are not only interested in each iterate (as was the case for the natural numbers) but also in what we may call the end product of this process, namely the number $A/(1 - 0.2)$.

Pr.: Is it reasonable to write $A + 0.2\,A + 0.2^2 A + 0.2^3 A + \cdots = A/(1 - 0.2)$?

St.: (uncomfortable) As an equation? I do not know. ... I mean ... if I write equation $1 + 1 = 2$, the left hand side is an ordinary sum and 2 is the result of performing that sum; 2 does not come out of the blue.

Pr.: Do you mean that, in our situation, the left and the right hand sides of the equality may be different entities? Is it like comparing apples and pears?

St.: Well, it could be. The left hand side is the symbol for a process and the right hand side is a number.

Pr.: But surely the process has something to do with numbers ...

St.: Sure, the process symbol is a kind of short writing for many numbers, what you called iterates ... a collection of numbers ... which are somehow in constant movement ... growing as a matter of fact since they are related by progressive addition ... but not a single number.

Pr.: But can you think of a single number with a tight connection with of all them in terms of the process?

St.: A single one as a kind of representative of what is going on? Surely it is not one of the iterates ... there is no reason to prefer one over another! ... Well, if we speak of a process, it is understood that the process is leading somewhere at the end, right? ... then, perhaps I should think of the number which appears as the result of the process ... what you called its end product ... which, in this case, I suppose it should be $A/(1 - 0.2)$.

Pr.: I propose to you that, in order to make sense of the equality, we may write an equality sign after the symbol 'process' to denote its end product.

St.: Are you saying that "$A+0.2\,A+0.2^2A+0.2^3A+\cdots$" and "$A+0.2\,A+0.2^2A+0.2^3A+\cdots=$" should mean different things: ongoing process and end product respectively?

Pr.: Well, it is for us to decide if it is useful to make this distinction. If it makes sense of the equality and doesn't contradict any known fact, it could be the solution to our symbolic troubles.

St.: I didn't realize we were in trouble but if you insist, once an equality sign appears, the process reverts to a number which is its end product ...

Pr.: Borrowing your words, as if something in movement stabilizes ...

St: OK, in that case, the equality now makes sense ...

Pr.: But not in the ordinary sense?

St.: No, the unending stuff makes the difference. But something is bothering me: back to my former example, I don't see much difference between writing "$1+1$" and "$1+1=$".

Pr.: I might be nitpicking here, but there is a slight difference if you decide to understand the equality sign as a kind of command over $1+1$; perform the addition!

St.: Agreed, there is nothing new on the presence of the equality sign, but "$A+0.2\,A+0.2^2A+0.2^3A+\cdots=$" and "$1+1=$" are different, the difference being the impossibility of executing whatever command is implicit in the first one.

Pr.: But does this impossibility affect the validity of the corresponding equality?

St.: No, because I know the end product for it ... because I have information ... in a way, it is as if I can execute the command, without properly doing so.

Pr.: Are we in the presence of something new to you?

St.: You bet we are! This is not plain algebra, for sure; it is more like a twisted mind game.

Intrigued by closely resembling symbols and whether they mean something different, he asks if there is any relationship between iterative processes and decimal expressions.

5.5. Are we talking about something essentially new?

Pr.: I happened to mention the expression 1.111 ... Has this symbol meaning by itself?

St.: It is a decimal expression and hence a number ... it can be understood as a length if you want.

Pr.: A measurable one?

St.: The more '1' I consider, the more approximate can I measure this length.

Pr.: But not an exact one? Isn't it also the symbol that appears when calculating 10/9?

St.: (checks with his calculator) No three dots are to be seen in the display: 1.1111111 is what appears ... but that has to do with size of the window ... if more digits were allowed to show, I suppose I would see more '1' ...

Pr.: How can you be sure? There has to be a reason why precisely those digits appear in that order.

St.: (takes paper and pencil and performs the usual algorithm) Well, division is the reason, since every division produces a '1' ... which makes reasonable to write 1.111 ... for the result of division ... the dots are there to say that a '1' comes after another '1' and not a different digit. Then, I can write $1.111\ldots=10/9$ and it is an exact length.

Pr.: The ellipsis and an equality sign, again! Is there any relation with what we have done so far?

St.: Uh-huh! If you want me to see those three dots as some kind of iteration again, why don't you say so?

Pr.: Is 1.111 ... the result of a kind of process consisting in repeating the same operation on and on?

St.: This is obviously the case here, as division is performed on and on? But I do not see the connection to the symbol $A+0.2\,A+0.2^2A+0.2^3A+\cdots$.

Pr.: Unless there is some reason to believe that our old symbol 1.111 ... has the same meaning as our recently defined symbol $1 + 1/10 + 1/10^2 + 1/10^3 + \cdots$.

St.: (checks with paper and pencil again and thinks) That is the connection, the progressive accumulation of amounts which, if added, are 1, 1.1, 1.11, 1.111 and so on ... It makes sense to see 1.111 ... as a process.

Pr.: And, if we introduce an equality sign after the process?

St.: Hmm ... it reverts to its end product.

Pr.: 10/9 stands for ... St.: Its end product, right? I can write $1.111\ldots = 10/9$, understanding $1.111\ldots$ as $1 + 1/10 + 1/10^2 + 1/10^3 + \cdots$. What we have been seeing so far is not that strange after all!

Pr.: Yes. In the context of the money problem, what is the relevance of $1.111\ldots = 10/9$?

St.: A should be 1 and VAT is 10% and you should give me 10/9 EUR.

Pr.: Observe that, again, the end product did not appear as the result of adding without end.

5.6. A first try in extending the equality and exploring the limits of this extension

Now we agree that the choice of VAT's amount was arbitrary and we could write $A + rA + r^2A + r^3A + \cdots = A/(1 - r)$ if $0 < r < 1$ without further explanation. We recapitulate that we have seen iteration as an algebraic operation such as applying a function to get a complete description of the process and we remind him of the dual role played by the function $h(x)$ to generate the left and the right-hand side of this equality.

Pr.: Does this equality make sense if $r = 0$?

St.: If we go back to the original problem, if VAT is 0, it is enough you provide me with A EUR which explains the right-hand side.

Pr.: But not the left-hand side?

St.: If, after giving me A EUR, you insist in providing me with 0 EUR indefinitely, I guess I am stuck with A EUR.

Pr.: What if $r = 1$?

St.: 100% VAT? From the point of view of the money problem, no matter how much you give me, I am left empty handed.

Pr.: From the symbolic point of view?

St.: The equality makes no sense: $A + A + A + \cdots = A/0$; the left-hand side grows without bound ... I mean is heading nowhere ...

Pr.: And the right-hand side?

St.: It is no known number, since division by 0 is not allowed.

Pr.: Taking $A = 1$, is the iteration $1 + 1 + 1 + \cdots$ essentially different from $1 + 1 + 1$?

St.: The second one is a step in the iteration $1+1+1+\cdots$ producing the iterate 3 as $1+1+1+1$ produces 4 and so on but $1 + 1 + 1 + \cdots$ can be made larger than any number.

5.7. Is the equality context-bound? Developing a gut feeling for the end product

A geometric procedure is asked to be mentally executed and, by relating it to iteration, he will develop a feeling for the idea of proximity.

Pr.: Suppose you are given a segment of length 2. You bisect it and remove the left half. Proceed that way with the leftovers indefinitely. It is clearly a process of iteration. What is left of the initial segment?

St.: (takes paper and pencil) Hmm ... , I don't know; the moment I stop the procedure of halving and throwing out something is left.

Pr.: Let's put numbers in our construction. Even if you can do it physically, do it mentally in terms of lengths of the segments which have been removed.

St.: I am removing 1, $1+1/2$, $1+1/2+1/4$ (hesitates) ... well, I could write $1+1/2+1/4+\cdots$ to represent the ongoing procedure.

Pr.: Does $1+1/2+1/4+\cdots$ stand for the symbol of a process as before? Does our former equality apply to this situation?

St.: It is indeed the symbol for a process of iteration, but it is clearly not a tax problem.

Pr.: Is there a tax problem which leads to the same symbol?

St.: (takes pencil) A should be 1 and $r = 1/2$ and, when writing an equality sign for the end product ... voilá! ... $1+1/2+1/4+\cdots = 2$. I should get 2 EUR. But, even so, can I say that 2 is the end product here?

Pr.: Can you even say that there is an end product to the process $1+1/2+1/4+\cdots$? If you have $1+1/2+1/4+\cdots$, do you have also $1+1/2+1/4+\cdots = ?$

He is concerned about the equality $1+1/2+1/4+\cdots = 2$ being true independently of the context and rightly so. We tell him that we are not in position to give a clear cut answer removing all doubts until we can free the symbol from its origins; he recognizes that all troubles come from the fact that the symbol has no algebraic personality, that is, it is unclear what command is supposed to be executed, when equality is written after process. But a certain confidence can be gained if, assuming the equality true in the different removal procedure, we reach conclusions expected to happen: since the end product of the process $1+1/2+1/4+\cdots$ should stand for the total length of removed segments in the procedure and since 2 is the number provided by the equality, once the procedure is completed nothing of the original segment should be left and that is consistent with expectations. Thus the equality seems to hold, at least, in two different contexts. But something of more importance is to be gained by introducing a different setting: the idea of the end product as being in the progressive (numerical) proximity of iterates.

Pr.: Returning to process, can you explain what happens in terms of iterates?

St.: Assuming the equality works, 2 is the end product of the process $1+1/2+1/4+\cdots$ (Using a pocket calculator) Each iterate is smaller than 2 but larger than the preceding one. Moreover, 2 acts as a kind of frontier to which iterates approach ... as close as I want.

Pr.: 'As close as I want' means?

St.: The difference or distance between 2 and iterates becomes as small as I want.

Pr.: You changed 'close' by 'small', but you should explain to me the 'I want' part.

St.: Well, is not personal. What I mean is that the process advances inexorably to its end product.

Pr.: I need you to rephrase your comment by providing a more precise statement. Let us try it this way: returning to the removal procedure, suppose that I establish a small 'security zone' around 2 not to be interfered with by your procedure of discarding segments. Is this possible?

St.: If I am allowed to halve indefinitely, every security zone you establish however small is going to be touched ... no way you can stop me from eating away the space between us ...

Pr.: Turning again to the process and speaking in terms of iterates ...

St.: You cannot put any obstacle between iterates and end product ...

Pr.: By obstacle, do you mean a number?

St.: Yes, eventually one iterate will jump over any numerical obstacle.

Pr.: Could you synthesize your statement by saying 'for every number smaller than the end product, there exists an iterate which is larger than that number'

St.: (irritated) I don't think it is important if you say it one way or the other; what you said is exactly what I meant, just a different syntax was used.

5.8. Visualizing iteration as a way of escaping context

We inform him that our interest is shifting from context (taxes, removal procedures) to a context-free equality, even if the context was useful in providing meaning for the process, a gut feeling for the end product and even extra arguments to calculate it. We turn to visualization to produce an explanation of the connection between process and end product and hence the possibility of executing the process in some way in which the visual and algebraic argumentative steps interact with each other. For the sake of simplicity we suppose $A = 1$. A short recapitulation is in order: we arrived at the formula $1 + 0.2 + 0.2^2 + 0.2^3 + \cdots = 1/(1 - 0.2)$ where this last number appeared as the solution of the equation $h(x) = x$, $h(x)$ being the function $1 + 0.2x$ which described the repeated operation to be performed in the iteration process expressed as $1 + 0.2 + 0.2^2 + 0.2^3 + \cdots$.

Pr.: Can we obtain a graphical representation of both process and end product which makes sense of the equality obtained? Start with the end product.

St.: Since the end product is the solution of the equation $h(x) = x$, it has to be the intersection point of two graphs; those arising from x and $h(x)$.

Pr.: (we activate GW to produce both graphs, see Figure 6) A point described as two coordinates or its abscissa?

St.: Well, yes it has to be an abscissa ... even if both are equal; I take the abscissa 1.25.

Pr.: Right. How do we graph in abscissas something showing $1 + 0.2 + 0.2^2 + 0.2^3 + \cdots$?

St.: I don't know; I could go step-by-step drawing in abscissas, 1, then $1 + 0.2$ and so on and seeing the process as an ever growing segment.

Pr.: Too laborious! It is better that the tool does the job for us. Let us try to picture our problem in two dimensions in a screen and then go to one dimension to see of process, the same way we did when jumping from the intersection point to its abscissa.

St.: How?

Pr.: In the first place, step-by-step is the right way to go because it is the essence of an iteration process. Consider $h(x)$ and ask yourself which value of x has to be fed on it to obtain the first term 1.

St.: $x = 0$ produces $h(x) = 1$, but in ordinates.

Pr.: (we use GW to produce a vertical line on the y-axis, see Figure 7) Abscissas and ordinates! That's why you need two dimensions. But, since 1 is your next feed for $h(x)$, you need to put this value in abscissas.

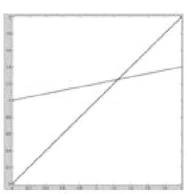


Figure 6

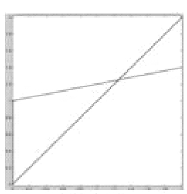


Figure 7

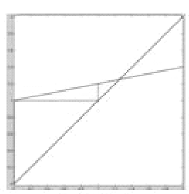


Figure 8

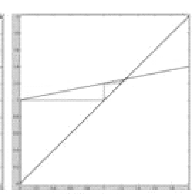


Figure 9

We want him to see explicitly 'former output as next input' in the screen without further complications in order to make our chain reaction visible. We formulate our question as follows: instead of calculating anything and placing it on the screen, think of the surest path to follow from an ordinate output to land on an equally valued abscissa as next input. We ask him to place his finger on ordinate 1 and we suggest that he uses the bisectrix line as support for the intended purpose.

St.: From ordinate 1, I move horizontally until I meet the bisectrix and then go I go down vertically.

Pr.: (using GW to perform the next iteration, see Figure 8). Now we go up to $h(x)$ to obtain the ordinate $1 + 0.2$ which in turn ...

St.: It becomes the next feed for $h(x)$. (Pointing finger on the screen) I go back to the x-axis as I did previously.

Pr.: Please, help yourself with the computer. The edit box 'N iterations' allows you to see the process of taking as many iterations as deemed appropriate. Start with a few.

St.: (performs the required task for several small numbers of them, see Figure 9) I am drawing a staircase.

Pr.: If you could project the staircase step-by-step on the x-axis, what do you obtain?

St.: (gesticulating) First 1; then 1+0.2; then $1+0.2+0.2^2$ and so on; which are also the progressive addition of horizontal steps of the staircase.

Pr.: Where is the staircase heading to?

St.: Apparently, to the intersection point of both lines, if the trend seen in Figure 9 continues.

Pr.: And where is $1 + 0.2 + 0.2^2 + 0.2^3 + \cdots$ to be seen?

St.: (pointing at the screen) The screen shows a growing staircase when I perform more and more iterations. If I could project it into the x-axis, the process would appear as a kind of a growing segment in the x-axis with right end approaching 1.25.

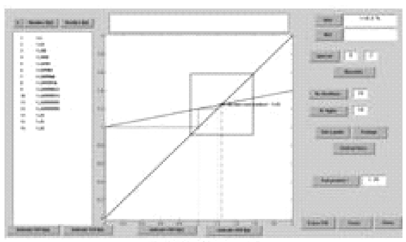


Figure 10

Pr.: (producing figure 10 by activating 'vertical lines' and 'end product'). Something like this?

St.: Uh-Huh! Do the green verticals point at iterates in the x-axis?

Pr.: Right. Approaching means ... ?

St.: As I said before, if I concentrate in the x-axis, iterates get closer and closer to the abscissa of the intersection point in the same way as it happened in the 'removal procedure' situation.

Pr.: Right. Thus we have a process $1 + 0.2 + 0.2^2 + 0.2^3 + \cdots$ and its end product 1.25. What is the idea connecting process and end product?

St.: The idea that the process produces iterates depleting the distance between them and the end product. No security zone around 1.25!

Pr: If you could draw the whole staircase—which you can't—and project it, what would you see in the x-axis?

St.: (stretches his hand) A segment starting in 0 and ending in 1.25.

We check how he reacts to a closely related problem.

Pr.: Imagine we were dealing now with $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots$ What changes?

St.: I have to consider a different function ... $h(x) = 1 + 0.8x$ (manipulates GW, Figure 11). There is also a staircase heading to the intersection point too.

Pr.: (activates GW with both staircases in different colors, Figure 12) Compare both processes.

St.: Since now the intersection point is further away, getting closer takes a longer staircase.

Pr.: Can you be more precise?

St.: I need the tool to perform more iterations, which means more steps to be seen in the screen which in turn it comes to more iterates in the x-axis approaching the end product.

Pr.: Why is it further away?

St.: (looking at Figure 12) The intersection point, do you mean? The black line being the same, it depends on the inclination of $h(x)$ ... the closer to the inclination of the bisectrix, the more distant the intersection point is ... it seems to me that if both lines are parallel, there is no intersection point and hence no end product.

Pr.: Right! But still true that $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots = 1/(1 - 0.8) = 5$?

St.: (we activate CW explaining the appearing column of numbers as iterates, see Figure 13) Yes, the numbers in the table confirm what is seen in the graphic window.

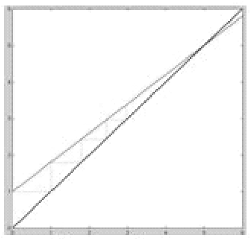


Figure 11

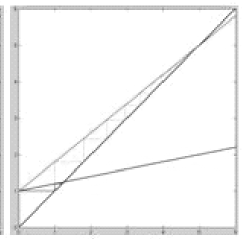


Figure 12

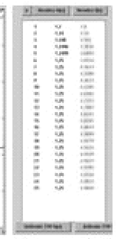


Figure 13

5.9. Further explorations of the equality: from staircases to spirals

Once equality $1 + r + r^2 + r^3 + \cdots = 1/(1 - r)$ for $0 < r < 1$ is accepted as plausible, we wonder if this equality might make sense for negative r, that is $-1 < r < 0$ or, if we want to deal with positive r, we may ask if $1 - r + r^2 - r^3 + \cdots = 1/(1 + r)$, $0 < r < 1$ holds true. We move to a visual context by avoiding possible interpretations of the tax problem which generated the former equality and therefore we ask him to concentrate in the search for a suitable function to ensure that we are still dealing with an iteration process.

St.: (checks with paper and pencil) I have to consider a different $h(x)$ ... let us see ... (after several attempts) I think $h(x) = 1rx$ does the trick.

Pr.: Try the tool for $r = 0.2$.

St.: (activates GW, see Figure 14) No staircase now ... but a kind of ... spiral.

Pr.: But still approaching visually the intersection point?

St.: (performing several iterations and zooming, see Figure 15) If the equality remains true, the process should head to $1/1.2$ which equals $0.8333\ldots$ what I see is that the spiral encroaches the intersection point ... really ... each iterate is better than the preceding one from the point of view of being closer to $0.8333\ldots$ in the x-axis; in this sense approaches.

Pr.: Again, where is the process $1 - 0.2 + 0.2^2 - 0.2^3 + \cdots$ to be seen?

St.: (fingering at the bottom end of GW) Here is 1, there is $1 - 0.2$ and so on: we advance an amount 1, we retreat to $1 - 0.2$, and then we advance to $1 - 0.2 + 0.2^2$ and so on. Counting from 0 onwards, it is a kind of segment which expands and contracts without end.

Pr.: Is each advance larger than the following retreat?

St.: (pensive) Quite so, and each retreat is larger than the following advance.

Pr.: (activating and pointing at CW, see Figure 16) Observe again that those numbers are the result of successive iterations performed by you, that is, iterates.

St.: (checking on the numbers on the lower side of the screen) Yes, they correspond to projecting the spiral. I could have used also the green verticals. Anyway, the numbers approach $0.8333\ldots$, but in a different way; they come closer and closer to it but as jumping from one side to the other ... as if estimating by defect and excess.

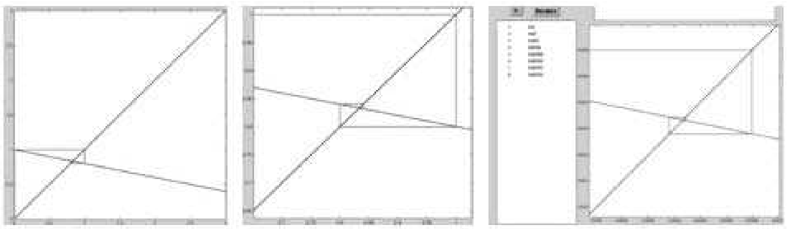


Figure 14 Figure 15 Figure 16

Now he seems convinced that $1-r+r^2-r^3+\cdots=1/(1+r)$, $0<r<1$ holds true. We want him to decide whether the equality holds for $r=1$. GW is activated and it demands his attention.

St.: Shall I check with $h(x)=1x$? (Manipulates GW) The graph of $h(x)$ meets the bisectrix, hence there is intersection point with abscissa 1/2 ... but ... no spiral here ... just a self-repeating square (see Figure 17), no matter how many iterations I ask to be seen ... does this happen because both lines are perpendicular?

Pr.: (activating CW) Good observation! Please, have a look at the arithmetical process itself.

St.: (pointing at the equality) The process is $1-1+1-1+\cdots$, CW shows that iterates are 1,0,1,0, ... as it should be and 1/2 is the number the equality suggest as end product should it be valid ... but there is no way those numbers approach 1/2 in the depleting sense ... of course, you may say that iterates approach both 1 and 0 but not a single number.

Pr.: Since "$1-1+1-1+\cdots=$" should stand for a single end product if there is one ...

St.: But there is not such a one! The equality has to be not valid, then!

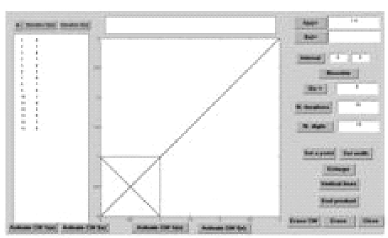


Figure 17

Thus, we can conclude that $A+rA+r^2A+r^3A+\cdots=A/(1-r)$ if $-1<r<1$ makes sense.

5.10. Linking iterates and end product: towards a verbal formulation of convergence

Again it is time for a further recapitulation: first, we discuss whether this equality is of the usual algebraic type. He points out that this is not the case since the left-hand side is a non-ending addition as that is why the three dots are written. We want to be sure about the meaning assigned to them. A process where a single repeated action is performed indefinitely starting with one value and using this number to produce the next one (which is the addition of two numbers) and so on, he replies. This process can't be done physically but can be imagined such as the construction of the natural numbers. What is the purpose of performing such a process? Getting at equality with some number, he answers. But one thing is a process and other one is a number; how can they be equal? Once an equality sign is written after the process, he says, the process comes to signify something slightly different, what is called the end product of the process, that is, a concrete number. End product understood as ... ? He feels uncomfortable and he mentions tool-generated staircases and spirals heading to somewhere. We ask what do they have in common and he replies that, going to the x-axis, iterates generated by the process come closer and closer, either unilaterally or from both sides, as depleting distances to a number which is already in the picture. Since it is visualization, not arithmetic, responsible for triggering an intuitive notion of convergence, we continue our experience visually. Now we consider the process $1 + 0.5 + 0.5^2 + 0.5^3 + \cdots$.

5.10.1. Making a conjecture and failing

Pr.: (we activate GW with a partial staircase but without both lines, see Figure 18) Could you've predicted which number the end product is, even if the equality wasn't there to give you a clue?

St.: Equality present means I know the intersection point (stares at GW while manipulating it to draw a larger staircase, see Figure 19). Without help and working in the screen, I could use the cross to select a point ... and reading its abscissa, I get the end product (selects a point which will turn out to be not exactly what is wanted).

Pr.: You just made a guess. How can you be sure that your choice is the correct one?

St.: For starters, since I know the solution to be on the bisectrix line, I have to be sure that abscissa and ordinate of the cross is the same (checks it). I could draw the bisectrix to be more comfortable.

Pr.: Right, but it still is only a guess.

St.: It looks as if the staircase is heading there. To be sure I should have a closer look to the point, because it could happen that an even longer staircase may overtake my guess.

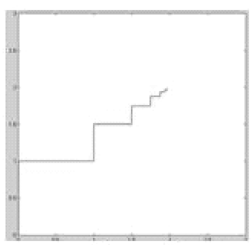


Figure 18

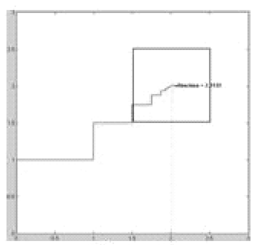
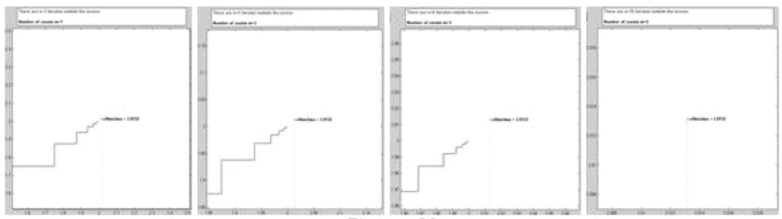


Figure 19

We encourage him to use of the zooming iteration capability as later on will reveal itself as a key ingredient when we ask to elaborate on a precise verbal notion of convergence. We point out that a closer look can be obtained by iterating the zoom facility centered at the chosen point (pushbuttons 'Set a point' and 'Enlarge', Figure 19). We recall our earlier exchange about the zooming capability and its relation to choosing different scales; we remind him that a zoom operation means equal scales on both axis and we indicate where to look to see the number of executed zooming operations.

St.: (does so, simultaneously increasing every time the number of steps to be seen, see Figures 20) Right. As I perform zoom iterations, I can see ... a portion of the staircase in the screen ... and those steps which are not visible are accounted for (pointing at the message above the screen) ... but wait! ... I do not see the staircase now.



Figures 20

Pr.: Why do you think is this happening?

St.: Either I need to put visually even more steps (tries in vain) ... or my guess was not right in the first place; I'll try with a slightly different one.

Pr.: Trial and error then. It could take some time. Let us cheat: why don't you look at the equality to get a good guess?

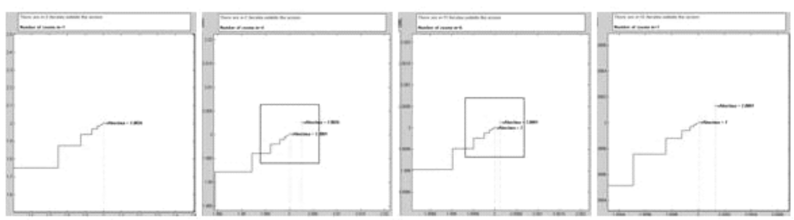
St.: You are right: 2 is what the equality indicates and hence I should choose the point $(2, 2)$ in the bisectrix as zooming center (starts doing again the zooming iterations, see Figures 21). Now it seems right.

Pr.: In absence of the equality, how could you be absolutely sure that your choice was correct?

St.: It should happen exactly what is happening here; the trend observed with a few iterations has to continue indefinitely: the staircase should appear no matter how many zooms I perform, provided the staircase is long enough. But, since I can't perform those operations indefinitely ...

Pr.: Not to mention the fact that by choosing so small scales you may break the resolution of the screen.

St.: There is no way to be absolutely sure. Something seems to be missing ...



Figures 21

He is aware that extra arguments on which to rely upon are needed in order to cement the conquests of intuition boosted by visualization, which by itself does not provide certainty. What it is needed is refinement and a more systematic articulation of what we see and talk about through algebra, which is supposed to be the lingua franca of college mathematics courses, which is used precisely to avoid the vagueness of words. Algebra, then, could give us the solution if we were able to provide the algebraic equivalent of the symbolic three dots, or its visual realization as progressive zooming or its verbal translation of 'getting closer and closer'. If successful, we could proceed with all the power of algebraic machinery. We promise him to fulfill this promise when we shall return to the classroom, but before we go there and in order to fix the key ingredients of this translation by focusing in the visual perspective, could he explain in a natural way and with plenty of detail what we have being doing lately by projecting staircase, guess and zooming on the x-axis?

5.10.2. Trying not to fail: using the tool to magnify and project

St.: Projecting the staircase (pointing at Figure 22) produces a kind of rain of iterates …

Pr.: Which is basically a visualization of the information CW provides.

St.: Right. Projecting the staircase means placing iterates in the x-axis and projecting the guessed point produces the end product of the process.

Pr.: Since we are interested in the 'tail' of the staircase more than in how it starts, the repeated use of the zooming capability seems to be the key to check on your guess.

St.: Yes, that and also the possibility of adding more and more steps. It allows us to have a closer look of where the staircase is heading to confirming or refuting our guess and also how many steps are out of sight.

Pr.: Going back to the original scale and having drawn a very long staircase, what means projecting successive zoom windows?

St.: (gesticulating) Each zooming window projects in an interval (Figure 23) where part of the rainfall is visible, the one corresponding to part of a tail of the staircase.

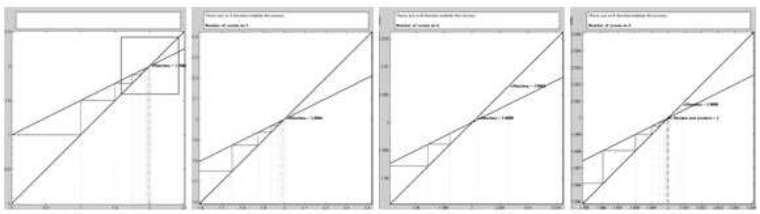


Figure 22 Figure 23 Figure 24 Figure 25

Pr.: Can you explain the relevant information shown by that figure? Start with the message above.

St.: The message tells us the first n iterates which are outside view …

Pr.: Is this number n related to the number of steps in the staircase?

St.: n staircase steps project in precisely n iterates.

Pr.: Basically, n stands for the first n steps of the staircase which …

St.: I am not able to see. What the message also indicates is how many zooms m we have performed so far; in this case, only one.

Pr.: Where does the rain fall? Is the x-axis present in the figure?

St.: Not necessarily. The rain falls on a segment of the size of the projection of the zooming square.

Pr.: The numbers written on the segment allow us to estimate values for iterates and end product, although there is always the possibility to know them with precision …

St.: We activate CW choosing the number of digits to be seen.

Pr.: Right. If we keep performing zooming operations, it amounts to what?

St.: Basically, it amounts to taking intervals around the end product which are successively smaller …

Pr.: As m gets …

St.: Bigger.

5.10.3. If your guess is the right one, what do you see?

Pr.: And, staying in the x-axis, what should happen no matter how many zooming operations you perform, if your guess is the correct one?

St.: First iterates are outside the visual field as indicated by the message, but it comes the moment where some iterate should come forward.

Pr.: If it doesn't …

St.: It is because more steps in the staircase than those originally provided are needed: I do so until I see a staircase step appearing in the picture and then I project to estimate the first visible iterate.

Pr.: Is it the same first visible iterate for every zooming operation?

St.: No, each zooming operation produces a screen and hence an interval where a first iterate should be visible but obviously a different one for each operation ...

Pr.: Being economical in the use of language, what you just said is: each selection of m provides a different number n of iterates outside our view. Where is the evidence on this affirmation?

St.: (zooming again several times, see Figure 24) Well, the message at the screen is clear about that ... the message alerts you of the number n of steps of the staircase projecting outside this interval, that is the number of iterates outside view.

Pr.: The bigger m is or equivalently the smaller the interval ...

St.: The larger the number of iterates n lying outside the interval, that is, they are not visible ... if I know this number I obviously know then the first visible iterate, namely the $(n+1)$-th iterate.

Pr.: Only this one?

St.: No, that one and all that follow in order; well, I may need to ask the tool to draw a staircase long enough (see Figure 25), that is, the number of steps being larger than $n+1$.

We ask him if the fact that the choice of m dictates the value of n is everything he needs to know to complete his mental picture of what is going on, either in one or two dimensions. He agrees and we ask him to write down a summary of what ensures that a certain conjecture for end product is the right one. It reads: *on one hand, different number of zooms around our point of choice executed leave different number of iterates outside view (once a zooming operation is executed, say the m-th, some n first iterates lie outside view for a certain n) and, on the other hand, once n is known, the next iterate and all that follow should be visible if desired and that should happen no matter how many times a zooming operation is executed. Hence m and n are all that matters as relevant data and there is a connection between them: as m changes, so does n.* Asked where this essential information is stored, he refers to the message above GW which contains that information and nothing else; moreover, as the message changes every time he pushes the button 'Enlarge', it is possible to construct a double-column table showing the evolution of n in function of m and, with luck, a formula relating both integers.

5.10.4. Verbalizing a notion of convergence

We proceed to our last recapitulation in order to force a verbal definition as economical in terms of language as possible keeping what is indispensable and avoiding what is not. We started with an algebraic iteration process with a plausible end product which has to be conjectured. We provide a bi-dimensional visualization of this process (a staircase) and we ask to make a judicious visual guess (a point) putting another iteration process at his disposal, namely the capability of zooming around the conjectured point. If the choice turns out to be the right one, the projection of staircase and point stand for algebraic process and end product respectively.

Pr.: What does it mean that the choice of end product is the right one? I know we just covered that topic but I want you to express it as succinct as possible using the terms zooming iteration and algebraic iteration.

St.: The choice is good if each zooming iteration results in an algebraic iterate which is visible in the window produced by the zooming iteration as well as all forthcoming ones if the staircase is long enough.

Pr.: (handing over a paper sheet and insisting in him writing down what is demanded) Suppose I wanted to have the last part of your affirmation codified in a pre-established form: for every (blank space) there exists (blank space) such that (blank space) and try to keep essentials only.

St.: For every zooming iteration, there exists an (algebraic) iterate such that it is visible.

Pr.: Very good! Be even more economical: use only letters m and n!

St.: (impatient) For every m, there exists n, such that the $(n+1)$-th-iterate should be visible.

Pr.: (activating GW, see Figure 26) Should a spiral appear instead of a staircase, does your last sentence explain what can be seen?

St.: Although iterates appear at the right or left of the end product (see Figures 27, 28 and 29), they still should be visible since they lie in the interval determined by the zooming iteration: thus the answer is yes.

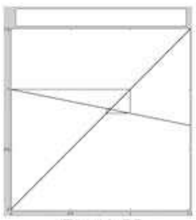


Figure 26

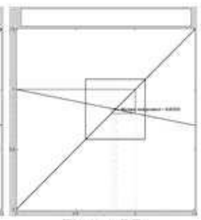


Figure 27

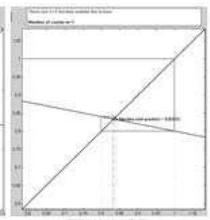


Figure 28

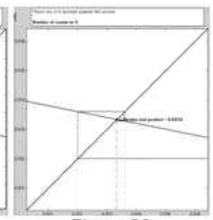


Figure 29

6. Discussion

Neither mild nor severe dysfunctions show in the former transcript, which has been heavily edited and restricted to successful students, but they were very present indeed and we shall attempt to provide a more accurate picture of the experience by including them. For the sake of brevity we limit ourselves to the presentation of the major difficulties encountered along the experience.

No trouble was expected at our introductory remarks concerning their understanding of time (or space) as flowing continuously since this conception adapts well to their intuitions suggesting that any temporal interval or length, as small as it may be, can be subdivided. No distinction between rational 'continuum' and real continuum came forward since irrational numbers were not part yet of their thinking processes or intuitions. No resistance to accept this model was expected to arise when we dealt with variables which were to be fed to functions, with no restriction on the number of times this evaluating procedure can be performed on them. The idea of process as a chain of actions posed no challenge and it was accepted that a function understood either as an algorithm, the product of an algorithm or an input-output description, could be considered a process in a finite number of algebraic steps.

When we dealt with infinite sets and processes that never end the first epistemological obstacle which was bound to appear was the passage from finite to infinite. When confronted with unending processes of arithmetical or geometrical nature in discrete time, reluctance to accept them was observed at first glance. It was natural to expect that a tiny part of our interviewees would reject the infinite as a matter of principle, but it took no much effort to convince the vast majority to accept the task of performing an unending operation repeatedly as theoretically feasible in their imaginations as opposed to executed physically (which can't be done in practice, they will argue). Their acceptance of performing such a process did not include the possibility that such a process may actually lead to something.

The 'mechanics' of an iterative process was natural and easy to grasp to them, starting with the construction of the natural numbers or the repeated use of the zooming capability. The training in iterative processes along the interview eased the aforementioned transition of acceptance from finite to infinite, although the last ounce of suspicion was never completely removed based in deep held feelings that '*mathematics is completely abstract and far from reality*'. Once we jumped to a concrete but more delicate process, such as the tax 'word problem', other difficulties arose: how to turn a word problem into iteration by searching a concrete function which stands for the repeated action in the process, nothing to be surprised of because it is well documented that, for many students, the transformation of word problems into arithmetic or algebra causes difficulty and a number of studies have addressed the linguistic and mathematical sources of that difficulty from a psychological point of view (see [2]). Our next port of call was to suggest that some of those processes may have a numerical end product' to them and some not. Since for those having it, its existence was guaranteed by the word problem and not by the process itself, they were initially accepted as feasible, but not as approximation devices or estimations to some desired result whose actual value can be deduced from the process itself.

When pushed to translate the word problem in algebraic terms, students produced a new symbol which had to explained and justified to see how it acted as support of internal cognitive processing. To enable communication it seemed convenient to distinguish symbolically between *process* (a finite addition followed by three dots) and *end product* (finite addition followed by three dots followed by an equality sign leading to an equation, albeit not of the usual algebraic type since the right hand side seems not to be generated by the left hand side). We had to deal with difficulties arising from connections between perceptions and symbols (these obstacles have been thoroughly investigated in the literature) to avoid the malpractice of many students who tend to employ symbolic manipulations with no coherent understanding of their meanings. Particularly, a certain sense of uneasiness was detected because the main symbol was pictorially close enough to a known entity (a finite addition) but added an ellipsis (three dots) which moved it outside familiar territory, illustrating the change from a static to a dynamic situation, the difference being a symbol standing for a specific command which can be executed in practice and another perceived as a theoretically executable but non-executable in finite time, and hence frustrating, command. The symbol was finally accepted as a representation of the idea of a repeated action, especially when confronted with decimal expressions which, not without a certain difficulty, made clear that such a description had been already used and an end product was obtainable. A further difficulty arose when the accepted meaning of the symbol kept changing to adapt to new needs, namely the passage from an operational conception to a structural one; that was really arduous since students were lacking an unavoidable ingredient which will require construction: the notion of convergence.

A detour: since this experience can also be seen as an early introduction to the notion of convergence of numerical series disguised as unending geometric progressions, the advantage of our strategy is that convergence appears in a well grounded context and inconsistencies in students' minds such as '*infinitely many addends, infinitely great sum*' are, at least, avoided. Cognitive obstacles derived from the wild goose chase triggered by asking to provide arithmetical meaning to expressions such as $1+1+1+\cdots$ and $1-1+1-1+\cdots$, which can be algebraically manipulated at will to produce the most unexpected answers, can also be dealt with when those series are presented in a context such as ours which most of them can relate to. In our study, both series appeared as borderline cases in a certain context and therefore the validity of a formula relating them to an end product was easily under suspicion. Negating the existence of end product for the first one was unproblematic in the context given. Concerning the more delicate second series, a context to motivate a rejection of end product could be created by re-interpreting what a negative tax may mean, even if their imaginations can be a bit stretched, but we did not take this path in the interview because (i) we did not want to risk reinforcing previous prejudices on *mathematics as a game with no connection to reality*, Sierpińska [8] and Bagni [1], and (ii) our next goal is to force the need to see both process and end product out of context. For those students who failed to complete the interview due to their lack of understanding of whatever ideas followed, we proceeded after hours to a complementary tutorial providing a full explanation of this context, how an end product may arise and which the borderline cases were and what they meant, hence leaving them, at least, with a picture of how the formula providing the sum of a geometric series may arise.

Now that a precise meaning had been attached to the symbol representing the process, we were willing to develop a feeling for what the end product may stand for since, at this stage, it has appeared as a *deus ex machina*. We introduced our interviewee to a segment removal procedure, quickly identified as an iterative process, for which a symbolic expression similar to the one appearing in the tax problem was produced by him without our intervention. Whether the symbol by itself implied the existence of end product and whether the end product of the tax process was the same here, were questions provoking a discussion centered on whether the equality was context-independent or not. For the time being, we asked him to assume it is and see what follows from there: the geometric context triggered the mental image of the end product as some number which was close, in an intuitive sense, to the numbers the process generates, its iterates. Physical gesticulation and all sorts of metaphorical verbal expressions accompanied his intuitive explanation of proximity. When we tried to extract equivalent formulations somewhat closer to mathematical language, our inquiries were met with muted hostility, but finally an effort was made of being verbally more precise.

In order to free the equality from context, we turned to visualization as a technological way of producing pictures of our algebraic processes, allowing him to grasp the end product as coming from the process itself by playing the idea of proximity. A sound verbalization of this idea as coming from the perspective of a visual thinker is what was left in our agenda. Visualization, understood as the construction on a computer screen of objects which the interviewee identifies with objects or processes in his mind, contributed to intuitive cognition and served purposes other than just presenting information in a new format. Although not without limitations, it enjoyed the advantage of organizing all given information hence reducing cognitive load and complexity; it also provided a monitoring and evaluating device to assess progress by revealing further information and implications. Our manipulative tool, used from the very beginning of the interview, was our representation device of choice.

The relationship between visualization and mathematical performance has been an area of interest in mathematics education [7], and we were aware that the educational use of visual registers and the ability of interpretation and reflection upon pictures, images or diagrams may need non-algorithmic training in order to capitalize on their strengths and may be problematic for ill-equipped students without the ability to coordinate representation or switch between different ones. The fact that a vast majority of students do not like thinking in terms of pictures unless explicitly required to do so, complaining that the passage from picture to words disturbs their reasoning, adds to complicate the situation. What was of importance here, and it can be counted as an added obstacle, was the need to develop an understanding that a single picture was not enough, because a dynamic process needed to be depicted, a sense of 'forward movement' had to be appreciated on the tool's screen corresponding to the dynamical image developed in his mind when dealing with processes from an algebraic point of view. In spite of the aforementioned difficulties, visualization's considerable advantages were worth the try: sequences built by numeric iteration provided fertile ground for exploring graphing and for understanding the relationship between the data generated by an iteration rule and the shape of a graph of that data. The graph of a sequence became another lens through which students could understand the operation they were performing to generate data. Using the tool to build iterations helped students to think about them at a more abstract level as they figured out how to explicitly specify the seed and the operation to be performed on the seed and make them realize that the symbol of the process dictated the shape of the figures generated and not the context in which the process was presented.

Now that a point was reached in our arguments which made distinctions necessary and once visualization of processes was at his disposal, we lead him to the activity of judicious guessing taking advantage of the tool's visual capabilities. The lack of unpleasant consequences or manifested impatience when guesses were made and testing proved them wrong was encouraging for them, especially for the least abstract-minded students which seemed to benefit more from introducing exploration in instruction. Reflecting on what constitutes a good guess, we asked gently to make a first try in explaining what 'getting close' should mean. Although intuitive enough, it was a bumpy road to make them express with precision what they observed in an accepted syntax giving priority where it was due (first, zooming and then counting on the terms of the process). Once their verbal formulation agreed with the meaning negotiated, we kept asking to be economical with language and keep only essentials; we arrived finally to a verbalized notion of convergence of the process to its end product which can act as a meaningful definition, when properly formalized; this will come easily when logical and algebraic maturity is an asset.

It is our claim that this highly visual approach with strong symbolic support motivates and guides, although it does not replace, analytical thinking which will enter when a formalized definition is available. Last but not least, our visual approach opened the door to the study of the notion of convergence independently of the setting chosen, insofar as it is possible to obtain a dynamic representation of an unending process and that it is possible to conjecture convergence when the end product is not previously provided. For a study of the convergence of series of positive numbers we refer the interested reader to [5]; an experience attempting the passage from a verbal to a formal definition of convergence is described in [6]. Both studies are deemed at more advanced students than those who participated in our present study.

Report. Three general observations to begin with: first, it was clear to us that verbal explanation presented serious challenges to many students, although the most gifted and curious ones were able to refine their language in terms of verbal accuracy along the interview even if a certain disdain regarding precision in verbal communication skills was noticeable: once an idea was understood, no much value was placed in expressing it unambiguously as if insight was only relevant to the individual and no communication abilities were deemed relevant. Second and curious enough, lack of familiarity with a computer-generated tool presented no problems whatsoever and they quickly adapted to their use, some of them with remarkable proficiency. Third, and not to be assumed lightly, we were blessed with students trying to understand.

Although conceived simply as a quick, uncomplicated and simultaneous introduction to the idea of iteration and the graphical capabilities of the tool, the appearance of the tangent line to a curve after a repeated use of the zooming capability (*as if you look through a microscope*, most of them said) triggered different interesting reactions. From our pool of twenty interviewed students, *two* of them had to be discarded from the interview when the word 'unending' intruded as they were convinced 'finitists' and no appeal to their imagination was successful: for them infinity was metaphysical and *one should avoid speaking about it*'. The rest accepted the process as legitimate: a vast majority of them trusted what they saw at the screen without further thought and made no objections whatsoever, but a very few of them called our bluff. They argued that such a line cannot be obtained after a finite number of steps because what we have done is simply to observe a curve and this action should not alter its shape. Since line and curve are essentially different shapes, if a line is what really appears, this line cannot be reached in a finite number of steps and hence unending iteration is the only possibility remaining. Even accepting that this process can be done theoretically, they were puzzled by the consequences … they started to grasp how infinite procedures can change the rules of the game and that visualization without thought can be deceptive.

The word problem caused severe difficulties to four of our remaining eighteen students either because the dynamic aspect of the formulation was not entirely understood or because they lacked the algebraic ability to find the iterative function. As expected, the remaining *fourteen* dealt with the symbolic representation of the process without trouble but reluctantly when it ended up as part of an equation; acceptance was finally negotiated with the help of their previous understanding of decimal expressions when explained as progressive additions (symbol) and arising as the result of repeated division between integers (equality). Doubts arose when confronted with the algebraic formulation of particular situations such as $1 + 1 + 1 + \cdots$ and $1 - 1 + 1 - 1 + \cdots$, particularly the latter.

Visualization of iterative processes animated the experience triggering their natural curiosity and they enjoyed taking partial command of the interview. Executing commands in the tool and producing graphs was an easy task for them but when interpretation was demanded, it showed that Cartesian functional representations are hardly spontaneous since conventions on how to code mathematical information in them exist and are easily ignored; five of them were unable to exhibit clear understanding of what was going on: either the visual construction was not understood as related to the problem in question, the relationship between bidimensional staircases and spirals to one-dimensional processes and end products was unclear or convergence was not perceived as the clue to our problem. Lack of visual imagination resulted in lack of mental realization. From the remaining *nine* students understanding the need for an explicit notion of convergence, two of them failed to provide sound verbal and written explanations on how convergence should be formulated in terms of iterating zooming operations and number of iterates to be seen in the screen. Seven students were able to complete the experience, although only five of them produced precise economical verbal explanations, showing that implicit logical quantification was part of their reasoning armory.

Concerning students who started but failed to complete the interview, we drew personalized attention to their individual misunderstandings, logical flaws and errors explicitly and we provided, as said, extra tutorials when demanded. We sensed that they benefited from the open door to higher mathematics provided by the experience and from their first encounter with the tool. Not unexpectedly, it generated excitement and its manipulation became the unintended focus of their efforts, but efficient graphical capabilities without flexible power of interpretation lead them to failure, albeit at different stages of the interview. The rate of failure exhibited by students sailing through our last considerations on how to put words into the images they observed to formulate convergence verbally, showed that there are limits to what one hopes to achieve with High School students when confronted with delicate mathematical ideas without the benefit of a period of incubation. One may contend that some deficiency of our design interfered with its purpose, thus limiting its success, but similar former experiences dealing with limiting processes leads us to believe that this is not the case.

7. Coda

Once the experience was over, students were invited to deliver their questions and comments. Most frequently heard student's comments are below and were addressed by us individually and as a group:

'It seems to me that theorizing about actions which cannot be executed in reality is a complete waste of time'

'You and me have spent more than an hour talking and talking ... well, I have done most of the talking ... is this really what math is all about? ... in the classroom nobody except the teacher talks'

'The interview is exhausting. If I had to do the thinking I have been doing here for every topic in the math curriculum I will never get my degree'

'I hate word problems; I prefer to solve equations which is what math is about, isn't it?'

'Is a symbol something you can interpret at will?'

'So many pictures in what seems to be an algebraic problem! Is algebra part of geometry?'

'It is not easy to look at a picture and draw conclusions from it'

'It is amazing how many things can be done in math by executing a simple operation on and on ... a routine, right?'

'It looks as if many things we have learnt can be done automatically by a machine ... It makes one wonder why are we obliged to learn so many algebraic operations'

'Are techniques and tedious calculations in math really indispensable?'

'The tool is cool and really helpful ... no way I could have done most things by hand!'

'Why is the computer not used more often in the classroom? ... If I can see what I am supposed to do it makes it easier to understand it'

'Math teachers seem to enjoy complicating things, for instance getting close is intuitive enough. What is all the fuss about it?'

'Why do you insist in straight jacketed verbal expressions when the idea is clear enough?'

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