THE TEACHING OF MATHEMATICS
2013, Vol. XVI, 1, pp. 6–11

ON COMPUTING THE DERIVATIVE OF A FUNCTION

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Abstract. This note discusses a fact and its application on examining the existence of a derivative of a function at a point. The application provides a relatively easier method while avoiding laborious computations when standard computing is used. This may be introduced as a part of applications of the derivative in a Calculus course.

MathEduc Subject Classification: I45

AMS Subject Classification: 97I40

Key words and phrases: functions; derivatives; mean value theorem.

0. Introduction

With all the basic rules in hands, students may want to get things easier with their works on computing a derivative of a function with or without its definition. While some of the works are easily judged to be correct or otherwise, some others are not that easy to be put simply as true or false, and hence need a little discussion. Two problem-solution below exemplify our concerns, which in turn motivate us to resolve the issues by providing a better method.

PROBLEM 1. Find the derivative of the function f, where $f(x) := \cos \sqrt[3]{x}$.

Solution. Write $f(x) = \cos x^{1/3}$. Then $f'(x) = (-\sin x^{1/3})(\frac{1}{3}x^{-2/3})$, that is

$$f'(x) = \frac{-\sin x^{1/3}}{3x^{2/3}}. \quad \blacksquare$$

PROBLEM 2. Find the derivative of the function g, where $g(x) := x \cos \sqrt[3]{x}$.

Solution. Write $g(x) = x \cos x^{1/3}$. Then

$$g'(x) = (x \sin x^{1/3}) \left(\frac{1}{3}x^{-2/3}\right) + \cos x^{1/3} = \frac{1}{3}x^{1/3} \sin x^{1/3} + \cos x^{1/3}. \quad \blacksquare \tag{1}$$

Despite the fact that the answers are true, both works missed to check the case $x = 0$. Perhaps, the reason is because one simply claims that $f'(0)$ does not exist, since the formula $f'(x) = (-\sin x^{1/3})/3x^{2/3}$ does not work for $x = 0$, while $g'(0)$ exists, since the formula $g'(x) = \frac{1}{3}x^{1/3} \sin x^{1/3} + \cos x^{1/3}$ works for $x = 0$. As a matter of fact, both formula of f' and g' (as can be seen from (1) provide the derivative for all x but 0, meaning that the case $x = 0$ has yet to be examined. The part of the work that is left undone for the function f would be

$$\text{(2)} \qquad \lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0}\right| = \lim_{x\to 0}\frac{|\cos x^{1/3}-1|}{|x|} = \lim_{x\to 0}\frac{1-\cos x^{1/3}}{|x|}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x^{1/3}}{|x|(1+\cos x^{1/3})} = \lim_{x\to 0}\frac{\sin^2 x^{1/3}}{|x|(1+\cos x^{1/3})}$$

$$= \lim_{x\to 0}\left|\frac{\sin x^{1/3}}{x^{1/3}}\right|\left|\frac{\sin x^{1/3}}{x^{1/3}}\right|\left(\frac{1}{|x^{1/3}|(1+\cos x^{1/3})}\right)$$

$$= \infty$$

or alternatively, by showing that $\lim_{x\to 0^-}(f(x)-f(0))/(x-0) = -\infty$, or $\lim_{x\to 0^+}(f(x)-f(0))/(x-0) = \infty$ (as that of calculation (2), each of these seems to be not obvious), from which we conclude that $f'(0)$ does not exist. The missed part to be checked for the function g would be

$$\text{(3)} \qquad \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{x\cos x^{1/3}}{x} = \lim_{x\to 0}\cos x^{1/3} = 1$$

so that $g'(0)$ exists and equals 1. The idea of removing (2) as a part of the work for the function f is evidently misleading. For instance, in finding the derivative of the function $h(x) := \cos\sqrt[3]{x^2}$, if the same idea is used, then one would end up with the answer that

$$h'(x) = \frac{-2\sin x^{2/3}}{3x^{1/3}}$$

leaving $h'(0)$ unchecked, as is perceived from the formula that h' is undefined at $x = 0$. In fact, using a similar technique with that of (2), we have

$$\text{(4)} \qquad \lim_{x\to 0}\frac{h(x)-h(0)}{x-0} = \lim_{x\to 0}\frac{\cos x^{2/3}-1}{x}$$

$$= \left(\lim_{x\to 0}\frac{-\sin x^{2/3}}{x^{2/3}}\right)\left(\lim_{x\to 0}\frac{\sin x^{2/3}}{x^{2/3}}\right)\left(\lim_{x\to 0}\frac{x^{1/3}}{\cos x^{2/3}+1}\right)$$

$$= (-1)(1)\left(\frac{0}{2}\right) = 0$$

so that $h'(0)$ exists and equals 0. Likewise, removing (3) as a part of the work for the function g is subject to question: as the formula of $g'(x) = \frac{1}{3}x^{1/3}\sin x^{1/3} + \cos x^{1/3}$ is derived from (1) for all x but 0, how do we know that $g'(0) = \frac{1}{3}0^{1/3}\sin 0^{1/3} + \cos 0^{1/3}$? Although the assertion is true, such a work does not seem to be well-sounded.

We have a bad and a good news. If the basic rules of derivatives are all that we have at this stage, then there is no other way to examine each $f'(0)$, $g'(0)$ and $h'(0)$ except using its definition as shown above. This is a bad news, since computing a derivative using its definition can be very laborious and even difficult. The powerful Mean Value Theorem, however, turns out to be helpful in fulfilling our need to reduce computations, as by this we have an elegant and a relatively easier way of examining the derivative at a point, while removing parts of tedious works such as those of (2),(3), and (4). This is apparently a good news—the thing that we want to discuss here.

1. Computing the derivative

Recall that there is a differentiable function on an open interval containing a point a such that the limit of the derivative function at a does not exist (e.g. $f(x) := x^2 \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$), showing that the existence of the derivative of a function at a point does not imply the existence of the limit of the derivative function at that point. Yet the converse is true, as we have the following fact. First note that we write the left and right derivative of a function f as f'_- and f'_+ respectively.

FACT 3. *Let f be a continuous function defined on an open interval I containing a point a. Suppose that f is differentiable, except possibly at a.*

(a) If $\lim_{x\to a^+} f'(x)$ exists, then $f'_+(a)$ exists and equals $\lim_{x\to a^+} f'(x)$, and if $\lim_{x\to a^-} f'(x)$ exists, then $f'_-(a)$ exists and equals $\lim_{x\to a^-} f'(x)$.

(b) If either $\lim_{x\to a^+} |f'(x)| = \infty$ or $\lim_{x\to a^-} |f'(x)| = \infty$, then $f'(a)$ does not exist.

Proof. First, we shall show part (a) only for the case when $\lim_{x\to a^+} f'(x)$ exists, as the other case is similar. By Mean Value Theorem, there exists a function γ on some interval (a, b) such that for every $x \in (a, b)$, $\gamma(x)$ is between a and x, and

$$f'(\gamma(x)) = \frac{f(x) - f(a)}{x - a} \tag{5}$$

Since $\gamma(x) \neq a$ for every $x \in (a, b)$ (that is, γ is "eventually distinct" from a, by the definition introduced in [1]) and $\lim_{x\to a^+} \gamma(x) = a$, it follows that $\lim_{x\to a^+} f'(\gamma(x))$ exists and equals $\lim_{x\to a^+} f'(x)$. Therefore, (5) implies

$$\lim_{x\to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a^+} f'(\gamma(x)) = \lim_{x\to a^+} f'(x)$$

and so $f'_+(a)$ exists and $f'(a) = \lim_{x\to a^+} f'(x)$.

For part (b), it suffices to consider the case $\lim_{x\to a^+} |f'(x)| = \infty$ only, as we have the same argument for the other case. Let $b > a$ such that $|f'(x)| > 0$ for all $x \in (a, b)$. Now let $g(x) := 1/|f'(x)|$ for all $x \in (a, b)$, so that $\lim_{x\to a^+} g(x) = 0$. Let γ be as in part (a) above. By the same argument, it follows that $\lim_{x\to a^+} g((\gamma(x))$ exists and equals $\lim_{x\to a^+} g(x)$, that is $\lim_{x\to a^+} g((\gamma(x)) = 0$. Thus, noting (5), we have

$$\lim_{x\to a^+} \left|\frac{x - a}{f(x) - f(a)}\right| = \lim_{x\to a^+} \frac{1}{\left|\frac{f(x)-f(a)}{x-a}\right|} = \lim_{x\to a^+} \frac{1}{|f'(\gamma(x))|} = \lim_{x\to a^+} g(\gamma(x)) = 0$$

and therefore

$$\lim_{x\to a^+} \left|\frac{f(x) - f(a)}{x - a}\right| = \lim_{x\to a^+} \frac{1}{\left|\frac{x-a}{f(x)-f(a)}\right|} = \infty$$

so that $\lim_{x\to a}(f(x) - f(a))/(x - a)$, hence $f'(a)$, does not exist. ∎

Notice that the condition in Fact 3(a), that $\lim_{x\to a^+} f'(x)$ or $\lim_{x\to a^-} f'(x)$ exists, is weaker than the condition that $\lim_{x\to a} f'(x)$ exists. Also, the condition in

Fact 3(b), that $\lim_{x\to a^+}|f'(x)|=\infty$ or $\lim_{x\to a^-}|f'(x)|=\infty$, is weaker than each condition $\lim_{x\to a}|f'(x)|=\pm\infty$, $\lim_{x\to a}f'(x)=\pm\infty$, $\lim_{x\to a^+}f'(x)=\pm\infty$, and $\lim_{x\to a^-}f'(x)=\pm\infty$.

As applications of Fact 3, consider the function f, g and h in Section 0. Having the formula $f'(x)=(-\sin x^{1/3})/3x^{2/3}$ for all $x\neq 0$, by Fact 3(b), $f'(0)$ does not exist, since

$$\lim_{x\to 0}|f'(x)|=\lim_{x\to 0}\frac{1}{3|x^{1/3}|}\left|\frac{\sin x^{1/3}}{x^{1/3}}\right|=\infty.$$

Having the formula $g'(x)=\frac{1}{3}x^{1/3}\sin x^{1/3}+\cos x^{1/3}$ for all $x\neq 0$, by Fact 3(a), $g'(0)=1$, as it is easy to see that $\lim_{x\to 0}g'(x)=1$. Since $1=\frac{1}{3}0^{1/3}\sin 0^{1/3}+\cos 0^{1/3}$, we can simply write the formula for g' as

$$g'(x)=\frac{1}{3}x^{1/3}\sin x^{1/3}+\cos x^{1/3}$$

by letting x without an exception. Having the formula $h'(x)=\dfrac{-2\sin x^{2/3}}{3x^{1/3}}$ for all $x\neq 0$, by Fact 3(a),

$$h'(0)=\lim_{x\to 0}h'(x)=\lim_{x\to 0}\frac{-2\sin x^{2/3}}{3x^{1/3}}$$

provided the limit exists. In fact, we can immediately see that the limit is

$$\lim_{x\to 0}(-2/3)(x^{1/3})\frac{\sin x^{2/3}}{x^{2/3}}=(-2/3)(0)(1)=0$$

so that $h'(0)=0$. Thus, the practical use of Fact 3 removes tedious computations (2), (3) and (4).

It is particularly interesting to state the following corollary. Contrasted with what it looks, the proof of this property is not obvious without the aid of Fact 3.

COROLLARY 4. *Let f and g be functions defined on an open interval I containing a point c. If f is differentiable on I except possibly at c, g is continuous on I, and $f'(x)=g(x)$, for all $x\in I\setminus\{c\}$, then f is differentiable on I, and $f'(x)=g(x)$, for all $x\in I$.*

Proof. It suffices to show that $f'(c)$ exists and $f'(c)=g(c)$. Since g is continuous at c,

$$\lim_{x\to c}f'(x)=\lim_{x\to c}g(x)=g(c).$$

It follows from Fact 3 that $f'(c)$ exists and equals $g(c)$, as is desired. ■

As a nice application of Corollary 4, consider the function g in the solution of Problem 2. Since $g'(x)=\frac{1}{3}x^{1/3}\sin x^{1/3}+\cos x^{1/3}$, for all $x\neq 0$, it follows immediately from the corollary that $g'(x)=\frac{1}{3}x^{1/3}\sin x^{1/3}+\cos x^{1/3}$ for all x. Thus, while removing tedious computation (3), the corollary gives a very concise solution. Notice that, applying the corollary means we immediately conclude that the formula of the derivative works at the indicated point, without evaluating the limit of the function at the point as it was done earlier when Fact 3 was used.

Below are other illustrations showing how effective and compelling Fact 3 and Corollary 4 in reducing computations. Let each continuous function f be defined as follows:

(a) $f(x) := \begin{cases} 3x-2, & -2 \le x < 1 \\ x^3, & \text{otherwise} \end{cases}$

(b) $f(x) := (x-1)^{2/3} \sin(x^2-1)^{1/3}$

(c) $f(x) := (x-1)^{1/3} \cos x^{2/3}$

(d) $f(x) := (\sin x^3)^{1/3}$

(e) $f(x) := \cos(x^{2/3}+x)^2$

For f in (a), $f'(x) = \begin{cases} 3x^2, & x < -2 \\ 3, & -2 < x < 1 \\ 3x^2, & x > 1. \end{cases}$ We still need to examine f' at $x=-2$ and $x=1$. Since the left and right limits of f' at -2 are 12 and 3 respectively, it follows from Fact 3(a) that $f'_-(-2) = 12$ and $f'_+(-2) = 3$. Therefore $f'(-2)$ does not exist. Now since the left and right limits of f' at 1 are both equal to 3, by Fact 3(a), $f'_-(1)$ and $f_+(1)$ both exist and equal 3, so that $f'(1)$ exists and equals 3. Thus, we can write

$$f'(x) = \begin{cases} 3x^2, & x < -2 \\ 3, & -2 < x \le 1 \\ 3x^2, & x > 1. \end{cases}$$

For f in (b),

$$f'(x) = \frac{2x}{3(x+1)^{2/3}} \left(\cos(x^2-1)^{1/3} + (x+1)\frac{\sin(x^2-1)^{1/3}}{(x^2-1)^{1/3}} \right)$$

for all $x \ne \pm 1$. Therefore

$$\lim_{x\to 1} f'(x) = \frac{2}{3(2^{2/3})} (1 + (2)(1)) = \sqrt[3]{2}$$

so that by Fact 3(a), $f'(1) = \sqrt[3]{2}$. Also, we can easily evaluate that $\lim_{x\to -1} f'(x) = -\infty$ and hence by Fact 3(b), $f'(-1)$ does not exist.

For f in (c),

$$f'(x) = \left(\frac{2x^{1/3}(x-1)^{1/3}}{3} \right) \left(\frac{\sin x^{2/3}}{x^{2/3}} \right) + \frac{\cos x^{2/3}}{3(x-1)^{2/3}}$$

for all $x \ne 0$ and $x \ne 1$. Then, in view of Fact 3(a), since

$$\lim_{x\to 0} f'(x) = \frac{0}{3}(1) + \frac{1}{3(1)} = \frac{1}{3}$$

it follows that $f'(0) = 1/3$. It is easy to check that $\lim_{x\to 1} f'(x) = \infty$, so that by Fact 3(b), $f'(1)$ does not exist.

For f in (d),

$$f'(x) = \left(\frac{x^3}{\sin x^3} \right)^{\frac{2}{3}} \cos x^3$$

for all $x \neq \sqrt[3]{n\pi}$ $(n = 0, \pm 1, \pm 2, \dots)$. In view of Fact 3, since

$$\lim_{x\to 0} f'(x) = (1^{\frac{2}{3}})(1) = 1$$

and $\lim_{x\to\sqrt[3]{n\pi}} |f'(x)| = \infty$, it follows that $f'(0) = 1$, and $f'(\sqrt[3]{n\pi})$ (for each $n = \pm 1, \pm 2, \dots$) does not exist.

For f in (e), $f'(x) = -\sin(x^{2/3} + x)^2 \cdot 2(x^{2/3} + x)((2/3)x^{-1/3} + 1)$, or simply

$$f'(x) = -2\left(1 + x^{1/3}\right)\left(\frac{2}{3}x^{1/3} + x^{2/3}\right)\sin(x^{2/3} + x)^2$$

for all $x \neq 0$, hence for all x, by Corollary 4.

Notice that, in contrast with the calculations above, examining each f' at each of those points by bringing into play its definition is indeed quite laborious.

2. Conclusion

In the realm of Calculus, people avoid a certain part of the work they think as unimportant and tedious, hence not practical to be explicitly shown, or even shortly discussed, as a part of it. In fact, things might not be just simple because of this reason. As for our case from each solution of Problem 1 and Problem 2, we have an issue of leaving the case $x = 0$ unchecked. The serious thing with removing this part from the solution of Problem 1 is that, not only does it undermine the concept thoroughly introduced in the early stage (where the concept of the derivative is firstly discussed), but worse, as was argued, the idea behind the work is misleading. The issue with the solution of Problem 2, by leaving that part undone, is even more profound; it is not well-sounded, though sometimes mistakenly to be true. For each of those issues, our argument suggests that there is no other way but writing up that part of the work through some method of clarifications.

The standard way to clarify such a part of the work is by examining the derivative using its definition, which in most cases can be impractical and difficult. It turns out we have Fact 3 that, along with Corollary 4, facilitates the part of the computations by aiding a relatively easier method. With Fact 3 we have a sound basis that provide a practical way to examine the derivative of a function at a point where technical and laborious computations that might appears, when standard computing is used, can be avoided. Fact 3, therefore, as was demonstrated through some examples, also allows us to explore illustrations that are more varying and sophisticated than those of its absence. As a part of applications of the derivative, it may conveniently be discussed in a section where the Mean Value Theorem is introduced.

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