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A GEOMETRIC MAXIMIZATION PROBLEM

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Abstract. We consider an area maximization problem to illustrate the importance of analytical work before solving an equation numerically.

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1. Introduction

The problem we consider is the following (see Figure 1): given a circle of radius r with center C and an external point P at a distance $d = |PD|$ from the circle, find the angle θ that maximizes the area inscribed in the circle by rotating the rays $\overline{PA}$ and $\overline{PB}$ through 2θ. Such a problem occurs when approximating polynomial zero exclusion regions in the complex plane of the form $|z^k - a| < \rho$ (with k a positive integer, a a complex number, and ρ a positive real number such that $\rho < |a|$) by optimally inscribed annular sectors which are much simpler.

At first sight, this appears to be just another calculus example. However, its solution, which involves geometry, trigonometry, and calculus, as well as numerical methods, has something in it for everyone, and is a good illustration of the analytical work necessary to construct an efficient numerical method. The latter is the main objective of this note.

Although we will limit ourselves to this particular problem, an unlimited supply of similar problems is obtained by varying the geometry. One could use a half-circle facing towards or away from the point P, or one could replace the circle with an ellipse, diamond, or any other regular (or even irregular) geometric shape. One can also change the way the area is inscribed by using straight lines instead of arcs and so on. Moreover, the whole construction can be revolved around the horizontal axis passing through P, turning it into a maximum volume problem. One can also change the variables, e.g., fix θ and maximize the area as a function of d. Some of these problems have analytical solutions, but many do not, and the equations defining the optimal angles and their properties vary considerably. They are quite well suited for small class projects.

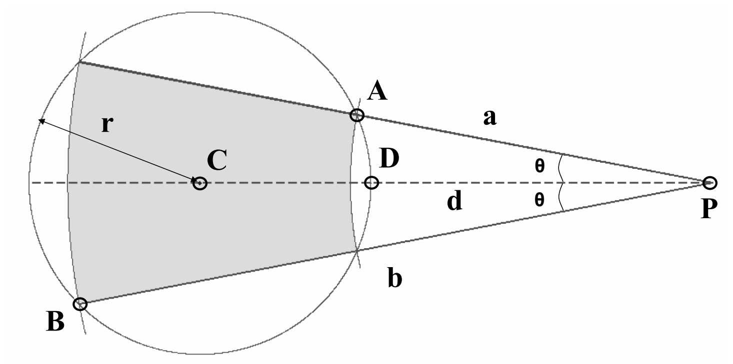

Fig. 1. Area inscribed in a circle by two rays

2. Area equation

We begin by computing the area of the shaded area in Figure 1 as a function of the angle θ, for which we need the quantities $a = |PA|$ and $b = |PB|$. Using the cosine rule for the two triangles $\triangle CPA$ and $\triangle CPB$ with $|PC| = |PD| + |DC| = d + r$, $|PA| = a$, and $|PB| = b$, we obtain the two equations

$$a^2 + (d+r)^2 - 2a(d+r)\cos\theta = r^2 \quad \text{and} \quad b^2 + (d+r)^2 - 2b(d+r)\cos\theta = r^2,$$

i.e., a and b are the solutions of the quadratic equation in x

$$x^2 - 2(d+r)\cos\theta\, x + (d+r)^2 - r^2 = 0.$$

A little bit of algebra leads to

$$a = (d+r)\cos\theta - \sqrt{r^2 - (d+r)^2\sin^2\theta} \quad \text{and} \quad b = (d+r)\cos\theta + \sqrt{r^2 - (d+r)^2\sin^2\theta}.$$

Note that $(d+r)\sin\theta$ is the orthogonal distance from C to the ray $\overline{PB}$, which is obviously less than the radius r, so that the expression under the square root is positive.

The shaded area $S(\theta)$ inscribed in the circle in Figure 1 is then given by $\theta(b^2 - a^2)$, which can be written as

$$S(\theta) = \theta(b-a)(b+a) = 4(d+r)\theta\cos\theta\sqrt{r^2 - (d+r)^2\sin^2\theta}.$$

Defining $\tau = d/r$, the area can be expressed as

$$S(\theta) = 4r^2(1+\tau)\theta\cos\theta\sqrt{1 - (1+\tau)^2\sin^2\theta}. \tag{1}$$

The angle θ lies in $[0, \theta_{\max}]$, where $\theta_{\max} = \arcsin(r/(r+d)) = \arcsin(1/(1+\tau))$, namely the angle for which $\overline{PA}$ and $\overline{PB}$ are tangent to the circle. Clearly, $S(0) = S(\theta_{\max}) = 0$. To avoid notation overload and because its meaning will be clear from the context, we will write $\theta_{\max}$ instead of $\theta_{\max}(\tau)$, and treat most other quantities that also depend on τ similarly. We are now ready to find the angle θ that maximizes $S(\theta)$.

3. Area maximization

The solution for $d = 0$ will be obtained later on as a limit, so from here on we assume that $d > 0$ (or $\tau > 0$), which means that

$$\theta_{\max} = \arcsin\left(\frac{1}{1+\tau}\right) < \frac{\pi}{2}.$$

Since $S(0) = S(\theta_{\max}) = 0$, $S(\theta)$ must reach its maximum for an angle in $(0, \theta_{\max})$ where its derivative vanishes. For $\theta \in [0, \theta_{\max})$, $S'(\theta)$ is given by:

$$\frac{S'(\theta)}{4r^2(1+\tau)} = \cos\theta\sqrt{1-(1+\tau)^2\sin^2\theta} \tag{2}$$

$$- \theta\sin\theta\left(\sqrt{1-(1+\tau)^2\sin^2\theta} + \frac{(1+\tau)^2\cos^2\theta}{\sqrt{1-(1+\tau)^2\sin^2\theta}}\right)$$

$$= \frac{\cos\theta\left(1-(1+\tau)^2\sin^2\theta\right) - \theta\sin\theta\left(1-(1+\tau)^2\sin^2\theta + (1+\tau)^2\cos^2\theta\right)}{\sqrt{1-(1+\tau)^2\sin^2\theta}}. \tag{3}$$

It is easily verified that $S'(0) > 0$ and $\lim_{\theta\to\theta_{\max}^-} = -\infty$, indicating that S' vanishes at least once on $(0, \theta_{\max})$, but further analysis of S' is complicated because of its cumbersome form (both (2) and the numerator in (3)), which also makes it difficult to construct a numerical method to solve $S'(\theta) = 0$ (there is no analytical solution) that should preferably be simple, fast, and guaranteed to converge from an appropriate starting point. Using $\sec^2\theta = 1 + \tan^2\theta$, we therefore rewrite (3) as follows:

$$\frac{S'(\theta)}{4r^2(1+\tau)}$$

$$= \frac{\cos\theta\left(1-(1+\tau)^2 + (1+\tau)^2\cos^2\theta\right) - \theta\sin\theta\left(1-(1+\tau)^2 + 2(1+\tau)^2\cos^2\theta\right)}{\sqrt{1-(1+\tau)^2\sin^2\theta}}$$

$$= \frac{\left(1-\tau(\tau+2)\tan^2\theta\right) - \theta\tan\theta\left(1+(1+\tau)^2 - \tau(\tau+2)\tan^2\theta\right)}{\sec^3\theta\sqrt{1-(1+\tau)^2\sin^2\theta}}.$$

This means that, on the interval $(0, \theta_{\max})$, $S'(\theta) = 0$ *if and only if*

$$\psi_\tau(\theta) \equiv \theta\tan\theta - \frac{1-\tau(\tau+2)\tan^2\theta}{1+(1+\tau)^2 - \tau(\tau+2)\tan^2\theta} = 0, \tag{4}$$

where

$$\tan^2\theta_{\max} = \frac{\sin^2\theta_{\max}}{\cos^2\theta_{\max}} = \frac{\sin^2\theta_{\max}}{1-\sin^2\theta_{\max}} = \frac{1/(1+\tau)^2}{1-1/(1+\tau)^2} = \frac{1}{\tau(\tau+2)}, \tag{5}$$

so that $1-\tau(\tau+2)\tan^2\theta \geq 0$ for $\theta \leq \theta_{\max} < \pi/2$, which shows that ψ_τ is well-defined on the closed interval. To see if this is any better than what we had before, we examine the derivatives of ψ_τ. To do this efficiently we first define

$$g_\tau(x) \equiv -\frac{1-\tau(\tau+2)x}{1+(1+\tau)^2-\tau(\tau+2)x} = -1 + \frac{(1+\tau)^2}{1+(1+\tau)^2-\tau(\tau+2)x},$$

which is an increasing convex function of x for $\tau(\tau+2)x < 1+(1+\tau)^2$. From (4) we then have that $\psi_\tau(\theta) = \theta\tan\theta + g_\tau(\tan^2\theta)$, and it becomes a straightforward exercise to show that ψ_τ is an increasing convex function on $[0,\theta_{\max}]$. Moreover, one verifies with the help of (5) that $\psi_\tau(0) = -1/(1+(1+\tau^2)) < 0$ and $\psi_\tau(\theta_{\max}) = \theta_{\max}/\sqrt{\tau(\tau+2)} > 0$, implying that ψ_τ has a unique root $\theta^\star$ on the interval $(0,\theta_{\max})$, at which the inscribed area S achieves its unique maximum.

The root $\theta^\star$ decreases monotonically as τ increases since $\partial\psi_\tau/\partial\tau > 0$, so that $\psi_{\tau_1} > \psi_{\tau_2}$ for $\tau_1 > \tau_2$. Figure 2 shows ψ_τ for $\tau = 3/2, 2, 3$ from right to left, respectively.

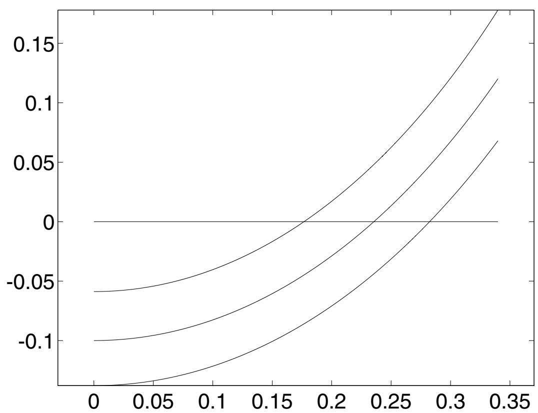

Fig. 2. The function ψ_τ for $\tau = 3/2, 2, 3$ from right to left, respectively

In the limit case when $\tau \to 0^+$ (or $d \to 0^+$), we have $\theta_{\max} = \pi/2$. On the other hand, $\psi_\tau(\pi/4) = \pi/4 + (1+\tau)^2/2 - 1 > 0$ for any $\tau \geq 0$, implying that $0 < \theta^\star < \pi/4$. This means that, in this case, $\psi_\tau(\theta) = 0$ has a unique solution on $(0,\pi/4)$, an interval on which $\tan\theta$ is finite. Consequently, as $\tau \to 0^+$, the equation for $\theta^\star$ becomes $\theta\tan\theta = 1/2$, whose (numerical) solution is $\theta^\star \approx 0.6533$ (37.43°).

The function ψ_τ turned out to be useful in the analysis of S, but that is not its only advantage: its convexity also suggests a simple and fast numerical method to compute its root, namely, *Newton's method*, defined by the iteration

$$z_{k+1} = z_k - \frac{\psi_\tau(z_k)}{\psi'_\tau(z_k)}.$$

Here this method converges monotonically from any starting point in $[0,\theta_{\max}]$ that is larger than the root, such as $\theta_{\max}$. However, a much better starting point can be determined from bounds on the optimal angle, which is our next objective.

4. Bounds on the optimal angle

Yet another advantage of ψ_τ is that it provides a convenient way to determine upper and lower bounds on $\theta^\star$. We will use the standard results that $\theta < \tan\theta$ for $\theta \in (0, \pi/2)$ and $x < \arcsin x$ for $x \in (0, 1]$. Since θ, $\tan\theta$, and $\psi_\tau(\theta)$ are all increasing convex functions of θ on $[0, \theta_{\max}]$, replacing θ by $\tan\theta$ in the first term of ψ_τ turns it into the equally increasing and convex function $f_\tau(\tan^2\theta)$, where

$$f_\tau(x) \equiv x - \frac{1-\tau(\tau+2)x}{1+(1+\tau)^2-\tau(\tau+2)x} = x - 1 + \frac{(1+\tau)^2}{1+(1+\tau)^2-\tau(\tau+2)x},$$

with $f_\tau(\tan^2\theta) > \psi_\tau(\theta)$ for any $\theta \in (0, \theta_{\max})$, implying that it has a unique root θ_1 on $(0, \theta^\star)$. Note that solving $f_\tau(x) = 0$ amounts to solving a simple quadratic equation. Likewise, replacing $\tan\theta$ by θ in both terms of ψ_τ turns it once again into an increasing convex function $f_\tau(\theta^2)$ with $f_\tau(\theta^2) < \psi_\tau(\theta)$ for any $\theta \in (\theta^\star, \theta_{\max})$, implying that it has a root $\theta_2 > \theta^\star$. Figure 3 shows the functions $f_\tau(\tan^2\theta)$, $\psi_\tau(\theta)$, and $f_\tau(\theta^2)$ for $\tau = 1$.

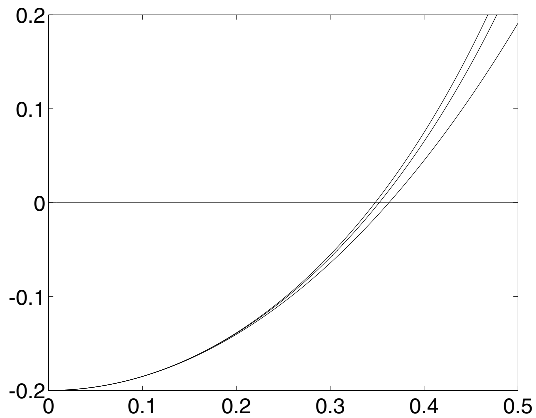

Fig. 3. The functions $f_\tau(\tan^2\theta)$, $\psi_\tau(\theta)$, and $f_\tau(\theta^2)$ for $\tau = 1$

It is not immediately clear that $\theta_2 < \theta_{\max}$, so let us have a closer look. For ease of writing, we set $\gamma = (1+\tau)^2$. The solutions of $f_\tau(x) = 0$ are then given by the solutions of $(\gamma-1)x^2 - 2\gamma x + 1 = 0$, which are the positive numbers

$$x_1 = \frac{\gamma - \sqrt{\gamma^2-\gamma+1}}{\gamma-1} \quad\text{and}\quad x_2 = \frac{\gamma + \sqrt{\gamma^2-\gamma+1}}{\gamma-1}.$$

Clearly, $x_1 < x_2$, which means that $\tan^2\theta_1 = \theta_2^2 = x_1$, or $\theta_1 = \arctan\sqrt{x_1}$ and $\theta_2 = \sqrt{x_1}$. Furthermore, we have that

$$\begin{aligned}\theta_2 &= \left(\frac{\gamma - \sqrt{\gamma^2-\gamma+1}}{\gamma-1}\right)^{1/2} = \left(\frac{1}{\gamma + \sqrt{\gamma^2-\gamma+1}}\right)^{1/2} \\ &< \frac{1}{\sqrt{\gamma}} < \arcsin\frac{1}{\sqrt{\gamma}} = \theta_{\max}.\end{aligned}$$

As a result, we conclude that

$$0 < \arctan\left(\frac{1}{\gamma+\sqrt{\gamma^2-\gamma+1}}\right)^{1/2} < \theta^\star < \left(\frac{1}{\gamma+\sqrt{\gamma^2-\gamma+1}}\right)^{1/2} < \theta_{\max},$$

$\gamma = (1+\tau)^2$. To have an idea of the quality of these bounds, Figure 4 shows the ratio $(\theta_2-\theta_1)/\theta_1$, (which is larger than $(\theta_2-\theta_1)/\theta^\star$) for $\tau \in [0,5]$. That ratio is never more than 0.15 and decreases rapidly as τ increases.

If we compute $\theta^\star$ with Newton's method starting from θ_2, then at most five iterations are necessary to compute it to 16 correct significant digits, regardless of the value of τ.

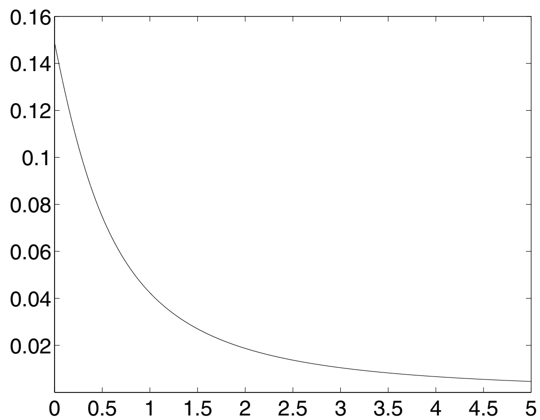

Fig. 4. The ratio $(\theta_2-\theta_1)/\theta_1$

5. Concluding remarks

- The expressions for θ_1 and θ_2 are written in terms of γ (and therefore in terms of τ), but they can also be expressed in terms of the angle $\theta_{\max}$. Since $\sin\theta_{\max} = 1/\sqrt{\gamma}$, we have

$$\begin{aligned}
\gamma+\sqrt{\gamma^2-\gamma+1} &= \csc^2\theta_{\max} + \sqrt{\csc^4\theta_{\max} - \csc^2\theta_{\max} + 1} \\
&= \csc^2\theta_{\max}\left(1+\sqrt{1-\sin^2\theta_{\max}+\sin^4\theta_{\max}}\right) \\
&= \csc^2\theta_{\max}\left(1+\sqrt{1-\sin^2\theta_{\max}+\sin^2\theta_{\max}\left(1-\cos^2\theta_{\max}\right)}\right) \\
&= \csc^2\theta_{\max}\left(1+\sqrt{1-\sin^2\theta_{\max}\cos^2\theta_{\max}}\right) \\
&= \frac{1+\sqrt{1-\sin^2\theta_{\max}\cos^2\theta_{\max}}}{\sin^2\theta_{\max}} = \frac{1+\sqrt{1-\frac{1}{4}\sin^2 2\theta_{\max}}}{\sin^2\theta_{\max}}.
\end{aligned}$$

From the expressions for the bounds, we obtain

$$\theta_2 = \frac{\sin\theta_{\max}}{\left(1+\sqrt{1-\frac{1}{4}\sin^2 2\theta_{\max}}\right)^{1/2}} \quad \text{and} \quad \theta_1 = \arctan\theta_2.$$

Since $0 \le |\sin 2\theta_{\max}| \le 1$, we obtain after some algebra that

$$\theta^\star < \theta_2 \le \frac{\sqrt{2}\sin\theta_{\max}}{\left(2+\sqrt{3}\right)^{1/2}} = \left(\sqrt{3}-1\right)\sin\theta_{\max} = \frac{\sqrt{3}-1}{1+\tau} < \left(\sqrt{3}-1\right)\theta_{\max},$$

with $\sqrt{3}-1 \approx 0.7321$. This inequality can also be obtained directly from the original expression for θ_2. Using its upper bound $(\sqrt{3}-1)(1+\tau)^{-1}$ – instead of θ_2 itself – as a starting point for Newton's method still reaches the same accuracy as before in at most five iterations. One similarly obtains a lower bound on θ_1, resulting in

$$0 < \arctan\left(\frac{\sqrt{2}/2}{1+\tau}\right) < \theta^\star < \frac{\sqrt{3}-1}{1+\tau} < \theta_{\max}.$$

Although these bounds on $\theta^\star$ are cruder than θ_1 and θ_2, their advantage is that they are very simple.

- As $\tau \to +\infty$ then both $\theta^\star$ and $\theta_{\max}$ go to zero, but it might be interesting to see what happens to their ratio. This does not turn out to be difficult and we proceed with a moderate amount of handwaving. As $\tau \to +\infty$, then $\theta_{\max} \approx 1/\sqrt{\gamma}$, and $\tan\theta \approx \theta$ on $[0, \theta_{\max}]$, so that $\theta^\star \approx \theta_1 \approx \theta_2$. This implies that

$$\lim_{\tau\to+\infty} \frac{\theta^\star}{\theta_{\max}} = \lim_{\gamma\to+\infty} \left(\frac{1}{\gamma+\sqrt{\gamma^2-\gamma+1}}\right)^{1/2} \sqrt{\gamma} = \frac{\sqrt{2}}{2}.$$

In fact, a slightly more careful analysis using Taylor series shows that

$$\lim_{\tau\to+\infty} \frac{\theta^\star}{\theta_{\max}} = \frac{\sqrt{2}}{2}\left(1+\frac{1}{8(1+\tau)^2}+\mathcal{O}\left(\frac{1}{\tau^4}\right)\right).$$

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