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ON MEANS, POLYNOMIALS AND SPECIAL FUNCTIONS

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Abstract. We discuss how derivatives can be considered as a game of cubes and beams, and of geometric means. The same principles underlie wide classes of polynomials. This results in an unconventional view on the history of the differentiation and differentials.

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1. On derivation and derivatives

In *On Proof and Progress in Mathematics* William Thurston describes how people develop an "understanding" of mathematics [31]. He uses the example of derivatives, that can be approached from very different viewpoints, allowing individuals to develop their own understanding of derivatives.

"The derivative can be thought of as:

(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.

(2) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f \circ g$ is $f' \circ g \times g'$, etc ...

(3) Logical: $f'(x) = d$ if and only if for every ϵ there is a δ, such that $0 < |\Delta x| < \delta$ when

$$\left| \frac{f(x + \Delta x) - f(x)}{\Delta x} - d \right| < \delta$$ [1]

(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.

(5) Rate: the instantaneous speed of $f(t) = d$, when t is time.

(6) Approximation: The derivative of a function is the best linear approximation to the function near a point.

[1] This is taken literally from [31].

(7) Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.

... ...

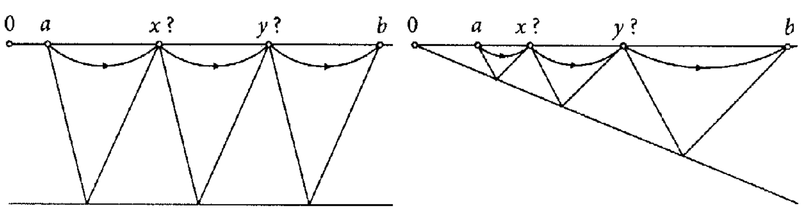
(37) The derivative of a real-valued function f in a domain D is the Lagrangian section of the cotangent bundle $T^*(D)$ that gives the connection form for the unique flat connection on the trivial $\mathbb{R}$-bundle $D \times \mathbb{R}$ for which the graph of f is parallel."

Thurston provides this partial list but states that this list can be still extended. With respect to 'individual' understanding, *"one person's clear mental image is another person's intimidation. Human understanding does not follow a single path, as a computer with a central processing unit; our brains are much more complex and capable of far more than a single path"* [31]. In addition, we should not forget that it has taken mathematicians thousands of years to come to a good understanding of the concept.

In this article we will use (2) and show how this reduces to a game of cubes and unit elements. The same procedure underlies important special polynomials in mathematics, as recent research shows. Essentially the game component of cubes and beams, very clear to mathematicians from the 16^{th} and 17^{th} century, and the unified approach for polynomials are the same. Many of the other different definitions of derivatives simply follow from these observations.

2. The geometry of means

Throughout the article we will only use very simple arguments, starting from geometric means, beams and cubes and unit (or neutral) element. To develop the argument we use a method proposed by the second author, a geometrical representation of the n^{th}-arithmetic, n^{th}-geometric and n^{th}-harmonic means [32–34].



$$x = \frac{2a+b}{3} = AM_{\frac{1}{3}} \qquad x = \sqrt[3]{a^2 b} = GM_{\frac{1}{3}}$$

$$y = \frac{a+2b}{3} = AM_{\frac{2}{3}} \qquad y = \sqrt[3]{ab^2} = GM_{\frac{2}{3}}$$

Fig. 1. Division of an interval $[a, b]$ into three parts according to AM or GM

For $n = 2$ the geometric mean GM and arithmetic mean AM are geometry in the sense that they are solutions to one of the oldest optimization problems: For a given rectangle with sides a and b, AM between a and b is the side of a square with the same *perimeter* as the given rectangle and GM between a and b is the side of a square with the same *area* as the given rectangle with sides a and b. Coefficient $n = 2$ refers to the use of a square root in GM. In general on any straight line, an interval $[a, b]$ can be divided in 2 or more parts following the construction in Figure 1, where the number of intervals is $n = 3$ with the relations given by the equations in Figure 1.

The intervals in the left graph show a composition of translations with itself. In the case of GM the subsequent divisions define homothetic transformations. An interval can be divided in n intervals with $n =$ any natural number. The second author studied the inverse question: *"Can x and y, z ... be determined graphically using only parallel lines?"* This is straightforward in the case of AM but impossible in the case of GM. However, a construction using only parallel lines yields the harmonic mean HM which is the ratio between AM and GM (Figure 2).

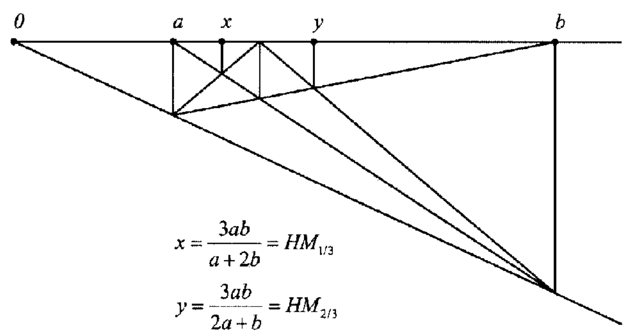


Fig. 2. Graphical construction of the harmonic means HM for $n = 3$

This leads to the following relations for the division of an interval in three parts (Figure 2)

$$AM_{\frac{1}{3}} HM_{\frac{2}{3}} = ab, \quad AM_{\frac{2}{3}} HM_{\frac{1}{3}} = ab, \quad GM_{\frac{1}{3}} GM_{\frac{2}{3}} = ab. \tag{1}$$

In general:

$$\begin{gathered} AM_{\frac{i}{n}} = \frac{(n-i)a + ib}{n}, \quad GM_{\frac{i}{n}} = \sqrt[n]{a^{n-i}b^{i}}, \quad HM_{\frac{i}{n}} = \frac{nab}{ia + (n-i)b} \\ AM_{\frac{n-i}{n}} = \frac{ia + (n-i)b}{n}, \quad GM_{\frac{n-i}{n}} = \sqrt[n]{a^{i}b^{n-i}}, \quad HM_{\frac{n-i}{n}} = \frac{nab}{(n-i)a + ib} \\ AM_{\frac{i}{n}} HM_{\frac{n-i}{n}} = ab, \quad AM_{\frac{n-i}{n}} HM_{\frac{i}{n}} = ab, \quad GM_{\frac{i}{n}} GM_{\frac{n-i}{n}} = ab \end{gathered} \tag{2}$$

These relationships can be proven using the similarity of triangles, or from a projective point of view, using double ratios as follows. The double ratio $[a, b, p, x]$ is harmonic, so $[a, b, p, x] = -1$. If we take p and q at infinity, then $x = \frac{a+b}{2} = AM_{\frac{1}{2}}$

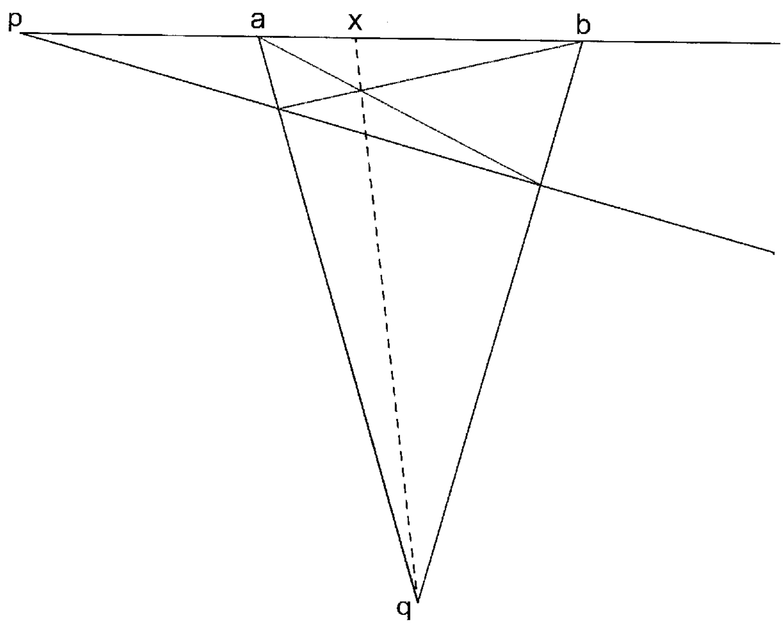


Fig. 3. Projective method for $n = 2$

and $\left[a, b, \frac{a+b}{2}, \infty\right] = -1$. If we take $p = 0$ and q at infinity, then $x = \frac{2ab}{a+b} = HM_{\frac{1}{2}}$ and $\left[a, b, \frac{2ab}{a+b}, 0\right] = -1$ (Figure 3).

In general the double ratio $[a, b, x, y]$ is calculated as the quotient of the share ratio's $[a, b, x]$ and $[a, b, y]$, so $[a, b, x, y] = \frac{[a,b,x]}{[a,b,y]}$. The share ratio $[a, b, x]$ is calculated by $\frac{x-a}{x-b}$.

If the abscissa of x towards (a, b) is $\frac{i}{n}$ then $x = AM_{\frac{i}{n}}$ for $\frac{x-a}{b-a} = \frac{i}{n}$ so $x = \frac{(n-i)a+ib}{n}$ and $[a, b, x, \infty] = [a, b, x] = \frac{x-a}{x-b} = \frac{i}{i-n} = k$. If we take now $[a, b, y, 0] = \frac{i-n}{i} = \frac{1}{k}$ then $[a, b, x, \infty]\,[a, b, y, 0] = 1$ and $y = HM_{\frac{n-i}{n}}$, for $[a, b, y, 0] = \frac{i-n}{i} = \frac{[a,b,y]}{[a,b,0]}$. Since $\frac{y-a}{y-b} = \frac{a}{b}\frac{i-n}{i}$, so $y = \frac{nab}{(n-i)a+ib} = HM_{\frac{n-i}{n}}$ and $AM_{\frac{i}{n}} HM_{\frac{n-i}{n}} = ab$.

This method provides for a recursive method for the calculation of roots [32–34]. The n-th root $\sqrt[n]{c}$ of a positive number can be interpreted as $GM_{\frac{1}{n}} = \sqrt[n]{1^{n-1}c^1}$ of the interval $[1, c]$. Let x be an approximation of $\sqrt[n]{c}$, smaller than $\sqrt[n]{c}$, then the interval $\left[x^i, \frac{c}{x^{n-i}}\right]$ includes $\sqrt[n]{c^i}$. For this interval

$$AM_{\frac{i}{n}} = \frac{(n-i)x^n + ic}{nx^{n-i}} \text{ and } HM_{\frac{n-i}{n}} = \frac{ncx^{n-i}}{(n-i)x^n + ic}, \quad AM_{\frac{i}{n}} HM_{\frac{n-i}{n}} = c.$$

So one could iterate on $AM_{\frac{i}{n}}$ as well as on $HM_{\frac{n-i}{n}}$ in:

$$\sqrt[n-i]{HM_{\frac{n-i}{n}}} = \sqrt[n-i]{\frac{ncx^{n-i}}{(n-i)x^n + ic}} < \sqrt[n]{c} < \sqrt[i]{\frac{(n-i)x^n + ic}{nx^{n-i}}} = \sqrt[i]{AM_{\frac{i}{n}}}. \tag{3}$$

Here x is smaller then $\sqrt[n]{c}$ so we iterate on the interval $\left[x^i, \frac{c}{x^{n-i}}\right]$ that contains $\sqrt[n]{c^i}$. If we take $i = 1$ then we obtain:

$$\text{(4)} \qquad \sqrt[n-1]{HM_{\frac{n-1}{n}}} = \sqrt[n-1]{\frac{ncx^{n-1}}{(n-1)x^n + c}} < \sqrt[n]{c} < \frac{(n-1)x^n + c}{nx^{n-1}} = AM_{\frac{1}{n}}.$$

On the right-hand side we recognize the formula of Newton for the zero value $\sqrt[n]{c}$ of the function $f(x) = x^n - c$. The derivative of f is $f'(x) = nx^{n-1}$. So with the tangent method we obtain:

$$\text{(5)} \qquad x - \frac{f(x)}{f'(x)} = x - \frac{x^n - c}{nx^{n-1}} = \frac{(n-1)x^n + c}{nx^{n-1}}.$$

This algorithm however, is the shortest of all possible algorithms of this kind. The speed of convergence is higher with a higher value of i. So the algorithm on the left side of the last expression is the fastest for the root exponent $(n-1)$ has the highest level.

3. Geometric means and Pascal's Triangle

It is remarkable that these formulae can be generated with simple geometry and algebra, represented in beautiful nomograms, without the sophisticated tools of analysis. It is thus possible to understand various means of different order n geometrically and algebraically. The arguments over which the n^{th}-root is taken are also the various entries of Pascal's Triangle wherein the normal rules of arithmetic are encoded. The coefficients for each term in the expansion of $(a+b)^n$ can be derived using the Binomial theorem of Newton. Every product between a and b in the Triangle is the argument of the geometric mean of some order between numbers a and b (Figure 4).

1

$1a \quad 1b$

$1a^2 \quad 2ab \quad 1b^2$

$1a^3 \quad 3a^2b \quad 3ab^2 \quad 1b^3$

$1a^4 \quad 4a^3b \quad 6a^2b^2 \quad 4ab^3 \quad 1b^4$

Fig. 4. First rows of Pascal's Triangle

We observe that the order n decreases from n to 0, from left to right for a and increases for b in the same direction. If we write $1a^2$ we understand at the same time that this is equal to b^0a^2. In the one direction we have a lowering of the exponent of a or b, in the other direction an increase according to the following rules:

$$x^n \rightarrow nx^{n-1} \tag{6}$$

$$(n+1)x^n \leftarrow x^{n+1} \tag{7}$$

The procedure with Equation (6) is also known as derivation (definition *(2) Symbolic*). Performing this in two directions and using proper normalization dividing by $n!$ the normal binomial coefficients come out.

$$\begin{array}{ccccccccc} 1a^4 & \rightarrow & 4a^3 & \rightarrow & 12a^2 & \rightarrow 24a^1 & \rightarrow 24 \\ 24 & \leftarrow 24b & \leftarrow & 12b^2 & \leftarrow 4b^3 & \leftarrow & 1b^4 \end{array} \tag{8}$$

Multiplying term by term

$$24a^4 \quad 96a^3b \quad 144a^2b^2 \quad 96ab^3 \quad 24b^4. \tag{9}$$

Adding all terms and dividing each term by $n! = 4!$

$$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4. \tag{10}$$

4. Geometrically: A game of cubes and beams

Geometrically, each entry in a given row of Pascal's Triangle has the same dimension. The fourth row for example, consists of cubes with side a and b (and respective volumes a^3 and b^3, one of each), beams with sides a, a and b (and volume a^2b, three of them) and beams with sides a, b and b (and volume ab^2, also three of them) [14].

Any row in the Triangle contains pure n-cubes a^n and b^n (Figure 4, numbers in bold) on the one hand, and n-beams on the other hand (Figure 4, in grey, non-bold), whose sum is equal to a hypercube $(a+b)^n$. Hypercubes or n-cubes are n-dimensional cubes (with $n > 3$, with side a or b), hyperbeams or n-beams are n-dimensional beams (with $n > 3$) of which at least one side is different from all other sides (for example a^nb^m).

Fig. 5. Simon Stevin of Bruges [29]

It is in fact also very easy to turn beams like $a^n b^m$ of dimension $(n+m)$ into cubes of the same dimension. For each hyperbeam $a^n b^m$ one can construct a hypercube with the same volume by taking the $(n+m)$-th root of $a^n b^m$. This gives the side of the $n+m$ dimensional cube. Which is the procedure discussed in part 2, on geometric means of a particular order, all of the same dimension. More specifically for $GM_{i/n}$ between two numbers and b, we specificy i, n, a and b (either a or b can be 1). So, $\sqrt[n]{a} = GM_{\frac{1}{n}}$ of the interval $[1, a]$ since $\sqrt[n]{1^{n-1}a^1} = \sqrt{a}$ and thus $\sqrt[n+m]{a^n b^m} = GM_{\frac{1}{n+m}}$ of the interval $[1, a^n b^m]$.

Simon Stevin (1548–1620) (Figure 5) reasoned and thought about *geometric numbers* (Figure 6). Stevin was one of the greatest mathematicians of the 16^{th} century and his work was both of a pure and applied nature providing a bridge between the old and the new sciences [27]. His equilibrium of forces and the parallelogram rule was the beginning of abstract algebra and of higher dimensional geometry [29, 32].

In Figure 6 the examples are given of the powers of 2 (upper row) and the powers of 1 (lower row). 2^3 is a cube, and 2^4 $(= 16)$ are two cubes of the size 2^3. Likewise 2^5 $(= 32)$ are 4 of these cubes. For a cube with all sides equal to 1, the results remain the same for any power. It is the neutral element and any number of multiplications of 1 by itself always yields the same result.

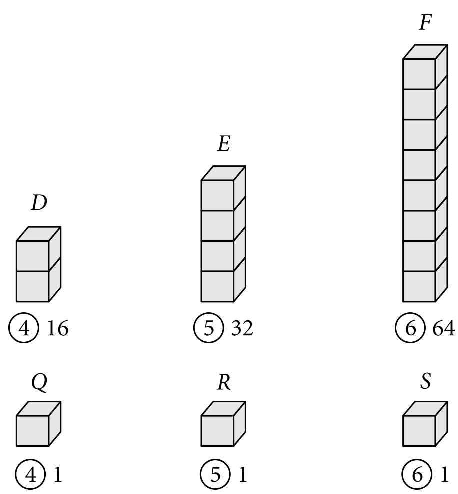


Fig. 6. Stevin's Geometric numbers [14, 29]

An object like b^4 (arithmetically the product $b \times b \times b \times b$) can be understood geometrically in many different ways. The object b^4 is not only a four-dimensional volume of a hypercube with side b, but it is also b times a three-dimensional volume b^3 (side of this cube is b). At the same time it can be seen as $b \times b$ times an area of b^2, but b^2 could also be interpreted as a beam of volume $1 \times b^2$. This beam can then be made into a cube with the same volume, but with side $(1 \times b^2)^{1/3}$, the one-third geometric mean between b^2 and 1, or the second geometrical mean of 1 and b, $GM_{2/3}$ of $[1, b]$ is $\sqrt[3]{1^{3-2} \times b^2} = \sqrt[3]{b^2}$, $GM_{1/3}$ of $\left[1, b^2\right]$ is $\sqrt[3]{1 \times b^2} = \sqrt[3]{b^2}$. And so on.

For the four-dimensional volume, it is important to be reminded how Pascal thought about the fourth dimension: *"Et l'on ne doit pas être blessé par cet quatrième dimension"* [3]. Which means that intelligent people should not be put off by something like the fourth dimension, because in reality it is about multiplication.

Stevin's approach of geometrical numbers was directly related to arithmetic and in Definition XXXI [29] he states that any number can be square, cube etc., or that also roots are numbers: *"Que nombres quelconques peuvent estre nombres quarrez, cubiques etc. Aussie que racigne quelconque est nombre"*. From this Stevin reaches the fundamental conclusion that there are no absurd, irrational, irregular, inexplicable or surd numbers, *"Qu'il ny a aucuns nombres absurdes, irrationels, irreguliers, inexplicables, ou sourds"* [29].

A contemporary of Stevin, François Viète (1540–1603) prefers to deal exclusively with numbers avoiding all geometrical connotations. In his *"Logistices speciosae canonica praecepta"* (canonical rules of species calculation [22]), the main law (*Lex homogeneorum*) states that only species of the same kind (*homogeneous* species) can be added or subtracted. In a typical row of Pascal's Triangle, all n-cubes and n-beams are of the same dimension or the same species (*speciosa*), but in general polynomials (for example in one variable) this is not the case and here the *Lex homogeneorum* rules, according to *"common sense"*. This *"in fact-not-so-common-sense"* is one of the main reasons for the split between arithmetic and geometry.

Contrary to Viète's *Lex homogeneorum* however, it is very easy to get the same dimension for any term of a polynomial using the unit element. For example, a polynomial like x^4+x^3+x can be written as $(x \times x \times x \times x)+(x \times x \times x \times 1)+(x \times 1 \times 1 \times 1)$; all of the same dimension and actually, all geometric means of different orders between x and the unit element 1 can be written this way [32]

$$x^3 + x^2 + x = x^3 + \left(\sqrt[3]{(1 \times x^2)}\right)^3 + \left(\sqrt[3]{(1^2 \times x)}\right)^3. \tag{11}$$

It is of interest to add here another definition of Stevin in his first book on arithmetic, namely Definition XXVI *"Multinomie algebraique est un nombre consistent de plusieurs diverses quantitez"*. This definition introduces the reader to algebraic multinomials or polynomials, *"Comme* $3z + 5y - 4x + 6$ *s'appele multinome algebraique. Et quand il aura de quantitez comme* $2x + 4y$ *s'appelent binomie, et de trois quantitez s'appellera trinomie, etc."* [29].

5. The decimal principle and fluxions

All this was very natural for mathematicians like Simon Stevin and his contemporaries. René Descartes wrote: *"Just as the symbol* $c^{1/3}$ *is used to represent the side of a cube* a^3 *has the same dimension as* a^2b*"* [19]. The relation between products and rectangles was frequently used for didactical reasons, for example by John

Colson in his "perpetual comment" to Newton's *Method of fluxions and infinite series – To which is subjoin'd: A perpetual comment upon the whole work, consisting of annotations, illustrations and supplements in order to make this treatise A Compleat Institution for the use of learners* [21].

This treatise brings out a nice historical connection between Stevin and Newton, going back to the complete arithmetical treatment with natural numbers by Stevin in the decimal system in 1585. Stevin showed in his book *De Thiende* that a complete arithmetical control of the real number system is achieved by explicitly demonstrating how all operations on and with real numbers can be carried out when expressing these numbers in the decimal system. Stevin added in an appendix to *De Thiende* [28] that the decimal principle should be advocated in *"all human accounts and measurements"*, thereby "anticipating the (partial) realization of this simple idea by two centuries". The importance of the decimal principle for geometry and for science cannot be overemphasized. It opened the way to Descartes algebraic geometry and inspired Newton to write his *Method of fluxions and infinite series with its application to the geometry of curve-line* [21] from the following motivation:

> *"Since there is a great conformity between the Operations in Species, and the same Operations in common Numbers; nor do they seem to differ, except in the Characters by which they are presented, the first being general and indefinite, and the other definite and particular: I cannot but wonder that no body has thought of accommodating the lately-discover'd Doctrine of Decimal Fractions in like manner to Species, . . . , especially since it might have open'd a way to more abstruse Discoveries. But since this Doctrine of Species, has the same relation to Algebra, as the Doctrine of Decimal Numbers has to common Arithmetick: the Operations of Additions, Subtractions, Multiplication, Division and the Extraction of Roots, may easily be learned from thence, if the Learner be but skilled in Decimal Arithmetick, and the Vulgar Algebra, and observes the correspondence that obtains between Decimal Fractions and Algebraick Terms infinitely continued. For as in Numbers, the Places towards the right-hand continually decrease in a Decimal or Subdecuple Proportion; so it is in Species respectively, when the Terms are disposed in an uniform Progression infinitely continued, according to the Order of the Dimensions of any Numerator or Denominator. And as the convenience of Decimals is this, that all vulgar Franctions and Radicals, being reduced to them, in some measure acquire the nature of Integers, and may be managed as such, so it is a convenience attending infinite Series in species, that all kinds of complicate Terms may be reduced to the Class of simple Quantities . . . "*

One of the major development in the book concerns infinite series. John Colson (1680–1760) who later became Lucasian professor of Mathematics, as one of the successors of Barrow and Newton, wrote in his Introduction [21]:

> *"As to the Method of Infinite Series, in this the Author opens a new kind of Arithmetick, (new at least at the time of writing this), or rather he vastly improves the old. For he extends the received Notation, making it*

completely universal and shews, that as our common Arithmetick of Integers received a great improvement by the introduction of decimal Fractions; so the common Algebra or Analyticks, as an universal Arithmetick, will receive a like Improvement by the admission of his Doctrine of Infinite Series, by which the same analogy will be still carry'd on, and farther advanced towards perfection. Then he shows how all complicate Algebraical Expressions may be reduced to such Series, as will continually converge to the true values of those complex quantities or their Roots, and may be therefor be used in their stead."

In Taylor and MacLaurin series, the same rules as above (Equation 6, $D(x^n) = nx^{n-1}$) are key. They give an operational definition of functions with for example, the MacLaurin series for e^x, *cosine* and *sine*. This is shown in Figure 7 for sine.

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}
\tag{12}
$$

A key to understand derivatives is putting the unit element back where it belongs.

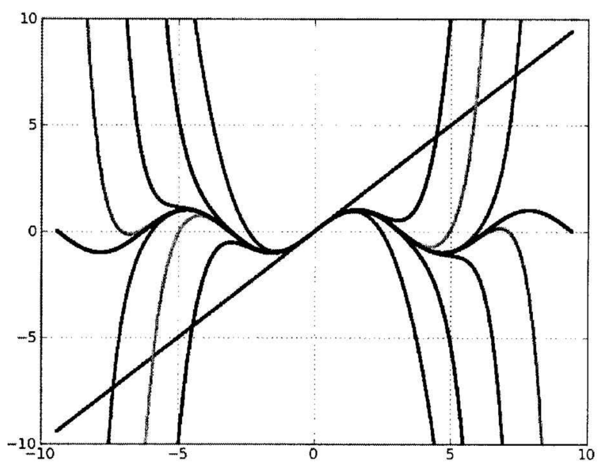


Fig. 7. Sine function for increasing number of terms in partial sums

Indeed, performing derivation is substituting x by 1, one at the time in a series. In Equation 6 in the process of derivation (e.g. of x^3) we observe $x^3 = (xxx) \rightarrow 3(xx1)$ or 1 cube of x^3 is compared to beams of sides x, x and 1, and we need three of them.

So, if we want to find the derivative of e^x we substitute in every term of the series one factor x by 1 (and if there is no x as in the first term it becomes 0;

the "lowering of the exponent operation") and put the original exponent up front lowering $n!$ to $(n-1)!$. The result is the same series, as expected. It is easy to show that the same occurs if we use the series expansions for sine and cosine, based on the relationship of Euler, $D(\sin x) = \cos x$ and $D(\cos x) = -\sin x$.


cub. = $f^3 + 3ffe + 3fee + e^3$

Fig. 8. From Newton's Principia [20]

In general, derivatives are related to (higher order) *geometric* means between two numbers f and e. Higher order geometric means between two pure numbers f and e involve expressions of the type $(m+n)$-th root of $f^m \times e^n$. For $e = 1$, $f^3 + f^2 + f + 1$ can be written as $f \times f \times f + f \times f \times e + f \times e \times e + e \times e \times e$ (Figure 8).

6. The geometry of parabolas

The use of the unit element allows for comparing n-volumes (converting m volumes into n-volumes of the 'same' dimension if needed). This, in our view, also shows that the Greeks were well aware of "units" in relation to conic sections. In a parabola ($y = x^2$), the variable y scales to the first power while some other variable x scales to the second, but geometrically a parabola *indicates* that for each coordinate x, one can construct (in Greek terminology to each line with length x it is possible to apply) a square with area equal to x^2, such that this area corresponds exactly to the area of a rectangle with width 1 and height y ($x \times x = y \times 1$). In this sense the parabola is an *"equiareal"* figure. This was known to Ancient Greek geometers and is at the basis of the conic sections.

Allometric equations and power laws, expressing the constancy of relative growth and generally depicted as straight lines in log-log plots, can be understood in the same geometric way [14], namely that these equations express some conservation law for n-volumes of n-cubes and n-beams, with the parabola and hyperbola for $n = 2$. The power law $y = x^{3/4}$ or equivalently $y^4 = x^3$ thus states that the 4-cube with side y is exactly the same as the $4-$beam with four sides x, x, x and 1 (with volume $x \times x \times x \times 1$). In our days $y = x^{3/4}$ is related often to "fractal dimension", but these are parabola's of the type $y^n = x^{n-1}$ (Eq. 6). In a general way the parabolas of the family $y^n = nx^{n-1}$ are nothing but derivatives. Reading the graphs the other way, $y^n/n = x^{n-1}$ stands for integration. Simple and pure.

An example from physics is Kepler's Law of Periods, which states that the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of the elliptical orbit. So the volume of a *beam* formed by the orbital period T_1 and the unit element is $T_1 \times T_1 \times 1$ (i.e. the volume of a *beam* with height and length T_1 and width 1), equals the volume of a *cube* $a_1 \times a_1 \times a_1$

(with sides equal to the semi-major axis a_1) up to a constant. On a weighing balance this constant can be interpreted as the length of the arms. This is also an *equi-areal* law.

$$T_1{}^2 = \frac{4\pi^2}{G(M_1 + M_2)} a_1{}^3. \tag{13}$$

This in our opinion at least, settles the question whether the Greeks understood derivatives and dynamics in an affirmative way.

The examples of parabola and ellipse show that they understood the unit element in a general way, not necessarily as a number. It was only in the Renaissance that it really became a number. Simon Stevin, was one of the first to state that the unit was a number *"Que l'Unité est Nombre"*. It becomes the neutral element for multiplication, but it certainly does not need to be "one", as long as the element selected is "neutral". If we look back at Figure 1 right and move the point zero as far as possible to the left, and define a neutral element close to zero, this element becomes as far as possible removed from a and b, having the least possible influence for computing AM or GM. One can call such an infinitely small number for example "dx". Obviously, one does not need to move zero; easier is to move the neutral element as close as possible to zero.

The 'rectangles' $x\,dx$ in Riemann sums for integrals are a generalization of parabolas, but viewed from a Greek's geometrical perspective, it does not add anything really new. The question whether $dx = 0$ or 1 becomes then pretty irrelevant. It needs to be neutral in multiplication (in which case $dx = 1$), and if it needs to be neutral for addition $(x + dx)$ it has the tendency to be rather close to zero or any number close enough.

Going back to Figure 1 and considering GM and AM (for $n = 2$) one understands that the geometric mean (GM) is strictly smaller than the arithmetic mean (AM). In order to have AM approach GM, the point 0 where the lines cross has to be moved as far to the left as possible. Equality is obtained only when lines run parallel. This is an arithmetical interpretation of Euclid's fifth postulate and an illustration of Shiing-Shin Chern's remark that *"Euclid's Elements are a geometrical treatment of the number system"* [5]: the fifth postulate ensures that AM and GM can be constructed and that, for any two numbers a and b on a line, GM is strictly smaller than AM, which is the cornerstone of our number system [14].

7. 1, 2, ..., 11, ..., 37: monomiality principle for polynomials

The explicit forms of the series expansions for the exponential, sine and cosine functions are:

$$e^x = \sum_{n=0}^{\infty} \frac{1^n}{n!} x^n, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \tag{14}$$

Sine and cosine are examples of simple polynomials, but in mathematics there are a large number of special polynomials in the theory of special functions, such as

Legendre, Hermite, Laguerre, Bell, Appell ... polynomials (Figure 9). Special functions play an important role in various field of mathematics, physics and engineering and during the last decades new families of special functions have been suggested in various branches of physics [7].

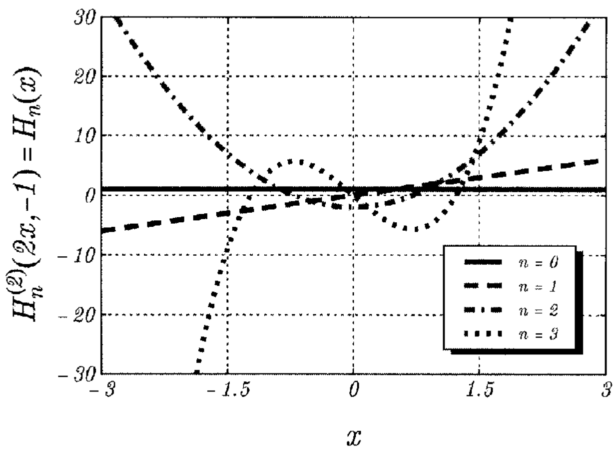


Fig. 9. Example of Hermite polynomials

Comparing many such special functions and their explicit form in contemporary pure and applied mathematics to functions of a more elementary nature (e^x, sine, cosine, ...) from the 17^{th} and 18^{th} century seems somewhat like jumping from older, simpler definitions of the derivative (list of Thurston 1 through 7) to definition 37. Indeed, at first sight contemporary special polynomials are a far cry from the simpler functions and power series, from parabolas and cubes and beams. But is that so?

Most families of special polynomials can be transformed by the same simple procedure of raising and lowering exponents, the same game of means and beams. By virtue of the so-called *monomiality* principle, all families of polynomials, and in particular special polynomials, can be obtained by transforming a basic monomial set by means of suitable operators P and M, called the derivative and multiplication operator of the considered family, respectively.

The definition of the *poweroid* introduced by J. F. Steffensen has been framed in the *monomiality* principle by G. Dattoli [6], providing a very powerful analytical tool for deriving properties of special polynomials, such as Hermite, Bessel, Laguerre, Bell and Legendre polynomials [2, 8, 9, 25].

Let us consider the Heisenberg-Weyl algebra with generators P and M satisfying the commutation relation $[P, M] = PM - MP = 1$, and the family of polynomials $\pi = p_n(x)$ $(n = 0, 1, 2, \dots)$. Then, π is quasi-monomial if the identities $P\left(p_n\left(x\right)\right) = np_{n-1}\left(x\right)$ and $M\left(p_n\left(x\right)\right) = p_{n+1}\left(x\right)$ hold true. In this case, the analytical properties of π can be obtained straightforwardly starting from those of the operators P and M. As an example, combining the mentioned identities yields immediately the governing equation of the general polynomial $p_n(x)$ belonging to π, namely $(MP - n)(p_n(x)) = 0$. Furthermore, it is worth noting that, under the assumption that $p_0(x) = 1$, $p_n(x)$ can be represented explicitly as $p_n(x) = M^n(1)$.

The simplest application of the monomiality principle regards the family of monomials $\{x^n\}$ in one variable with derivative and multiplication operators $P = D = d/dx$ and $M = x$ respectively, such that $[P, M] = 1$ and $M(x^n) = x^{n+1}$, $P(x^n) = nx^{n-1}$. These operations simply consist in raising and lowering the exponents of the general monomial belonging to the family. Additional relations are given by $x^0 = 1$ and $Dx^0 = 0$ (One can easily see that it is also possible to start from the antiderivative).

Using the monomiality principle, the analytical properties of special polynomials can be easily studied. Following this approach, the governing differential equation, recurrence relations and identities can be easily determined. As an example, let us consider the family of Hermite polynomials in two variables [1]:

$$H_n^{(m)}(x, y) := \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{n!}{k!(n-km)!} y^k x^{n-km}, \tag{15}$$

with $H_n^{(m)}(x, 0) := x^n$ and $m = 1, 2, \ldots, n$. As shown in [25] for the case $m = 2$, these polynomials are particular solutions of the generalized two-dimensional heat equation $\partial_y H_n^{(m)}(x, y) = \partial_x^m H_n^{(m)}(x, y)$.

Therefore, the general polynomial can be represented in a very compact way as $H_n^{(m)}(x, y) = (x + my\partial_x)^n(1)$, whereas the relevant governing differential equation is obtained as

$$my\partial_x^2 H_n^{(m)}(x, y) + x\partial_x H_n^{(m)}(x, y) = MP\left(H_n^{(m)}(x, y)\right) = nH_n^{(m)}(x, y).$$

Other recurrence formulae and identities involving $H_n^{(m)}(x, y)$ for $m = 2$ are derived in a similar way in [20], which provides an excellent overview on the application of operational techniques to special polynomials. Similar unifying approaches can be developed starting from the Pascal matrix [1].

8. −1, −2, −3, . . . : understand the legacy

For Thurston, the closest definition of mathematics is *"the theory of formal patterns"* and *"mathematicians are those humans who advance human understanding of mathematics"* [31]. In this article we showed that the formal pattern underlying various means, the normal rules of arithmetic, expansions and special functions is in principle a game of cubes and beams, going back to Greek foundations (−1, −2, −3 is counting down to some time, say −2500 years, when Pythagoreans mastered means, numbers and much more). With simple principles and elementary functions we can achieve already a very good understanding of mathematics and the relations among various fields. We hope that this article, with its didactic emphasis, can contribute to a better human understanding of these topics.

Newton's intuition (*"if the Learner be but skilled in Decimal Arithmetick, and the Vulgar Algebra ... "*), is still accurate today, but incomplete. *Basic geometry* is also required for a better and more flexible understanding. To understand higher dimensional cubes and beams as simple geometric numbers, cubes, beams and balances in equilibrium, lift the ban imposed by Viète [22] in the 17^{th} century and by Grassman/Peano [24] in the 19^{th} century. To this very day this ban continues to exist. Even Stephen Hawking [19] notes on La Géometrie of Descartes: *"At the time this was written a^2 was commonly considered to mean the surface of a square whose side is a, and b^3 to mean the volume of a cube whose side is b^3 while b^4, b^5 ... were unintelligible as geometric forms"*.

René Descartes (1596–1650) certainly knew about the works of Simon Stevin (1548–1620), not only because they were available in French, but also because of his collaboration with Isaac Beeckman, himself a student of Stevin. Anyway, two centuries later Monge, Gauss, Lamé and Riemann among others brought geometry back onto center stage [4].

With regard to the intimate relation of algebra and geometry and the various idle discussions on which of these two is more important, consider this end-of-20th-century attempt to define special functions [7]:

> *"It is also difficult to frame in an univocal way the concept of Special Function itself. Just to make an attempt, we can associate Special Functions with the solutions of particular families of ordinary differential equations with non-constant coefficients. During the end of the last century, Sophus Lie pondering on the deep reasons underlying the solution by quadrature of differential equations was led to the notion of group symmetry. This concept inspired the work of Cartan, who was the first to point out that Special Functions can be framed within the context of the Lie theory. This point of view culminated in the work of Wigner who regarded the Special Functions as matrix elements of irreducible representations of Lie groups."*

This is not the most general definition, since it leaves out some polynomials, but this quote is intended to illustrate the deep connections among fields. These very fundamental relations where summarized by Chern in the following: *"While algebra and analysis provide the foundations of mathematics, geometry is at the core"* [5]. This trinity is reflected in this article, with one underlying principle, understood very well in ancient Greece.

Greek and Hellenistic mathematics and science were advanced in every sense of the word [26]. The rebirth of Eudoxus mathematical findings into Dedekind's cut is one example. We referred earlier to Chern's statement that *Euclid's Elements are a geometrical treatment of the number system.* The influence of Stevin's Decimal fractions on the development of fluxions has been pointed out, but this was really based on this very same idea of commensurability. As D. J. Struik (editor of the mathematical parts of the Principal Works of Simon Stevin) writes:

> *"In his arithmetical and geometrical studies, Stevin pointed out that the analogy between numbers and line-segments was closer than was generally recognized. He showed that the principle arithmetical operations, as well*

as the theory of proportions and the rule of three, had their counterparts in geometry. Incommensurability existed between line-segments as well as numbers … ; incommensurability was a relative property, and there was no sense in calling numbers "irrational", "irregular", or any other name, which connoted inferiority. He went so far as to say, in his Traicté des incommensurable grandeurs, that the geometrical theory of incommensurables, in Euclid's Tenth Book had originally been discovered in terms of numbers, and translated the content of this book into the language of numbers. He compared the still incompletely understood arithmetical continuum to the geometrical continuum already explained by the Greeks, and thus prepared the way for that correspondence of numbers and points on the line that made its entry with Descartes' coordinate geometry [29]."

In some sense, all we are doing is adding some commas and punctuations here and there to what the Greek have done. Also on the historical side much is to be learned from Bacon's writings [16]:

"So that as Plato had an imagination that all knowledge was but remembrance; so Solomon giveth his sentence, that all novelty is but oblivion."

One example of such oblivion and "novelty" are the series expansions for cosine and sine, as well as their definitions, traditionally associated with names of 16^{th} and 17^{th} century Western mathematicians. In fact, trigonometric functions sine and cosine as well as their expansion were known already in the early 15^{th} century to the Indian mathematician Madhava of Sangamagramma (c. 1340–1425). Pascal's Triangle was not invented by Pascal, but was known long before in China and Persia. Many special polynomials were invented a century ago and then forgotten, until rediscovered by physics [7]. The history of mathematics is very old and its legacy extremely rich, transcending boundaries in space and time.

9. On landscapes and maps

Another goal of this article, other than pointing out that all these things are intrinsically linked spanning a period of at least 2,5 millennia, is to show how important it is to understand concepts in different ways. Thurston's list is *"a list of different ways of thinking about or conceiving of the derivative, rather than a list of different logical definitions"* [31]. It is important to be able to look at concepts, for which mathematicians spent thousands of years to come to an ever-better understanding, from very different perspectives, not only tailored to specific talents of individuals, but to safeguard the true spirit of mathematics. Thurston again: *"Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions"* [31]. One example is the development of algebra and its deviation from geometric numbers.

One of the defining characteristics of Greek mathematics was not only the development of mathematics, but also to understand that this has an intimate relationship to the workings of the world [26]. According to Feynman [11] *"To* *those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature … If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in. She offers her information only in one form".* Fortunately, simple rules are a basic feature of this language, by whatever name they are known, addition and multiplication, means, cubes or the monomiality principle [9].

Despite the simple basic rules from which a wide range of methods can be deduced, contemporary mathematics has evolved into very diverse landscapes, in pure and applied mathematics or in theoretical physics. Each uses one or more very specific high-level (increasingly abstract) languages, with various local dialects. A mathematical concept like curvature goes under a variety of different names in mathematics and physics, dependent on the field. There is the imminent danger of high-level languages for different landscapes (the languages as the maps or *Des Cartes*), taking precedence over these landscapes and territories themselves.

> *"The transfer of understanding from one person to another is not automatic. It is hard and tricky. Therefore, to analyze human understanding of mathematics, it is important to consider who understands what, and when. Much of the difficulty has to do with the language and culture of mathematics, which is divided into subfields. Basic concepts used every day within one subfield are often foreign to another subfield. Mathematicians give up on trying to understand the basic concepts even from neighboring subfields, unless they were clued in as graduate students. In contrast, communication works very well within the subfields of mathematics. Within a subfield, people develop a body of common knowledge and known techniques [31]."*

When it can be shown that simple rules underlie different mathematical landscapes, the general language (the art of mapmaking) need not be too abstract for a basic understanding or the development of a certain feeling for the matter.

The "maps" we spoke of are based on the Pythagorean theorem or triangles in general (including triangulation of surfaces), and geometric means between numbers. There are other ways to make maps to better understand nature. A generalization of the Pythagorean Theorem based on n-cubes (instead of squares) leads to the simplest cases of Minkowski-Finsler geometry and the curves associated with this generalization are Lamé curves, named after Gabriel Lamé (1795–1870). A subclass of Lamé curves are superellipses [12, 27], defined by:

$$\left|\frac{x}{A}\right|^n + \left|\frac{y}{B}\right|^n = 1. \tag{16}$$

This in fact considers cubes (and n-cubes) only, not using beams or geometric means between numbers and hence the relation of Lamé curves to the Last Theorem of Fermat. The coefficients of the expansion of $(2x+1)^n$ have a nice geometrical meaning [27, Chapter 4] for $n = 4$, $(2x+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$, a *tesseract* (a four dimensional cube) is composed of 16 points, 32 lines, 24 squares, 8 cubes and of course 1 *tesseract*. Again, this is a nice example of lowering of exponent with a clear geometrical meaning.

A recent generalization of Lamé curves provides a new way of studying geometry and natural shapes [13, 15, 17] (Figure 10 for some natural shapes), whereby Geometry becomes intimately connected with Growth and Form in nature [18]. A geometric study of natural shapes and phenomena can be proposed using only pure numbers a^n or variables x^n without geometric means.

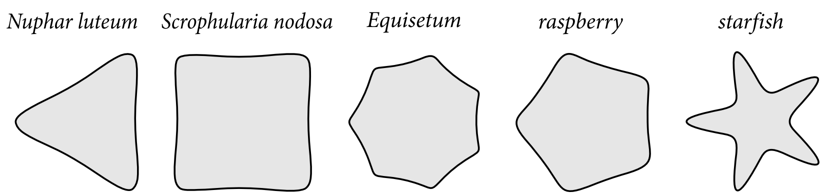


Fig. 10. Some natural shapes as transformations of a circle [12, 13]

As René Thom stated: *"Geometry is successful magic!"* [30][2]. Simon Stevin's motto was *"Wonder is no Wonder"* [10], meaning that geometrical understanding is the way to get rid of miracles and wonders[3]. This remains true to this very day and in the future. André Weil [35] wrote:

> *"Obviously everything in differential geometry can be translated into the language of analysis, just as everything in algebraic geometry can be expressed in the language of algebra. Whether one considers analytic geometry in the hands of Lagrange, tensor calculus at this of Ricci, or more modern examples, it is always clear that a purely formal treatment of geometric topics would invariably have killed the subject if it had not been rescued by true geometers, Monge in one instance, Levi-Civita, and above all Elie Cartan in another ... The psychological nature of true geometrical intuition will perhaps never be cleared up ... Whatever the truth of the matter, mathematics in our century would not have made such impressive progress without the geometric sense of Elie Cartan, Heinz Hopf, Chern and a very few more. It seems safe to predict that such men will always be needed if mathematics is to go on as before."*

Geometry improves not only the understanding of mathematics, but also connects with the foundations. Radu Miron wrote:

> *"If Mathematics could be torn from its foundations, it would become a series of formulae, recipees and tautologies that could not be applied any longer to the objective reality, but only to some rigid, mortified scheme of this reality." [23]*

[2] *"La Géometrie est magie qui réussit"*. Geometry is magic that works, successfully.

[3] The English translation of the biography of Simon Stevin by Van den Berghe and Devreese is "Miracle is no miracle" [10]. Personally I prefer wonder over miracle. Wonder in Dutch occurs in the words *wonderlijk* (strange, odd, surprising), *verwondering* (wonder, astonishment, surprise). Wonder has a much broader meaning and is therefore less miraculous than miracle (miracles would be a main thesis of Spinoza almost one century later, based on a Newtonian laws and, indirectly, Stevin's Motto.

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