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GEOMETRICAL VERSUS ANALYTICAL APPROACH IN PROBLEM SOLVING—AN EXPLORATORY STUDY

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Abstract. In this study we analyse the geometrical visualization as a part of the process of solution. In total 263 students in the first year of study at three different universities in three different countries (Poland, Slovakia and Spain) were asked to solve four mathematical problems. The analysis of the results of all students showed that geometrical visualization for problems where there is a possibility to choose different ways of solving is not as frequent as one would expect even if the problem is hardly solvable without it. Especially Spanish students prefer to solve the problem analytically. Visual reasoning is mostly regarded as an intuitive, preliminary stage in the reasoning processes and nothing needs to be done to develop it. On the basis of our results we consider it necessary to regain the use of geometry in the classroom and encourage visualization, the use of the figure, the physical and spatial intuition.

MathEduc Subject Classification: C30

MSC Subject Classification: 97C30, 97G99

Key words and phrases: Geometrical arguments; analytical arguments; visualization; problem solving.

1. Introduction

Several studies on cognitive process differences show the relationships among the problem-solving process, the success in it and the spatial abilities of students [8,12]. The most common classification of these differences consists of the following three items: the analytic type, based on reasoning, abstract formalisms and ideas; the geometrical type, trying to solve the problem using pictorial-visual elements; and the third one is the type that uses harmonic form balanced between these two types (see [1,10,14]). Other authors established similar classifications [11], however all these experts fail to determine whether any of these types is the most successful.

We analyse the geometrical visualization and geometrical intuition as part of the process of the construction of knowledge. For this reason, we do not use the classical tests of visualization like the Paper Folding Test [2] or the Vandenberg Rotations Test [19]. In addition, research on spatial intelligence [4] indicates that this type of test has the disadvantage that it provides results strongly dominated by the previous practice with similar tests by the student.

There are also examples in specific contexts, such as in medicine [5–7], meteorology [16] or mechanics [9], where the cognitive processes of spatial intelligence is studied in nonrealistic situations and contextualized visualization. Therefore, we consider the mathematical problem solving process as a contextualized and close to real situation in order to reproduce the process of building knowledge in mathematics. Increased knowledge of the cognitive profile will facilitate the adaptation of classroom activities by the teacher.

This work continues the first approach to the subject [17]. In that paper the authors focus their attention on Spanish students. In this paper we add results obtained in Poland and Slovakia by students with a comparable background.

In order to evaluate the results of the mathematical problem solving process we firstly focus on the type (analytic or geometrical) of arguments used by students. Then we try to find out whether good geometrical abilities have a significant influence and improve the results of students. We recognize the theoretical framework proposed by Van Hiele [18] but remark that our study cannot be embedded in it—Van Hiele's work categorizes the development levels of students' thinking about geometry while our work looks at their style in problem solving.

Comparing results from countries in which there are some differences in school curriculum (implemented, e.g., in textbooks, in the teaching styles and traditions) may show that the teaching process can influence the cognitive process of acquiring knowledge.

2. Methodology

The study sample consists of 263 students from three different universities. They have been organized in five groups according to fields of study. The groups are Technical Architecture (University of A Coruña, Spain) – 50 students, Computer Engineering (University of A Coruña, Spain) – 50 students, Educational Studies (University of A Coruña, Spain) – 50 students, Applied Informatics (Gdansk University of Technology, Poland) – 56 students and Informatics and Applied Mathematics (Pavol Jozef Šafárik University in Košice, Slovakia) – 57 students. When tested, the students were all in their first year of study. The Spanish students from Technical Architecture and Computer Engineering had prior training in drawing in high school.

The aim of our study is to collect information about the geometrical visualization as part of the process of solution. The students were asked to solve four mathematical problems. These problems were chosen after several informal discussions with high school teachers. From the written solutions we identify the way of thinking during the mathematical problem solving process by categorizing the solutions as being of geometrical or of analytical type. Participation was voluntary and answers were anonymous. The maximum time allowed was one hour.

The selected problems allowed the students to choose different ways of solving them. The following items were considered:

- We classify the method of solution as geometrical if the student solves the problem using a drawing, usually accompanied by an argument. Otherwise we classify the method of solution as analytic.
- We evaluate the method of solution even if the answer is wrong.

- We are interested in global results of each group.

3. The problems

We now briefly describe each of the problems and explain our approach to categorize the students' solutions either as geometrical or analytical.

P1: *Let us consider the following system of equations*

$$y = x^2$$
$$(x - a)^2 + y^2 = 1$$

For which values of the parameter a the system has 0, 1, 2 solutions?

The problem stated in algebraic terms invite an algebraic solution. Therefore, it is not surprising that many students approach the problem analytically without realizing a geometrical interpretation. The intention was to find out whether students who struggle with the analytic method change to geometrical arguments as an alternative, which would indicate that they have the insight that the problem has a geometrical facet too.

We evaluate the solution as geometrical if the student identifies the geometrical interpretation of at least one equation. That means he/she realized that geometrical arguments can be used to solve the problem. Figure 1 shows the geometrical solution which we considered to be correct.

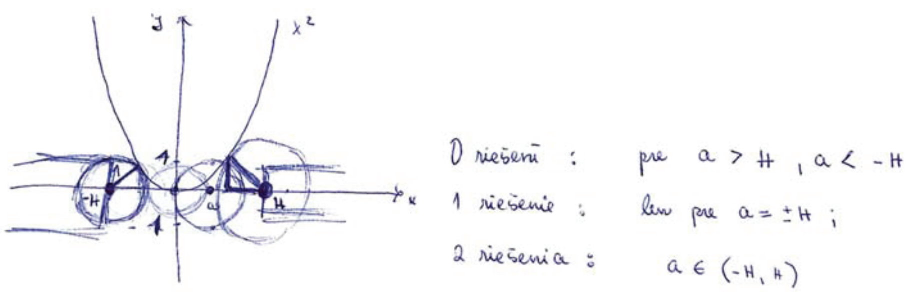
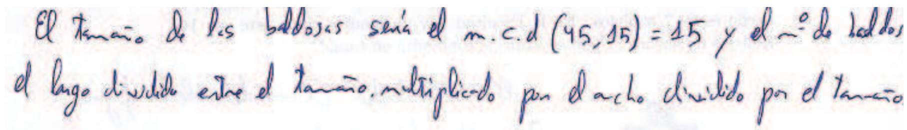


Fig. 1 – Student's solution of problem P1

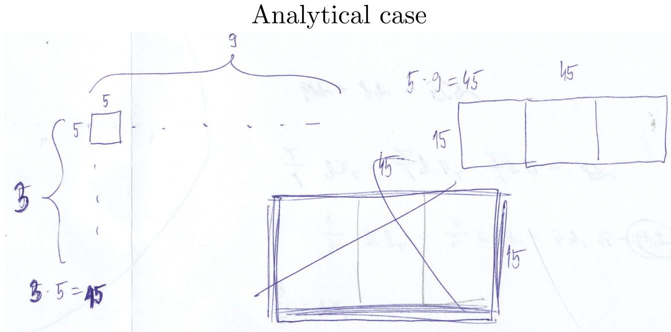
P2: *An entrepreneur is going to change the floor in his warehouse. He wants to put square tiles of equal size, the largest possible, without cutting any of them. The floor of the warehouse is* 45 m *long and* 15 m *wide. How long is the side of such a tile (in meters)?*

This problem is simple and includes an explicit reference to physical reality. The result of the problem is surprising—tile has 15 meters long side. Our interest here was to find out if students translate the problem into a geometrical one, either by making a suitable drawing or by convincing us that they could "see" and benefit from it without drawing it.

If a student gives a direct answer we treat it as an analytical answer, because if the student did not use the drawing for visualization this means that it is not fundamental for him. We also accept the analytical solution based on the concept of greatest common divisor, while it will be a geometrical answer if student makes a drawing which is a base for calculations performed later or is used to check the calculations carried before (Fig. 2).



Analytical case



Geometrical case

Fig. 2 – Geometrical and analytical solution to P2

P3: *The figure (Fig. 3) shows a model consisting of black and white tiles. Its width is* 7 *tiles. Consider the analogous model with a width of* 149 *tiles. How many tiles are needed to make this model?*

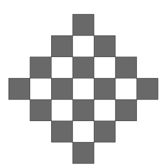
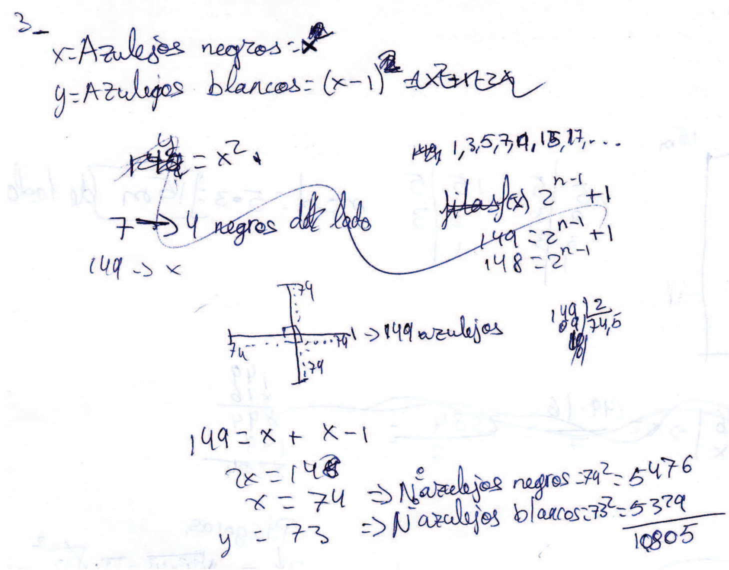


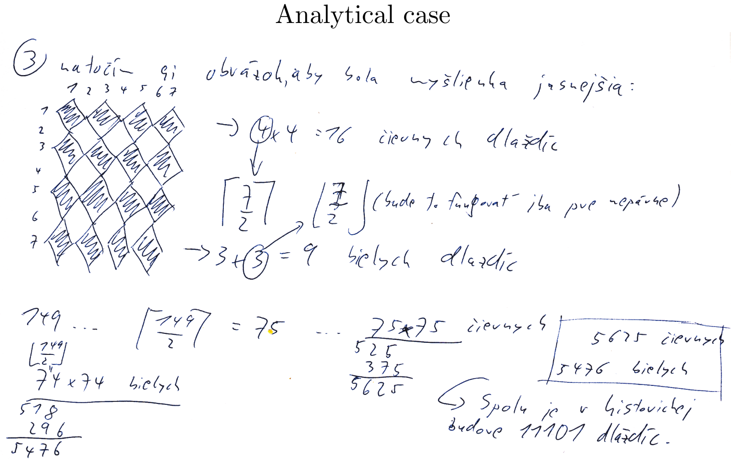
Fig. 3 – Model of black and white tiles with width 7 tiles

This problem already contains a drawing, which suggests using geometrical arguments. Again, the problem was chosen to see if students would choose analytic arguments or continue to work with the drawing and take into account the geometrical properties of the presented figure (Fig. 3).

If it is clear from the solution that a student worked with the drawing and studied its geometrical properties we consider the solution as geometrical. If the student worked only with the number of tiles and was looking for a pattern we consider the solution as analytic (Fig. 4).



Analytical case



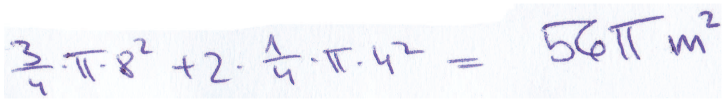
Geometrical case – Student used rotation of original drawing

Fig. 4 – Geometrical and analytical solution to P3

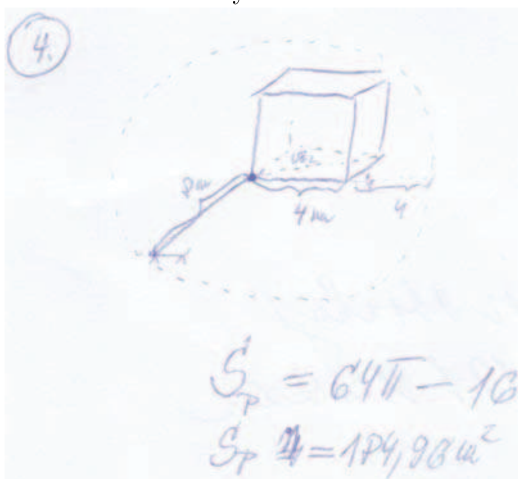
P4: *A dog is tied on a leash of a length* 8 m *in the corner of a house. The house has the shape of a square with a side of* 4 m. *What is the surface area in which the dog can move?*

It is natural to use geometrical arguments for this problem. Its difficulty is in the need to imagine spatial relations. That is why it is important for us to find out how many correct answers are obtained by each group of students.

We consider the solution of the problem to be geometrical if the student visualized the situation and then solved the problem. The solution without any



Analytical case



Geometrical case – bad spatial insight

Fig. 5 – Geometrical and analytical solution to P4

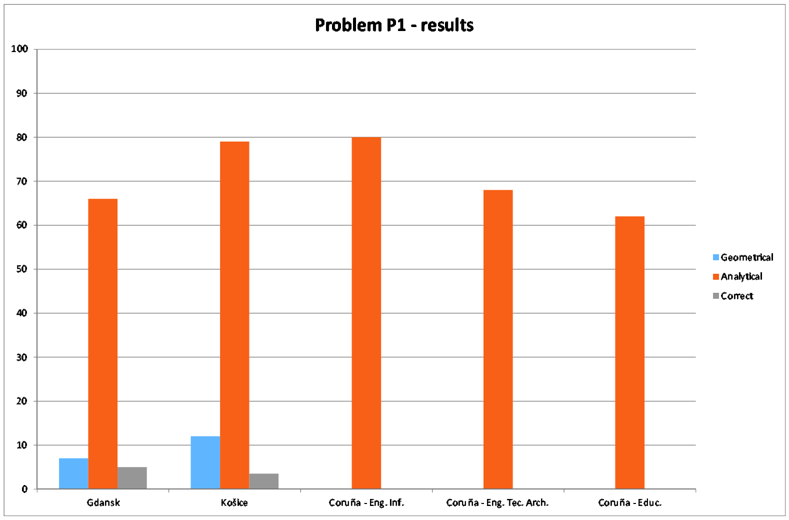
visualization we consider to be analytic (Fig. 5).

4. Results

Below is a compilation of the results of students. In the following four graphs different solution strategies are shown in different colours: the percentage of students who solved the problems in an analytical way is shown in red and those in geometrical way is shown in blue. The vertical axis represents the percentage of students and the horizontal one, the different fields of study. Total percentage of correct answers is indicated in gray.

Discussion of the results of Problem 1

Problem 1 was solved by most students in an analytical way. The students gave up after they found out that they cannot solve the problem using the standard method of solution for a system of equations. Students could have better insight into the problem if they started to solve the problem geometrically. This way of solution was chosen by only 11 students, 7 from Slovakia and 4 from Poland. Students who solved the problem geometrically were of two types. The first type saw at the beginning that the first equation is a parabola and the second is a circle with the centre on the x-axis. They realized that the parameter a translates the position of circle along the x-axis. Other students started to solve the problem analytically and when they realised that it does not lead to the goal they changed the strategy. Two of them made some conclusions.



Discussion of the results of Problem 2

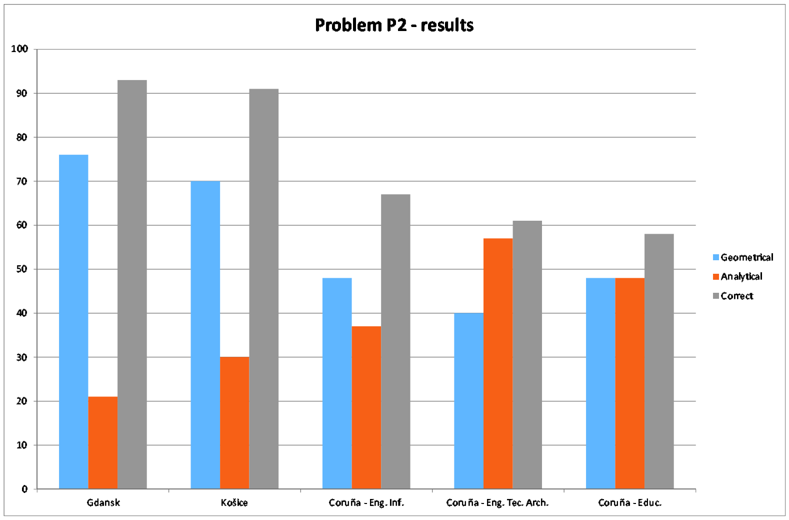
Of the combined group, Problem 2 was solved by most students geometrically. Many students drew a picture which reflects the statement of the problem and it helped them to solve it. In most cases it was a rectangle with an indication of its length and width (15 m and 45 m). They divided this rectangle into three squares with side lengths 15 m. Some of the students started with a smaller square – 5 m in length. Most of the students solved the problem correctly. Students from Poland and Slovakia showed a preference to solve the problem geometrically in contrast to Spanish students who showed a preference for an analytical solution.

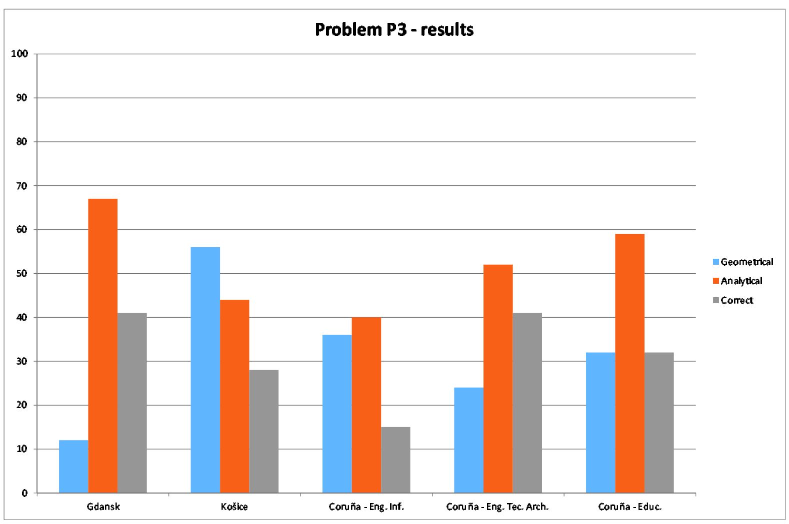
Discussion of the results of Problem 3

To solve Problem 3, the combined group of students used mainly analytical methods although the formulation of the problem contains a geometrical figure, which makes it easier to visualize the problem. Most of them tried to use properties of an arithmetic sequence to solve the problem. More than 50 percent of students from Slovakia attempted to solve the problem geometrically. Some of these students moved a part of the picture in order to create a rectangle. Other students used a rotation of the original drawing. Many of the students who used geometry, did not see the geometrical properties of the picture correctly.

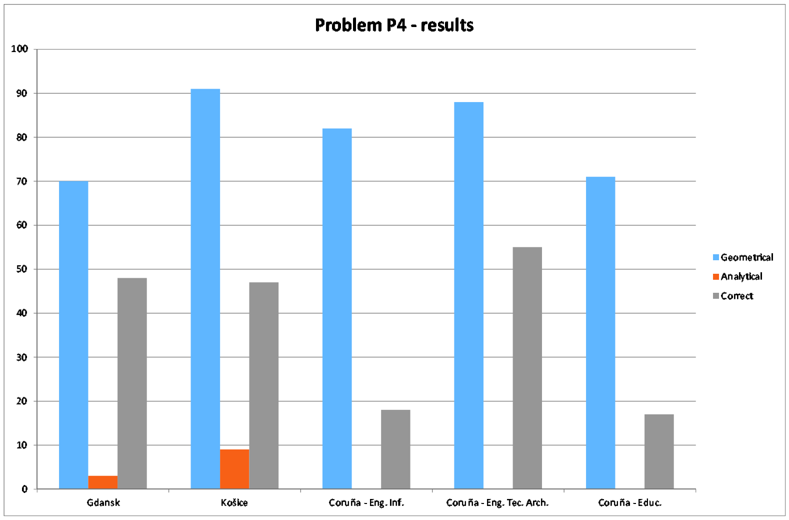
Discussion of the results of Problem 4

Finally, Problem 4 was solved generally in a geometrical way. The formulation of the problem (with references to everyday life and concepts which are often used in geometry problems like square and surface area) guided students to use geometry. The results show that only the students of Technical Architecture achieved a success rate of more than fifty percent. This result may indicate the lack of spatial ability





of the other students. Only 7 students solved the problem without picture, 2 of them correctly. These students wrote the formula for the surface area directly from the text, perhaps a mental picture of the situation.



5. Summary

All the results are summarized in the table presented on the next page. In this table the letter G refers to geometrical and the letter A to analytical solution. All numbers are percentages of students from the group marked in the first column. The column Correct shows the percentage of students with correct solutions of those who solved the problem geometrically and analytically.

From this table we can see that students who solved the problems geometrically have better results than students who solved the problems analytically. The only exception is P3, in which students from Computer Engineering in Spain and students from Poland who offered an analytical solution have better results than the students from the same group who offered geometrical solution.

6. Conclusions

Again we emphasize that the mathematical background of the 263 students who participated is similar. But according to the field of study there may be some differences between students. That is why we decided to evaluate students from Spain in three groups.

If we analyse the results of all Spanish students, we can conclude that the general tendency among these students is to solve problems in an analytical way. There are no significant differences between the groups of students: students of Computer Engineering are the most "geometrical" followed closely by the students of Technical Architecture. We note that in Problem 4 more students of Technical

		P1		P2		P3		P4	
		Total	Correct	Total	Correct	Total	Correct	Total	Correct
Polish students (56)	G	7	75	76	98	12	29	70	76
	A	66	0	21	83	67	55	3	0
Slovak students (57)	G	12	29	70	98	56	31	91	48
	A	79	0	30	76	44	24	9	40
Spain students, Comp.Eng. (50)	G	0	0	48	56	36	0	82	21
	A	80	0	37	68	40	10	0	0
Spain students, Arch. Tecn. (50)	G	0	0	40	79	24	49	88	58
	A	68	0	57	40	52	28	0	0
Spain students, Educat. (50)	G	0	0	48	60	32	42	71	23
	A	62	0	48	41	59	18	0	0

Architecture achieved the correct solution than students from the two other Spanish courses of study. Where the nature of the problem is geometrical, Technical Architecture students have a greater ability and a better use of geometrical techniques than the others. This difference is not visible when they solve the other problems; perhaps because of during the earlier education such methodologies were not used.

The results also show that Polish and Slovak students have a higher level of geometrical skills and a higher percentage of them solved the problems correctly than the Spanish students. The dominance of the geometrical profile of Polish students has an exception in problem 3. The possible reason may be that the picture, which is part of the text of the problem, is a rotated square, making it difficult to choose a geometrical method for its solution.

This research does not try to generalize the results to other students, other courses or other countries. It concerns a study on a sample of students from the University of A Coruña, P. J. Šafárik University in Košice and the Gdańsk University of Technology. We want to draw attention to the potential problems associated with the current trends in the teaching of mathematics. Since the middle of the last century the teaching of mathematics in Spain leaned further to logical rigor, emphasizing the cultivation of algebra and spatial intuition consequently almost disappeared in the classroom. The easy solution of exercises using geometry was replaced by complex and tautological sentences, so the students now have a lack of spatial intuition caused by the removal of the geometry from basic education programs. On the other hand, there is a longstanding tradition to develop geometrical abilities of students in Slovakia and Poland although in the last years the same tendency as in Spain is observed. Teachers have more freedom to choose the content of the mathematical lesson and they prefer to teach algebra or geometry without pictures (compute the measure of something). The ability to solve geometrical problems correctly gets worse.

We consider it necessary to regain the use of geometry in the classroom and encourage visualization, the use of the figure, the physical and spatial intuition. If the previous mathematical training of students with geometrical skills for solving problems is continued, the results in problems like the four we discussed would be better. It is also interesting to think about the necessity of adapting the university education to improve the geometrical skills, in line with what is showed in the works of Ridin and Sadler-Smith [15], Halford [3], Pitt-Pantazi and Christou [13]. If we analyse the results we can see that this problem is more pronounced in Spain.

The reasoning using visualization is often limited in mathematical classes. The reason of this can be that visualization is often seen as a student's inborn ability and the arguments based on visualization are usually considered to be less rigorous than analytical arguments. We believe that mathematical visualization is very important and that the skills of visualization can be developed during the process of education. Visualization is often an essential tool for solving geometrical problems.

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