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THE RELATIONSHIP BETWEEN THE CHANGE OF VARIABLE THEOREM AND THE FUNDAMENTAL THEOREM OF CALCULUS FOR THE LEBESGUE INTEGRAL

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Abstract. We discuss an interplay between five versions of the Change of Variable Theorem and the Fundamental Theorem of Calculus for the Lebesgue integral. We show that, under certain assumptions, they imply one another.

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1. Introduction

This note is concerned with interconnection between five versions of the Change of Variable Theorem for integrals (CVT) and the Fundamental Theorem of Calculus (FTC) for the Lebesgue integral. Here we assert that the FTC and five versions of the CVT (one is a weak version and the other four are standard), under certain assumptions, imply one another. For this, we present a proof that is independent from a certain property from which any of the three standard CVT can be derived.

Recall that a real function f is *absolutely continuous on* $[a,b]$ if, for $\epsilon > 0$ there exists $\delta > 0$ such that if $\{[u_i, v_i] : 1 \leq i \leq k\}$ is any finite collection of pairwise non-overlapping subintervals and $\sum_{i=1}^{k} |v_i - u_i| < \delta$, then $\sum_{i=1}^{k} |f(v_i) - f(u_i)| < \epsilon$. Throughout our discussion, we shall call the term absolutely continuous and almost everywhere as AC and a.e. respectively. Also, we assume that all integrability are in the Lebesgue sense.

Let f be a real-valued function on $[a,b]$, and g be a real-valued function defined on $J := f([a,b])$. Consider the following statements.

CVT 1. *If f is* AC *and nonzero, then f'/f is integrable on $[a,b]$, and*

$$\int_a^b \frac{f'(x)}{f(x)}\, dx = \ln|f(b)| - \ln|f(a)|. \tag{1}$$

CVT 2. *If f is* AC*, g is integrable, and either $f' \geq 0$ or $f' \leq 0$ a.e., then $(g \circ f)f'$ is integrable, and*

$$\int_a^b g(f(x))f'(x)\, dx = \int_{f(a)}^{f(b)} g(y)\, dy. \tag{2}$$

CVT 3. *If f is* AC, *g is bounded and integrable, then $(g \circ f)f'$ is integrable, and* (2) *holds.*

CVT 4. *If f is* AC, *g and $(g \circ f)f'$ are both integrable, then* (2) *holds.*

CVT 5. *Suppose that f is* AC, *that G is an* AC *function defined on J with $G' = g$ a.e., and that $G \circ f$ is* AC. *Then g and $(g \circ f)f'$ are both integrable and* (2) *holds.*

FTC. *If f is* AC, *then f' is integrable, and*

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{3}$$

Main Theorem. *Suppose we assume the fact that f' exists a.e. whenever f is* AC, *and without presupposing the property that, for $E \subseteq [a,b]$,*

$$f(E) \text{ has a measure } 0 \quad \Longrightarrow \quad f' = 0 \quad \text{a.e. on } E \tag{4}$$

provided f has a derivative on E. Then

(i) CVT 1 *is equivalent to the* FTC*;*

(ii) *any of* CVT 3, CVT 4, *and* CVT 5 *implies the* FTC*; and any version of* CVT *is equivalent to the* FTC *if we assume the monotone convergence theorem.*

The Main Theorem shows that we only need to assume the FTC, or equivalently,

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \ln|f(b)| - \ln|f(a)|$$

—while assuming the monotone convergence theorem—for the fully equipped CVT to be true.

While CVT 1 gives a weak version, CVT 2, 3, 4 and 5 are standard. As discussed in either [1] or [2], property (4) (i.e. Theorem 1 of [1] or Lemma 6.92 of [2]) has an essential contribution on deducing the four standard CVT. For this reason, the proof of Main Theorem should be independent from this property.

Prior to the proof we need the following fact to clarify that certain properties we will use in some part of the proof are derived under the assumption of the monotone convergence theorem (MCT). The proof of this fact is usually provided, or left as a typical exercise, on textbooks (part of this can be seen, for example, in [2]). Let m denote a set measure.

Fact 1. *Consider the following statements.*

(i) *If h is integrable on $[a,b]$, then for $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subseteq [a,b]$, if $m(E) < \delta$, then $\int_E |h| < \epsilon$.*

(ii) *If h is integrable on $[a,b]$, then the function $x \mapsto \int_a^x h(t)\,dt$, $x \in [a,b]$, is* AC.

(iii) *Given a measurable set* $X \subseteq [a.b]$*, then for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for every open set* $Y \supseteq X$*, if* $m(Y \setminus X) < \delta$*, then* $\left|\int_X h\right| \leq \epsilon + \left|\int_Y h\right|$.

(iv) *If* $\int_I h = 0$ *for every interval* $I \subseteq [a,b]$*, then* $h = 0$ *a.e. on* $[a,b]$.

(v) *If* $h \geq 0$ *a.e. on* $E \subseteq [a,b]$ *and* $\int_E h = 0$*, then* $h = 0$ *a.e. on* E.

Then MCT $\Rightarrow$ (i) $\Rightarrow$ [(ii) and (iii)]*; and* (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Now, the scheme of the proof of Main Theorem would be as follows:

$$\text{CVT } 1 \Rightarrow \text{FTC} \Rightarrow \text{CVT } 1$$

for part (i) of Main Theorem; then followed by

$$[\text{CVT } 3 \;\text{ or }\; \text{CVT } 4 \;\text{ or }\; \text{CVT } 5] \;\Rightarrow\; \text{FTC};$$

$$\text{CVT } 2 \Rightarrow \text{FTC} \Rightarrow \text{CVT } 5 \Rightarrow [\text{CVT } 2 \text{ and CVT } 3]; \tag{5}$$

and

$$\text{CVT } 3 \Rightarrow \text{CVT } 4 \tag{6}$$

for part (ii) of Main Theorem, where in part (5) and (6) we assume the monotone convergence theorem, in which Fact 1(ii)–(v) will be employed.

2. Proof of main theorem

CVT 1 $\Rightarrow$ FTC. Set $F(x) := e^{f(x)}$, $x \in [a,b]$. Since $x \mapsto e^x$ is Lipschitz on $[a,b]$ and f is AC, it follows that F is AC. By CVT 1, $F'/F = f'$ is integrable, and

$$\int_a^b f'(x)\,dx = \int_a^b \frac{F'(x)}{F(x)}\,dx = \ln|f(b)| - \ln|f(a)| = f(b) - f(a). \tag{7}$$

FTC $\Rightarrow$ CVT 1. Set $F(x) := \ln|f(x)|$, $x \in [a,b]$. Since $x \mapsto \ln|x|$ is Lipschitz and f is AC on $[a,b]$, F is AC, so that by the FTC, $F' = f'/f$ is integrable, and

$$\int_a^b \frac{f'(t)}{f(t)}\,dt = \int_a^b F'(t)\,dt = F(b) - F(a) = \ln|f(b)| - \ln|f(a)|. \tag{8}$$

[CVT 3 or CVT 4 or CVT 5] $\Rightarrow$ FTC. The FTC follows trivially from either CVT 3 or CVT 4 by substituting $g(y) := 1$, $y \in J$. Also, it follows immediately from CVT 5 by substituting $G(y) := y$, $y \in J$.

CVT 2 $\Rightarrow$ FTC. First note that substituting $g(y) := 1$ into equation (2) gives

$$\int_a^b f'(x)\,dx = \int_{f(a)}^{f(b)} dy = f(b) - f(a) \tag{9}$$

provided f is AC, either $f' \geq 0$ or $f' \leq 0$ a.e. We have two consequences from (9) for our purposes: (i) equation (9) applies for a function f that is AC and monotonic; and (ii) if f is AC and $f' = 0$ a.e., then f is a constant function.

Suppose now that f is AC (which does not necessarily satisfy that $f' \geq 0$ nor $f' \leq 0$). Since $f' + |f'| \geq 0$ and $f' - |f'| \leq 0$, it follows that the functions $f_1(x) := \int_a^x \frac{1}{2}(f'(t) + |f'(t)|)\,dt$ and $f_2(x) := \int_a^x \frac{1}{2}(f'(t) - |f'(t)|)\,dt$ are both monotonic. Since, in addition to be monotonic, f_1 is AC by Fact 1(ii), it follows from the first consequence above that

$$\int_a^x f_1'(t)\,dt = f_1(x) - f_1(a) = \int_a^x \frac{1}{2}\left(f'(t) + |f'(t)|\right)\,dt.$$

This gives

$$\int_a^x \left(f_1'(t) - \frac{1}{2}\left(f'(t) + |f'(t)|\right)\right)\,dt = 0,$$

for every $x \in [a,b]$. Therefore $\int_I (f_1' - \frac{1}{2}(f' + |f'|)) = 0$, for any interval $I \subseteq [a,b]$, so that by Fact 1(iv), $f_1' - \frac{1}{2}(f' + |f'|) = 0$, or $f_1' = \frac{1}{2}(f' + |f'|)$ a.e. Similarly, we obtain $f_2' = \frac{1}{2}(f' - |f'|)$ a.e. This yields

$$f' = f_1' + f_2' \quad \text{a.e.} \tag{10}$$

But then, we have $(f - (f_1 + f_2))' = 0$ a.e., where $f - (f_1 + f_2)$ is AC, so that by the second consequence above, $f - (f_1 + f_2)$ is a constant function. If for some constant L, $f(x) - (f_1 + f_2)(x) = L$, for all $x \in [a,b]$, then $L = f(a)$, since $f_1(a) = f_2(a) = 0$. Therefore $(f_1 + f_2)(b) = f(b) - f(a)$. Hence, applying (9) for f_1 and f_2 on (10) by the first consequence, gives

$$\begin{aligned}\int_a^b f'(x)\,dx &= \int_a^b f_1'(x)\,dx + \int_a^b f_2'(x)\,dx \\ &= f_1(b) - f_1(a) + f_2(b) - f_2(a) \\ &= (f_1 + f_2)(b) \\ &= f(b) - f(a)\end{aligned}$$

as is asserted.

FTC $\Rightarrow$ CVT 5. Since G and $G \circ f$ are AC, by the FTC, both $G' = g$ and $(G \circ f)'$ are integrable, and

$$\int_a^b (G \circ f)'(x)\,dx = (G \circ f)(b) - (G \circ f)(a) = G(f(b)) - G(f(a)) = \int_{f(a)}^{f(b)} g(x)\,dx.$$

The proof would be finished if we can show that

$$(G \circ f)'(x) = g(f(x))f'(x) \tag{11}$$

for a.e. $x \in [a,b]$. To facilitate this, we partition $[a,b]$ into the set P and Q, where

$$P := \{x \in [a,b] : G'(f(x)) \text{ exists, and } G'(f(x)) = g(f(x))\}$$

and $Q := [a,b] \setminus P$. Let $C := \{x \in [a,b] : (G \circ f)'(x) \text{ and } f'(x) \text{ both exist}\}$. Since f and $G \circ f$ are AC, $m([a,b] \setminus C) = 0$. Thus, to show that (11) holds a.e. on $[a,b]$, it suffices to show that it holds a.e. on both $P \cap C$ and $Q \cap C$.

If $x \in P \cap C$, since $f'(x)$, $G'(f(x))$, and $(G \circ f)'(x)$ all exist, and $G'(f(x)) = g(f(x))$, it follows from the chain rule that $(G \circ f)'(x) = G'(f(x))f'(x) = g(f(x))f'(x)$, so that (11) holds.

We now show that (11) holds a.e. on $Q \cap C$. This can be done if we can show that $f'(x) = 0 = (G \circ f)'(x)$, so that (11) holds, for a.e. $x \in D := Q \cap C$. Notice that $0 \leq m(f(D)) \leq m(f(Q)) = 0$, since G' exists and equals g a.e. on J. Hence $m(f(D)) = 0$. At this stage we need the following lemma, where it is important to emphasize that we assume only the FTC and the properties listed in Fact 1 (as consequences of the monotone convergence theorem), and avoid using (4).

Lemma 2. *Suppose that f is* AC. *Given a measurable set $X \subseteq [a, b]$, then for any $\epsilon > 0$ there exists $k_\epsilon > 0$ such that*

$$\left| \int_X f' \right| \leq \epsilon + k_\epsilon \; m\left(f(X)\right). \tag{12}$$

Proof. Given any $\epsilon > 0$. Since f is AC, there exists $\delta_1 > 0$ such that if $\{[u_i, v_i] : 1 \leq i \leq s\}$ is any finite collection of pairwise non-overlapping subintervals and $\sum_{i=1}^{s} |v_i - u_i| < \delta_1$, then

$$\sum_{i=1}^{s} |f(v_i) - f(u_i)| < \frac{\epsilon}{3}. \tag{13}$$

By Fact 1(iii), there exists δ_2 such that for any open set $Y \supseteq X$, if $m(Y \setminus X) < \delta_2$, then $\left|\int_X f'\right| \leq \frac{\epsilon}{3} + \left|\int_Y f'\right|$. Set $\delta := \min\{\delta_1, \delta_2\}$. Choose an open set $X_0 \supseteq X$ such that $m(X_0 \setminus X)) < \delta/2$. Since $m(X_0 \setminus X)) < \delta_2$, it follows that

$$\left| \int_X f' \right| \leq \frac{\epsilon}{3} + \left| \int_{X_0} f' \right|. \tag{14}$$

Now choose a closed set $X_1 \subseteq X$ such that $m(X \setminus X_1) < \delta/2$. Write $X_0 = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint open subintervals. Then $X_0 \setminus X_1 = \bigcup_{n=1}^{\infty}(I_n \setminus X_1)$, so that

$$m\left(\bigcup_{n=1}^{\infty} (I_n \setminus X_1)\right) = m(X_0 \setminus X_1) = m(X_0 \setminus X) + m(X \setminus X_1) < \delta \leq \delta_1. \tag{15}$$

As $I_n \setminus X_1$ is open, write $I_n \setminus X_1 = \bigcup_{i=1}^{\infty} J_i^{(n)}$, where $\{J_i^{(n)}\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint open subintervals. Let $u_i^{(n)} < v_i^{(n)}$ be two points in the closure $\overline{J_i^{(n)}}$ of $J_i^{(n)}$ where f attains the maximum and the minimum value on $\overline{J_i^{(n)}}$. If $n_1 \neq n_2$, then $I_{n_1} \cap I_{n_2} = \emptyset$, and hence $\overline{J_i^{(n_1)}}$ and $\overline{J_i^{(n_2)}}$ are not overlapping. Thus, for arbitrary k and l, the collection $\left\{\overline{J_i^{(n)}} : 1 \leq i \leq l,\ 1 \leq n \leq k\right\}$, hence the collection

$$\left\{[u_i^{(n)}, v_i^{(n)}] : 1 \leq i \leq l,\ 1 \leq n \leq k\right\}$$

is a finite collection of pairwise non-overlapping subintervals. Since, by (15),

$$m\Big(\bigcup_{n=1}^{k}\bigcup_{i=1}^{l}[u_i^{(n)}, v_i^{(n)}]\Big) \le m\Big(\bigcup_{n=1}^{k}\bigcup_{i=1}^{l} J_i^{(n)}\Big) \le m\Big(\bigcup_{n=1}^{\infty}(I_n \setminus X_1)\Big) < \delta_1,$$

it follows from (13) that

$$\sum_{n=1}^{k}\sum_{i=1}^{l} m(f(J_i^{(n)})) = \sum_{n=1}^{k}\sum_{i=1}^{l} |f(v_i^{(n)}) - f(u_i^{(n)})| < \frac{\epsilon}{3}.$$

Letting $l \to \infty$, we have for arbitrary k,

$$\sum_{n=1}^{k}\sum_{i=1}^{\infty} m(f(J_i^{(n)})) \le \frac{\epsilon}{3}. \tag{16}$$

Let $I_n := [u_n, v_n]$. By the FTC, we have

$$\begin{aligned}\left|\int_{I_n} f'\right| = |f(v_n) - f(u_n)| &\le m(f(I_n)) \qquad (17)\\ &\le m(f((I_n \setminus X_1) \cup X_1))\\ &\le m(f(I_n \setminus X_1)) + m(f(X_1))\\ &\le m(f(I_n \setminus X_1)) + m(f(X_1)). \qquad (18)\end{aligned}$$

Note that (17) and the absolute continuity of f imply

$$\sum_{n=1}^{\infty}\left|\int_{I_n} f'\right| \le \sum_{n=1}^{\infty} m(f(I_n)) < \infty,$$

and so we can choose some $k_\epsilon \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty}\left|\int_{I_n} f'\right| < \frac{\epsilon}{3} + \sum_{n=1}^{k_\epsilon}\left|\int_{I_n} f'\right|. \tag{19}$$

By (16), we have

$$\begin{aligned}\sum_{n=1}^{k_\epsilon} m(f(I_n \setminus X_1)) &= \sum_{n=1}^{k_\epsilon} m\Big(f\Big(\bigcup_{i=1}^{\infty} J_i^{(n)}\Big)\Big) \qquad (20)\\ &= \sum_{n=1}^{k_\epsilon} m\Big(\bigcup_{i=1}^{\infty} f\Big(J_i^{(n)}\Big)\Big)\\ &\le \sum_{n=1}^{k_\epsilon}\sum_{i=1}^{\infty} m(f(J_i^{(n)})) \le \frac{\epsilon}{3}.\end{aligned}$$

Then (14), (18), (19), and (20), along with the fact that $X_1 \subseteq X$, give

$$\begin{aligned}\left|\int_X f'\right| \le \frac{\epsilon}{3} + \left|\int_{X_0} f'\right| &= \frac{\epsilon}{3} + \left|\int_{\bigcup_{n=1}^{\infty} I_n} f'\right| \\ &\le \frac{\epsilon}{3} + \sum_{n=1}^{\infty}\left|\int_{I_n} f'\right| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sum_{n=1}^{k_\epsilon} m(f(I_n \setminus X_1)) + k_\epsilon m(f(X_1)) \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} + k_\epsilon m(f(X)) \\ &\le \epsilon + k_\epsilon m(f(X))\end{aligned}$$

as is asserted.

Now, substituting $D^+ := \{x \in D : f'(x) > 0\}$ and $D^- := \{x \in D : f'(x) < 0\}$ respectively for X in (12), while noting $m(f(D)) = 0$ and ϵ is arbitrary, yields $\int_{D^+} f' = 0 = \int_{D^-} f'$, so that by Fact 1(v), $f' = 0$ a.e. on both D^+ and D^-, hence on D.

Next, since $m(f(D)) = 0$, the absolute continuity of G implies that $m(G(f(D))) = m((G \circ f)(D)) = 0$. Thus, applying similar argument as above, since $G \circ f$ is AC, it follows that $(G \circ f)' = 0$ a.e. on D. This completes the proof. ■

CVT 5 ⇒ CVT 2. It is enough to show the case that $f' \ge 0$ a.e., as the other case is similar. Set $G(z) := \int_{f(a)}^{z} g(y)\, dy$, $z \in J$. Note that here we have the FTC, as we have shown earlier that CVT 5 ⇒ FTC. Since G is AC, by the FTC, G' is integrable, and $\int_{f(a)}^{z} G'(y)\,dy = G(z) - G(f(a)) = G(z) = \int_{f(a)}^{z} g(y)\,dy$. Thus $\int_{f(a)}^{z}(G'(y) - g(y))\,dy = 0$, for every $z \in J$. It follows from Fact 1(iv) that $G' - g = 0$, thus $G' = g$, a.e. on J. Now, since $f' \ge 0$, it follows from the FTC that $\int_{x_1}^{x_2} f'(x)\,dx = f(x_2) - f(x_1) \ge 0$ if $x_2 > x_1$, showing that f is non decreasing. Then, since G and f are AC, so is $G \circ f$, and therefore the assertion follows from CVT 5.

CVT 5 ⇒ CVT 3. As in the previous part, set $G(z) := \int_{f(a)}^{z} g(y)\, dy$, $z \in J$. We have shown there that the FTC and Fact 1(iv) imply that $G' = g$ a.e. on J. Since g is bounded, G is Lipschitz. Now, since f is AC, so is $G \circ f$, and hence the assertion follows from CVT 5.

CVT 3 ⇒ CVT 4. It suffices to reduce the case that $g \ge 0$. Set for each n a function g_n, with $g_n(x) := g(x)$ if $g(x) \in [0, n]$, and $g_n(x) := 0$ otherwise. Then by CVT 3,

$$\int_a^b g_n(f(x))f'(x)\,dx = \int_{f(a)}^{f(b)} g_n(y)\,dy. \tag{21}$$

Since (g_n) is monotonic and $g_n \to g$ pointwise, it follows from the monotone convergence theorem that

$$\int_{f(a)}^{f(b)} g_n(y)\,dy \to \int_{f(a)}^{f(b)} g(y)\,dy. \tag{22}$$

By letting $E^+ := \{x \in [a,b] : f'(x) \geq 0\}$ and $E^- := \{x \in [a,b] : f'(x) \leq 0\}$, the monotone convergence theorem, again, gives

$$\int_a^b g_n(f(x))(f'\chi_{E^+})(x)\,dx \to \int_a^b g(f(x))(f'\chi_{E^+})(x)\,dx$$

and

$$\int_a^b g_n(f(x))(f'\chi_{E^-})(x)\,dx \to \int_a^b g(f(x))(f'\chi_{E^-})(x)\,dx,$$

where χ_{E^+} and χ_{E^-} denote the characteristic function on E^+ and E^- respectively, so that

$$\int_a^b g_n(f(x))f'(x)\,dx \to \int_a^b g(f(x))f'(x)\,dx. \tag{23}$$

The assertion then follows from (21), (22) and (23). ■

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