THE TEACHING OF MATHEMATICS
2014, Vol. XVII, 1, pp. 34–38

A PROOF OF METHOD OF CYLINDRICAL SHELLS BASED ON A GENERALIZED INTEGRAL REPRESENTATION OF ADDITIVE INTERVAL FUNCTION

Yuanhong Zhi and Yongkun Li

Abstract. In this paper we provide a generalized integral representation of additive interval function based on a fundamental integral representation of additive interval function given in Zorich's textbook, Mathematical Analysis, Vol I. Then we use it to give a rigorous proof of the method of cylindrical shells for the evaluation of volume of solid of revolution about vertical line.

MathEduc Subject Classification: I55

MSC Subject Classification: 97I50

Key words and phrases: Additive interval function; method of cylindrical shells; Riemann integrable function.

1. Introduction

In most undergraduate calculus textbooks, the *method of cylindrical shells* is given, in order to evaluat easily the volume of solid of revolution about a vertical line (in many cases about y-axis in Cartesian coordinate system). But almost all the textbooks do not provide a rigorous proof of the validity of this method (for example, see [1–3, 5, 6]). Although it is obvious in the frame of measure theory, it is still vital to give a vigorous proof of this method on the background of Riemann integral. On Page 376 in Zorich's book [7], a proposition concerning an additive interval function to be expressed as an definite integral is provided, which we call a *fundamental integral representation of additive interval function*, that is

Proposition 1. *Suppose that an additive function $I(\alpha,\beta)$, defined for points α,β of a closed interval $[a,b]$, is such that there exists a function $g\in\mathcal{R}[a,b]$ connected with I as follows: the relation*

$$\inf_{x\in[\alpha,\beta]} g(x)(\beta-\alpha)\le I(\alpha,\beta)\le \sup_{x\in[\alpha,\beta]} g(x)(\beta-\alpha)$$

holds for any closed interval $[\alpha,\beta]$ such that $a\le\alpha\le\beta\le b$. Then

$$I(a,b)=\int_a^b g(x)dx.$$

Here an *additive interval function* means a function $(\alpha,\beta)\mapsto I(\alpha,\beta)$ that assigns a real number $I(\alpha,\beta)$ to each ordered pair of points (α,β) of a fixed bounded closed interval $[a,b]$, in such a way that the following equality holds for any triple of points $\alpha, \beta, \gamma \in [a,b]$: $I(\alpha,\gamma) = I(\alpha,\beta) + I(\beta,\gamma)$.

Using Proposition 1, the calculation of arc length, of area of a curvilinear trapezoid, and of volume of a solid of revolution about horizontal line (that is about x-axis in Cartesian coordinate system) are just the direct corollaries. But for the solid of revolution about y-axis in Cartesian coordinate system, it is not seemly the case, without resorting to measure theory. In this paper, we give a generalized version of this proposition (see the theorem 1. As a direct application of this generalized theorem, we provide a rigorous proof of the method of cylindrical shells.

2. Main result

A generalized version of Proposition 1 reads as follows.

THEOREM 1. *Suppose $V = V(\alpha,\beta)$ is an additive (oriented) interval function defined for any points α, β belonging to a bounded non-degenerate closed interval $[a,b]$. Let functions f and g be Riemann integrable on $[a,b]$. Assume that for any closed interval $[\alpha,\beta]$ such that $a \le \alpha \le \beta \le b$, there exist $\xi, \eta \in [\alpha,\beta]$, such that the following condition hold:*

$$f(\xi) \inf_{\alpha \le x \le \beta} g(x)(\beta - \alpha) \le V(\alpha,\beta) \le f(\eta) \sup_{\alpha \le x \le \beta} g(x)(\beta - \alpha). \tag{1}$$

Then $V(a,b)$ can be expressed as the definite integral

$$V(a,b) = \int_a^b f(x)g(x)dx.$$

Proof. Since f, g are Riemann integrable, they are bounded on $[a,b]$. Let $A = \sup_{x\in[a,b]} |f(x)| + 1, B = \sup_{x\in[a,b]} |g(x)| + 1$. For any partition $P : x_0 = a < x_1 < x_2 < \cdots < x_n = b$ with corresponding distinguished points $\eta_i \in \Delta_i$ (or $\xi_i \in \Delta_i$) according to condition (1), $i = 1, 2, \ldots, n$, let $\Delta_i = [x_{i-1}, x_i], \Delta x_i = x_i - x_{i-1}$ and $\omega(f;\Delta_i)$ be the oscillation of f on the subinterval Δ_i. Then by condition (1) we obtain

$$\begin{aligned}
V(a,b) &= \sum_{i=1}^n V(x_{i-1},x_i) \le \sum_{i=1}^n f(\eta_i) \sup_{x\in\Delta_i} g(x)\Delta x_i \\
&= \sum_{i=1}^n f(\eta_i)\big(\sup_{x\in\Delta_i} g(x) - g(\eta_i)\big)\Delta x_i + \sum_{i=1}^n \big(f(\eta_i) - f(\xi_i)\big)g(\eta_i)\Delta x_i \\
&\quad + \sum_{i=1}^n f(\xi_i)\big(g(\eta_i) - g(\xi_i)\big)\Delta x_i + \sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i \\
&\le 2\sum_{i=1}^n A\omega(g;\Delta_i)\Delta x_i + \sum_{i=1}^n \omega(f;\Delta_i)B\Delta x_i + \sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
V(a,b) &= \sum_{i=1}^{n} V(x_{i-1},x_i) \geq \sum_{i=1}^{n} f(\xi_i) \inf_{x\in\Delta_i} g(x)\Delta x_i \\
&= \sum_{i=1}^{n} f(\xi_i)g(\xi_i)\Delta x_i - \sum_{i=1}^{n} f(\xi_i)\big(g(\xi_i) - \inf_{x\in\Delta_i} g(x)\big)\Delta x_i \\
&\geq \sum_{i=1}^{n} f(\xi_i)g(\xi_i)\Delta x_i - \sum_{i=1}^{n} A\omega(g;\Delta_i)\Delta x_i.
\end{aligned}$$

Since f, g are Riemann integrable on $[a,b]$, we know that (for reference, see [7]): for any $\epsilon > 0$, there exists a number $\delta_1 > 0$, such that for any partition P^* on $[a,b]$ for which mesh $\lambda(P^*) < \delta_1$, the oscillation of f and g satisfy

$$\sum_{i=1}^{n} \omega(g;\Delta_i)\Delta x_i < \epsilon/(6A), \sum_{i=1}^{n} \omega(f;\Delta_i)\Delta x_i < \epsilon/(3B).$$

Now for the prescribed ϵ, choose $0 < \delta < \delta_1$, then for any partition with distinguished points (P,ξ) on $[a,b]$ for which mesh $0 < \lambda(P) < \delta$, we have from above that

$$-\epsilon < -\epsilon/6 < V(a,b) - \sum_{i=1}^{n} f(\xi_i)g(\xi_i)\Delta x_i < \epsilon/3 + \epsilon/3 < \epsilon,$$

that is

$$\left| V(a,b) - \sum_{i=1}^{n} f(\xi_i)g(\xi_i)\Delta x_i \right| < \epsilon,$$

which is equivalent to $V(a,b) = \int_a^b f(x)g(x)dx$. Thus the proof is finished. ■

Actually, Theorem 1 is a generalized version of Proposition 1, because the latter is just the same as the former in the special case of $f(x) = 1, \forall x \in [a,b]$.

3. Applications

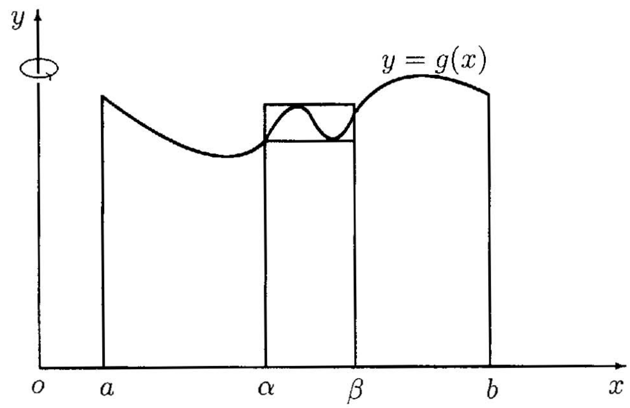
As a direct application of Theorem 1, we give a rigorous proof of the method of cylindrical shells, which is very useful to calculate the volume of the solid of revolution about vertical line.

Suppose a nonnegative Riemann integrable function $y = g(x)$ is defined on the bounded closed interval $[a,b]$, where $0 \leq a < b$. Let $D = \{(x,y) \in \mathbb{R} \mid a \leq x \leq b, 0 \leq y \leq g(x)\}$ be the ordinate set of g (for ordinate set, see [3]). Then revolve this set about the coordinate axis y, see figure below. As a consequence, the volume $V(a,b)$ of this solid of revolution about y-axis can be calculated by *the method of cylindrical shells*, and the formula of this volume is (see [1–6])

$$V(a,b) = \int_a^b 2\pi x \cdot g(x)dx, \text{ where } 0 \leq a < b.$$

It is intuitively and geometrically obvious that the above volume function $V = V(a,b)$ is an additive interval function defined on Cartesian product $[a,b] \times [a,b]$. By Theorem 1 we then give a rigorous proof of formula (2). Actually we see from the picture below that, for any α, β with $a \le \alpha < \beta \le b$,

$$\begin{aligned} V(\alpha,\beta) &\le \pi\beta^2 \sup_{\alpha\le x\le\beta} g(x) - \pi\alpha^2 \sup_{\alpha\le x\le\beta} g(x) = \pi(\beta^2-\alpha^2) \sup_{\alpha\le x\le\beta} g(x) \\ &= 2\pi\left(\frac{\beta+\alpha}{2}\right) \sup_{\alpha\le x\le\beta} g(x)(\beta-\alpha). \end{aligned}$$



Similarly,

$$\begin{aligned} V(\alpha,\beta) &\ge \pi\beta^2 \inf_{\alpha\le x\le\beta} g(x) - \pi\alpha^2 \inf_{\alpha\le x\le\beta} g(x) = \pi(\beta^2-\alpha^2) \inf_{\alpha\le x\le\beta} g(x) \\ &= 2\pi\left(\frac{\beta+\alpha}{2}\right) \inf_{\alpha\le x\le\beta} g(x)(\beta-\alpha). \end{aligned}$$

As a result, we see that the condition (1) of Theorem 1 is fulfilled, where $f(x) = 2\pi x$, with both ξ and η of Theorem 1 to be the middle point $(\alpha+\beta)/2$ of interval $[\alpha, \beta]$. Therefore we see the method of cylindrical shells is actually rigorous, that is the formula (2) is right, provided the function g is nonnegative Riemann integrable on interval $[a,b]$, where $0 \le a < b < +\infty$.

Acknowledgement. This work is supported by YNU Postdoctoral Science Foundation Project (W4030002).

REFERENCES

[1] H. J. Keisler, *Elementary Calculus, An Infinitesimal Approach*, second edition, Prindle, Weber and Schmidt, 1986.

[2] J. Marsden, A. Weinstein, *Calculus*, Vol II, Springer-Verlag, New York, 1985.

[3] M. Spivak, *Calculus*, 3rd edition, Vol I, Publish or Perish, 1996.

[4] J. Stewart, *Calculus*, Cengage Learning EMEA, 2009.

[5] G. B. Thomas, Jr., M. D. Weir, J. Hass, *Thomas' Calculus*, 11th Edition, Addison-Wesley, 2005.

[6] G. B. Thomas, Jr., R. L. Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley Publ. Co., 1998.

[7] V. A. Zorich, R. Cooke, *Mathematical Analysis*, Vol I, Springer-Verlag, 2004.

School of Mathematics and Statistics, Yunnan University, Kunming 650091, PR. China.

E-mail: yhzhi@ynu.edu.cn