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MATHEMATICAL PROBLEM POSING AS A LINK BETWEEN ALGORITHMIC THINKING AND CONCEPTUAL KNOWLEDGE

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Abstract. The paper proposes an idea of using problem posing as a link between conceptual and procedural mathematical knowledge. Two levels of conceptual understanding—basic and advanced—have been considered. Examples of the interplay between the two types of knowledge are presented. The paper is informed by the author's work with teacher candidates in different technology-enhanced mathematics education courses.

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Key words and phrases: problem solving; problem posing; technology; conceptual understanding; algorithmic thinking; reflective inquiry.

1. Introduction

This paper is motivated by William McCallum's talk [22] at the Mathematics Education workshop in Oberwolfach, Germany: *Mathematics in Undergraduate Study Programs: Challenges for Research and for the Dialogue Between Mathematics and Didactics of Mathematics*. One of the topics included in the talk highlighted the dichotomy of perspectives that mathematicians and mathematics educators have on the relationship between conceptual understanding and algorithmic thinking. This theme is not new, even within the community of mathematics educators only. The first publications concerned with the interplay between conceptual knowledge and procedural skills appeared in the early 1980s. At the preschool level of mathematics education, Gelman and Meck [13], based on their study of counting skills by preschoolers, argued that basic principles of counting have to be developed first in order to use counting as a skill, although the development of those principles does not imply that one has full conceptual understanding of ideas associated with counting at the level of abstraction (e.g., comprehending the infinity of the set of number names). These authors, following the studies by psycholinguists (e.g., [6]), made a distinction "between implicit and explicit knowledge of counting principles [similarly to how] young children are granted implicit knowledge of language rules well before they are said to have explicit knowledge of their grammar" [13, p. 344]. Indeed, just as a second grader (observed by the author), when measuring distance from a mark on the floor to the wall, intuitively puts a measuring tape perpendicular to the wall, "the child conjugates and declines correctly but without realizing it" [37, p. 109].

Freudenthal [12], apparently favoring learners' conceptual understanding in mathematics versus procedural performance, warned against nondiscriminatory use of algorithmic procedures without conscious thought as "sources of insight can be clogged by automatisms" (p. 469). For example, an automatic counting of points on a circle, in the absence of acumen that the first point counted has to be singled out in order to have one-to-one correspondence between the points counted and the number names used, may lead to either a confusion or an erroneous result. Nonetheless, Haapasalo & Kadijevich [14] found that in the early mathematics education procedural skills typically enable concept development and suggested to "call this position *developmental approach*" (p. 147, italics in the original). Perhaps the above example of automatically counting points on a circle may be used as a counterexample to the universal acceptance of this approach without a rather obvious codicil that no procedural skill is possible without having some basic (or implicit, as Gelman and Meck [13] called it) conceptual understanding of the skill.

The advent of technology into the mathematics classroom has given strong impetus to the theme that highlighted different pedagogic approaches to the integration of the procedural and the conceptual. Assuming the duality between the use of technology with minimal thinking and the importance of thinking in support of technological developments, Nesher [24] argued that "what remains for us to do is to consider which algorithms we will want to use to free ourselves from thinking, and which can be best used in order to further our thinking and understanding" (p. 8). As technology became more sophisticated, Kaput [19] has made an argument for the need of disciplined inquiry into "the relation between procedural and conceptual knowledge, especially when the exercise of procedural knowledge is supplanted by (rather than supplemented by) machines" (p. 549). To this end, Peschek and Schneider [25] put forth the notion of outsourcing algorithmic skills to new digital technology such as CAS, something that, nonetheless "presumes thorough basic knowledge in mathematics; ... the willingness and the ability ... to be precise when formulating one's own questions ... in a form which can be interpreted by CAS" (p. 17). As noted by Kadijevich [18], the use of CAS has the potential to facilitate links between procedural skills and conceptual understanding because even relatively unsophisticated symbolic computations "require the user to think conceptually before a procedure is used" (p. 72). That is, human-computer interaction in the context of outsourcing symbolic calculations to a digital tool does require users' conceptual understanding of the procedures involved.

Rittle-Johnson, Siegler & Alibali [27] contributed to the debate on the hierarchy of the two types of knowledge with a proposal "that conceptual and procedural knowledge [in mathematics] develop iteratively, with increases in one type of knowledge leading to increases in the other type of knowledge, which trigger new increases in the first" (p. 346). These authors defined the notion of "problem representation as the internal depiction or *re-creation* of a problem" [27, p. 348, italics added] formed by a problem solver each time the problem is solved. That is, solution of a problem precedes its re-enacting in a problem-solving setting. One's ability to re-enact a problem brings to mind the notion of reflective inquiry [8]—a way of learning through which experience leads to the growth of knowledge and, vice versa, knowledge makes experience profound. In mathematics, reflective inquiry is, in essence, an act of posing a new problem. So, considering the relationship between procedural skill and conceptual knowledge as an iterative alliance leads to the interpretation of problem posing as a recurrent reflection on a solved problem through the (potentially) never-ending cycle "solve-reflect-pose". The goal of this paper is to suggest that the two types of knowledge, procedural and conceptual, can be connected through posing problems. The ideas of the paper are informed by the author's work with future schoolteachers in different mathematics education courses.

2. Problem posing in mathematics education

Educators have been using problem posing for quite a long time, seeing it not only as a teaching method but, more generally, as an educational philosophy. Focus on educational problem posing can be found in Montessori's approach to schooling (developed in Italy at the end of the 19th century), nowadays referred to as a student-centered classroom [17], when "children create their own curriculum materials" [34, p. 130] as part of "the idea that the child is not just a smaller version of the adult [and] that children should be free to choose their own work" (ibid, p. 48). Likewise, Freire [11] introduced the problem-posing concept of education as a method which "does not dichotomize the activity of the teacher-student: she is not "cognitive" at one point and "narrative" at another" (p. 80), arguing that "problem-posing education ... corresponds to the historical nature of humankind ... for whom looking at the past must only be a means of understanding more clearly what and who they are so that they can more wisely build the future" (p. 84). This position is in agreement with teaching and learning that encourage reflection on what was already done in order to formulate and then find answers to new queries. In particular, the importance of problem posing as a method of encouraging reflective inquiry as a critical element of mathematical thought was emphasized through the so-called Socratic seminars [5]—a learning model that encourages learners to generate questions.

In mathematics education, problem posing appeared in a variety of didactic forms [30] as a way of providing students with experience in exploring mathematical ideas, investigating conjectures, and solving a variety of problems relevant to students' interest and background through the reformulation of those already solved. Reformulation of a problem can be seen as Freire's [11] "looking at the past" educational philosophy to understand the nature of the original problem in order to use this understanding to "more wisely build the future" learning of mathematics. Mathematics begins with posing problems and it evolves, using terminology introduced by Vygotsky [36], from concrete activities expressed through the first order symbols (e.g., pictures, diagrams, and manipulative materials commonly used in the modern classroom) to abstract concepts expressed through the second order symbols (e.g., numeric equalities or inequalities).

Therefore, mathematical problem posing may be considered as a subjectoriented element of a more general educational philosophy and teaching method. Consistent with the observation that "activities are much more effective than conversations in provoking problems" [16, p. 4], it utilizes the primary character of the first order symbols that "correspond to the historical nature of humankind" [11, p. 84] versus the secondary nature of the symbolic description of those objects emerging from teacher's role as 'a more knowledgeable other'. In that way, the modern student develops "the ability to *decontextualize* [from the first order symbols] and *contextualize* ... in order to probe into the referents for the [second order] symbols involved" [7, p. 6, italics in the original].

One can also distinguish between two types of questions that can be formulated to become a mathematical problem [15]: questions requesting information (e.g., asking about the number of rectangles within an $n \times n$ checkerboard or the number of cubes within an $n \times n \times n$ cube) and questions requesting explanation of the observed phenomenon (e.g., asking to explain the fact established in response to the request for information that the number of the rectangles and the number of the cubes coincide). Helping teacher candidates to move from questions seeking information to those seeking explanation that are reflections on the former ones, fosters their conceptual understanding of a mathematical method that produced the information sought. Just as one can distinguish between the first order symbols and the second order symbols (following the above-mentioned Vygotskian characterization), one can describe mathematical questions in terms of the first order questions (seeking information) and second order questions (seeking explanation). As will be shown below, whereas the first order questions may be easier to answer procedurally by using the second order symbolism, the first order symbols enable the resolution of the second order questions at the concept level.

Nowadays, mathematical problem posing can be enhanced by technology [3]. The process of making technology work for problem posing can contribute to conceptual understanding via developing an algorithm for making a problem dependent on a parameter so that one has a family of problems each of which can be posed to a student. According to Dubinsky and Tall [10], the use of a computer to develop conceptual understanding of calculus should precede practicing algorithmic skills. Likewise, conceptual understanding of pre-calculus mathematics can be developed through a computational experiment [1]. Through the engagement into problem posing, one develops conceptual understanding, something that can be interpreted in terms of turning a doing into an undoing [21]; that is, to use a solved problem as a means of asking a question about conditions that made it solvable. In analyzing an arithmetic/algebraic problem posed, one has to make certain that it is numerically coherent [3] and decide under what conditions its numeric data provides a solution. Numeric data generated by a computer can be used to pose a problem with a single answer as well as multiple (correct) answers.

Teacher candidates need to possess conceptual understanding of a problem regardless it can be solved by applying a well-defined algorithm or computational capabilities of a digital tool. For example, when finding the greatest common divisor between 144 and 89 through the Euclidean algorithm (using either pencil and paper or a spreadsheet), one can recognize in the resulting sequence of remainders consecutive Fibonacci numbers and ask whether there is a connection between the two concepts. Whereas, in general, using the Euclidean algorithm does not necessarily lead to its conceptual understanding, this special case may motivate, as it always happens within the zone of proximal development [37], making connection between well-developed procedural skills and emerging conceptual knowledge. And it is up to the teacher to explain that because the quotient from the division of the larger to the smaller consecutive Fibonacci numbers is always equal to one, the Euclidean algorithm, in the case of the unit quotients, follows the rule of Fibonacci recursion. (As an aside, this connection is discussed in a course taught by the author for prospective elementary teachers). Therefore, one way to connect two types of knowledge is through the recourse to special cases or the use of counterexamples. In the context of problem posing, such understanding enables the formulation of similar problems that can be utilized in assessing students' problem-solving skills and/or conceptual understanding.

3. Context and numeric modeling as means of conceptual understanding

Rosnick and Clement [28] reported about students' inability to correctly translate a statement of a word problem into an algebraic equation. These authors presented an example when the statement "there are six times as many students as professors at a university" is translated into an equation as $6S = P$ and they treated this result as a misconception stemming from the absence of conceptual understanding of the symbols involved (S – students, P – professors). However, not until the equation $6S = P$ is used to answer a quantitative question about the number of students and professors that one can understand that something is not right with this translation. Of course, one has to understand the question in terms of the context: a university may not have more professors than students. At the same time, the statement "there are six times as many apples as bananas in the basket" when being translated into a symbolic relation between apples and bananas as $6A = B$, the result of numeric modeling of this relation cannot be verified contextually as the quantities of fruits can go either way. This points at the importance of semantic clarity in formulating word problems.

An equation alone as a model of a word problem does not necessarily provide evidence of conceptual understanding or a lack thereof. For example, one can select a mathematical model for a complex physical or biological phenomenon by using a linear differential equation. Solving the equation may lead to results that are in contradiction with a physical or biological meaning of the phenomenon. Why do we need to write an equation if we then do noting with it? It is not until the equation is solved (whatever the problem-solving method) that one can achieve conceptual understanding of the relation among the symbols involved. In other words, procedural knowledge (which includes a skill in solving an equation) can facilitate conceptual understanding of a problem provided that its syntactic complexity [29] is reduced to the minimum.

4. Two levels of conceptual understanding

Just as Gelman and Meck [13] distinguished between implicit and explicit counting principles, one can talk about two types of conceptual understanding—basic conceptual understanding (BCU) and advanced conceptual understanding (ACU). Problem solving requires some basic level of conceptual understanding of the situation, both in terms of the first order symbols and the second order symbolism. For example, by knowing that the number of objects in a set is one more than in another set, one may decontextualize from the concreteness of this information and describe the two sets through the symbols x and $x+1$. For that, one has to see the symbol x as a variable (unknown) quantity and $x+1$ as a process of increasing this quantity by one. In other words, de-contextualization requires one to see the first order symbols as a concept and to think about the second order symbols as a process [32].

BCU is necessary in order to activate problem solving. Nonetheless, in many cases, BCU is not sufficient for either solving a problem or finding an efficient solution. For example, in the case of counting, knowing number names and a way of establishing one-to-one correspondence between the names and counted objects (implicit counting principles) does not imply that one possesses conceptual understanding of the explicit counting principle concerning the fact that invariance of the one-to-one correspondence between the first and the second order symbols enables the efficiency of counting. Such invariance can be discovered with the help of 'a more knowledgeable other' and this practical experience can lead to the growth of knowledge which, in turn, brings an intelligent insight into the experience and, through the emergence of ACU, making it profound.

5. Solving a problem seeking information

As an illustration of the interplay between BCU and ACU, consider the following slight modification of the problem discussed by McCallum [22] though under a different angle.

Problem 1. *The sum of three consecutive natural numbers is equal to 81. Find the numbers.*

This problem seeks information about certain type of integers three of which have a given sum. Its algebraic solution requires some BCU, namely, that any three consecutive integers form an arithmetic progression with difference one. Thus, the three integers can be written as x, $x+1$, and $x+2$ from where the equation $x+(x+1)+(x+2)=81$, then the value $x=26$, and, finally, the triple of integers $(26, 27, 28)$ follow.

Problem 1 can be solved differently, in a purely arithmetical way, without any explicit use of algebra. Instead, one needs to possess ACU of the problem structure[1].

[1]Such was the elementary school mathematics curriculum of the past when pupils were expected to solve word problems without explicitly using unknowns but rather, through asking

This understanding can be developed through requesting explanation as to why the sum of three consecutive terms of any arithmetic sequence (including natural, even, and odd number sequences) is a multiple of three. To this end, one can draw diagrams as a way of answering the second order questions by using the first order symbols. Consider the diagrams of Fig. 1 and Fig. 2. They show, respectively, that any 3-step staircase representing the sums of three consecutive natural numbers and, in general, integers in arithmetic progression can be rearranged into a 3-layer rectangular podium regardless of the length x of the upper step and, therefore, without loss of generality, x may be replaced by 1. That is, one can perceive abstract symbol x as a concrete (particular) concept embedded into the context of straightening out a 3-step staircase of a special type. This, of course, requires the ability to contextualize by probing into the referents provided by the first order symbols.

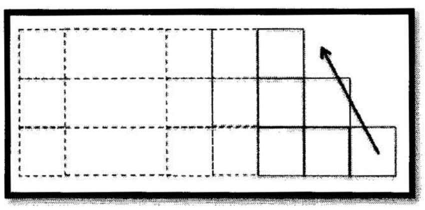


Fig. 1. Turning a 3-step staircase into a $3 \times (x+1)$-rectangle

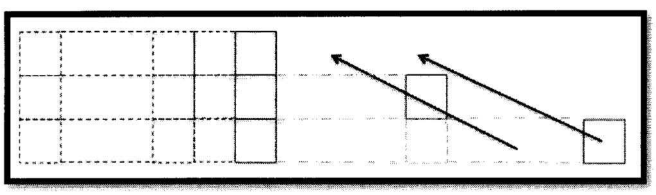


Fig. 2. Turning a 3-step staircase into a $3 \times (x+d)$-rectangle

In that way, ACU developed through the interplay between the second order questions and the first order symbols makes the corresponding solution purely arithmetical: divide 3 into 81 to get 27 which then has to be diminished and augmented by one yielding 26 and 28, respectively. That is, on the one hand, in order to answer a question that seeks information about three consecutive integers with a

conceptual questions to be answered in a purely numerical way. This kind of problem was famously described by Tchekoff [33, p. 70] in a story *Tutor*: "If a merchant buys 138 yards of cloth, some of which is black and some blue, for 540 roubles [*sic*], how many yards of each did he buy if the blue cloth cost 5 roubles [*sic*] a yard and the black cloth 3?" To solve this problem (something that the tutor could not do), one can partition $138 = 100 + 38$ as a guess (BCU) and then proceed using ACU that the difference between the actual and assumed payments has to be a multiple of the difference in prices for a yard of blue and a yard of black cloth. Indeed, one can see that $540 - (100 \cdot 3 + 38 \cdot 5) = 50$, $5 - 3 = 2$, $50 : 2 = 25$. This understanding makes it possible to offset the original guess (which may be any additive partition of 138 in two integers) to get the right partition through subtraction and addition: $100 - 25 = 75$, $38 + 25 = 63$.

given sum (the first order question), one transforms BCU of how these integers are related into an algebraic equation which yields a solution through a well-defined algorithm. Yet numeric information so obtained without reflective inquiry into why the algorithm works does not necessarily help one to develop ACU. On the other hand, by answering a question requesting explanation of the process through which numeric information was obtained, brings both an efficient numerical algorithm and ACU.

6. Problem posing leads to conceptual knowledge and collateral learning

ACU can serve another purpose: it is necessary for posing a similar problem through reflecting on the one previously solved. Indeed, BCU may not be enough to pose a numerically coherent problem by altering data without reflecting on the data through answering the (self-posed) second order questions. For example, replacing 81 by 80 yields no solution. ACU suggests that in the context of consecutive natural numbers one only needs to substitute 81 by any multiple of three greater than 5, a property that the sum of any three consecutive integers possesses, to have a similar problem which is solvable in integers. But 81 is not just a multiple of three, it is an odd multiple of three. So, whereas the sum of three consecutive odd numbers is also an odd multiple of three, the sum of three consecutive even numbers is an even multiple of three. In that way, in posing Problem 1, the quadruple of data source that includes the sum of numbers in arithmetic progression, their quantity, difference, and the type of numbers (determined by the first number) were selected to be in a conceptual bond. Fig. 3 (where the letters N, O, and E stand, respectively, for natural, odd, and even numbers) shows how one can go beyond BCU and through the above-mentioned cycle of actions "solve-reflect-pose" develop ACU. The far-left diagram of Fig. 3 shows Problem 1, the next diagram shows that in the case of three integers being consecutive odd numbers (difference $d = 2$) the sum 81 may be preserved, but in the case of consecutive even numbers the sum has to be replaced by an even multiple of three. At the same time, unless a counterexample is provided, an alteration of data leading to a solvable problem may be accidental and it is not due to conceptual understanding.

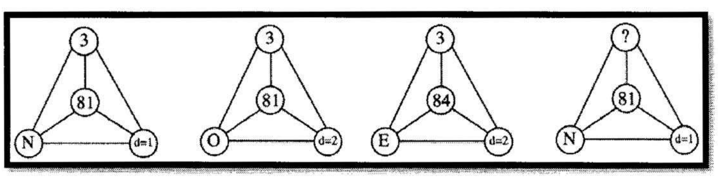


Fig. 3. The cycle "solve-reflect-pose" stemming from Problem 1

Finally, allowing the number of consecutive integers to vary (the far-right diagram of Fig. 3) leads to a problem with more than one correct answer all of which are shown in Fig. 4. The tetrahedron-like diagrams of Fig. 3 and Fig. 4 show how

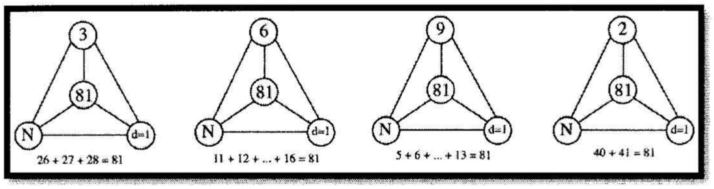


Fig. 4. Variation of the number of terms yields multiple solutions

the apex of the conceptual bond, the sum of numbers, holds the whole structure. Note that for any integer apex (a sum of an arithmetic series) one can always find three vertexes of the base enabling a numerically coherent problem. However, in the case when the sum is a prime number, only a trivial solution (partition into two consecutive integers) exists. The conceptualization of the last statement can be facilitated by the formula

$$S = \frac{2x + d(n-1)}{2}\, n, \tag{1}$$

an algorithm for finding the sum S of n terms of the arithmetic series with the first term x and difference d. Formula (1), in the case of prime S, yields $n = 2$ whence $x = \dfrac{S-d}{2}$. The last relation, obtained as a reflection on a formula defining a computational algorithm, shows that d can only be an odd number. For example, when $S = 79$ one possible solution is the representation $39 + 40 = 79$ related to $d = 1$. In particular, it follows that a prime number cannot be partitioned into three or more integers in arithmetic progression.

In connection with the last statement, a mathematical result due to Peter Gustav Lejeune Dirichlet, a notable German mathematician of the 19th century, is worth mentioning. Namely, if $d \geqslant 2$ and $x \neq 0$ are relatively prime, then there are infinitely many prime numbers among the terms of the sequence $x_n = x + dn$, $n = 0, 1, 2, \ldots$. One can see that whereas by adding three or more integers in arithmetic progression one cannot reach a prime number, there are arithmetic sequences that contain infinitely many primes. This is an example of how by appreciating the tenet "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time" [9, p. 49] one can become aware of mathematical results that belong to purely conceptual knowledge[2] through collateral learning. This kind of learning is especially important within a course for prospective teachers of mathematics enabling their familiarity with advanced subject matter context without the need to have the formal understanding of the context [31]. In that way, one develops connections among different mathematical concepts through the cycle "solve-reflect-pose" that includes the recurrent integration of procedural and conceptual knowledge. The above example of using formula (1), an element of mathematicians' tool kit, in developing ACU through problem posing may explain why, as it was noted by McCallum [22], many mathe-

[2] Indeed, Dirichlet provided no formula (algorithm) for generating prime numbers.

maticians believe that practicing algorithmic skills (that includes the creation and the use of formulas) can lead to conceptual understanding.

7. Conceptual bond as a tool for posing problems using technology

One can integrate the conceptual bond structured by formula (1) and an electronic spreadsheet in order to pose problems similar to Problem 1 through a computational experiment. This problem-posing experiment would not be possible without a reflective inquiry into the solved problem that seeks explanation of the success of problem solving. In particular, numeric data generated by the spreadsheet of Fig. 5 allows one to formulate

PROBLEM 2. *The sum of five integers forming an arithmetic sequence with difference seven is equal to 85. Find these integers.*

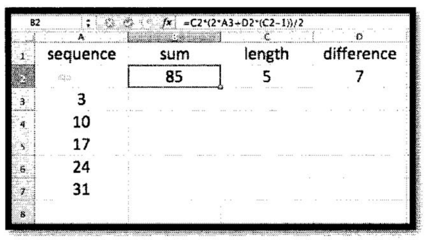


Fig. 5. Spreadsheet as a problem generator

Possessing ACU about a sum of an odd number of terms in arithmetic progression being a multiple of that number, one can divide 5 into 85 to get 17—the average of the five numbers, from where other four numbers result: 3, 10, 24, and 31. A multitude of similar problems can be posed using the spreadsheet shown in Fig. 5, where the entries of cells A3 (controlled by a slider enabling variation), C2, and D2 correspond to the vertexes of the base of the conceptual bond and the entry of cell B2, the sum, is computed through formula (1).

The tool can also be used to pose more complicated problems similar to the one represented through the tetrahedrons of Fig. 4. This requires certain level of procedural skills. To this end, setting $d = 1$ and $x+n-1 = m$ in formula (1) yields

$$(m + x)(m - x + 1) = 2S, \tag{2}$$

where x and m are, respectively, the first and the last terms of the sequence of consecutive natural numbers with the sum S. Formula (2) shows that any representation of its right-hand side as a product of two integers of different parity defines a distinct tetrahedron with the apex S. For example, when $S = 81$ there are four such products with the smaller factor representing the number of terms, namely, $162 = 81\cdot 2 = 54\cdot 3 = 27\cdot 6 = 18\cdot 9$. Note that each of the products includes an odd divisor of 81 except 1. That is, the number of such divisors (which, as shown in Fig. 6, can be found in the context of *Wolfram Alpha*) is equal to the number of ways an integer can be represented as a sum of consecutive natural numbers. Once again, the appropriate reflection on a formula, which was due to the application of a procedural skill, develops ACU of a rather complex mathematical structure referred to by Pólya [26] as a trapezoidal representation of an integer; in the case of 81 all trapezoidal representations are shown beneath the tetrahedrons of Fig. 4. Likewise, one can use conceptual knowledge developed through the unknown-free solution of the merchant problem in order to use a spreadsheet in posing similar problems.

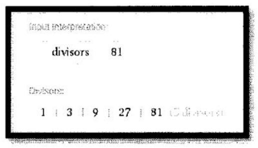
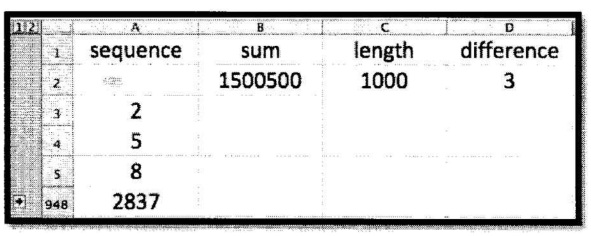


Fig. 6. Using *Wolfram Alpha* to find the divisors of 81

Finally, as a demonstration of Dirichlet's theorem mentioned in the previous section, one can also use this spreadsheet to generate arithmetic sequences of a sufficient length with the (relatively prime) first term and difference and check their terms for primality. The 946th term of the sequence with $x = 2$ and $d = 3$ is equal to 2837 (Fig. 7, cell A948). Uncomplicated preliminary analysis, requiring BCU, shows that 2, 3, 5, 7, and 11 are not the divisors of 2837. Once again, one can use the *Wolfram Alpha* to check this number for primality (Fig. 8), either asking the straightforward question (Fig. 8, left) or an indirect one. In both cases, the result confirms one's (procedurally supported) guess that 2837 might be a prime number. This is one of many examples of using integrated spreadsheets [22] as a paradigm of Type II technology application [20] in exploring mathematical ideas.



	A	B	C	D
1	sequence	sum	length	difference
2		1500500	1000	3
3	2			
4	5			
5	8			
948	2837			

Fig. 7. Is 2837 a prime number?

8. Concluding remarks

National Council of Teachers of Mathematics, the major professional organization of mathematics educators in North America, has strong belief that school

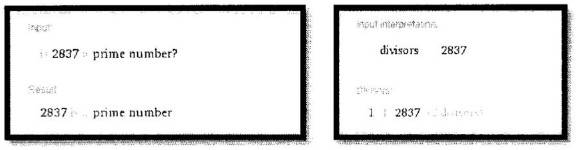


Fig. 8. *Wolfram Alpha* in deciding the primality of a number

mathematics curriculum must be "offering students opportunities to learn important mathematical concepts and procedures with understanding" [23, p. 3]. In another notable educational document in the United States, *Common Core State Standards*, mathematical understanding is referred to as "the ability to justify ... *why* a particular mathematical statement is true or where a mathematical rule comes from" [7, p. 4, italics in the original]. Whereas these documents do not attempt arranging mathematical understanding and procedural skills in the order of importance, there has been continuous debate regarding the hierarchy of the two types of mathematical knowledge. This paper was written to contribute to this debate. To this end, the paper proposed an idea of linking the two types of knowledge through problem posing, an activity that goes back to Montessori educational approach developed at the end of the 19th century and nowadays being elevated to the rank of educational philosophy [11].

Mathematical problem posing, appearing over the years in multiple didactic forms, can be seen as a recurrent reflection on a method used to solve a problem. In that sense, the proposed idea is congruent with the notion of iterative development of procedural and conceptual knowledge [27]. In the digital era, problem posing can be enhanced by computational experiments in search of friendly solutions. Such experiments allow for the integration of procedures that are at the core of computing and concepts that emerge from the reflection on the procedures.

In parallel with two types of semiotic mediation frequently employed by mathematics educators—questions with procedural/conceptual orientation [15] and symbols used as a frame of reference in support of contextualization and notation for decontectualization [36], as well as two types of technology application [20]—the paper introduced two types of conceptual understanding, basic (BCU) and advanced (ACU). It was argued that without BCU one is unable to initiate any problem-solving algorithm, even as simple as counting. However, once an algorithm starts working, ACU develops by trying to understand why the algorithm works, what its limitations are, and whether and how it can be applied in a similar situation.

The paper suggested that one way that algorithmic thinking can bring about conceptual knowledge (and, consequently, ACU) is to use a special case of the algorithm as a means of asking second order questions about the case observed. Through answering such questions conceptual knowledge develops. A particular example was the case of decomposing a prime number into a sum of consecutive integers from where understanding that such decomposition may only be a trivial one has resulted. By the same token, the paper demonstrated that ACU could be a source of an efficient problem-solving algorithm. Such was the case of purely numeric solutions to Problem 1 and to the classic merchant problem (the footnote in Section 6).

One of the common threads permeating the whole school mathematics curriculum (and consequently, course work for the future teachers of mathematics) is the representation of numbers as sums of other numbers. For example, integers (not all though) may be represented through the sums of consecutive natural numbers; squares of integers—through the sums of odd, triangular, and square numbers; unit fractions—through the sums of like fractions; integers—through irrational numbers; real numbers—through complex numbers; and so on. All such representations, although emerging from the algorithms of different levels of complexity, are often treated as the elements of procedural knowledge. What many teacher candidates (in the authors experience) don't appreciate is the significance of the fact that such representations may not be unique or do not exist at all. In the context of ideas brought to light by Problem 1, the following question can be asked: Why do the integers 13, 15, and 16 have, respectively, one, three, and zero representations as a sum of consecutive natural numbers? It is through asking questions of that kind that ACU of seemingly insignificant (through the lens of BCU) mathematical situations can be developed. In many cases, first order symbols can be used to facilitate such conceptual development in mathematics teacher candidates, something that they can later use in their own teaching of mathematics. Skills in asking conceptual questions about mundane procedures, something that often seen as the main mathematical experience of schoolchildren, are useful not only because of their relevance to the whole precollege curriculum, but, better still, such skills are important for preparing schoolchildren to study mathematics at the tertiary level.

In the case of a problem with an explicit numeric data, the paper suggested that there exists a conceptual bond that holds the data together allowing for a problem-solving algorithm to work. In turn, problem posing requires conceptual inquiry into the algorithm so that, by replacing data in the conceptual bond, one can formulate a numerically coherent problem. Whereas the creation of a formula, in many cases representing just a counting algorithm, can be achieved through the reflection on the formula by using BCU alone, ACU emerges and further matures through collateral learning. In that way, one begins the cycle "solve-reflect-pose" with only BCU and gradually develops ACU, which could be used as BCU at the next iteration of the interplay between the procedural and the conceptual. Therefore, problem posing enables a multi-level transition from BCU to ACU.

While the "solve-reflect-pose" cycle is a typical mathematical technique, one way to pursue it in mathematics teacher education with the minimal contribution of a ('more knowledgeable') instructor is through the process of reciprocal problem posing [3]. This process may involve a pair (or two groups) of teacher candidates who, using a basic problem (perhaps offered by the instructor) as a springboard, start posing different extensions of the problem for each other. By experiencing reciprocity in posing problems, teacher candidates can learn that one of the main difficulties involved in this process is finding the right balance between the challenge and the frustration, sentiments commonly associated with solving problems. Nowadays, the cycle can be enhanced procedurally through the use of computational environments specifically designed by the instructor for posing problems of a certain type (e.g., finding all ways of changing a quarter into dimes, nickels, and pennies) and conceptually through paying attention to the notion of didactic coherence of a problem [3] (e.g., thinking about students' familiarity with the names of the coins or about ways of reducing the multiplicity of answers). Also, the cycle can be sustained through the instructors competent guidance in engaging teacher candidates into the discussion of mathematical ideas leading to the development of the taste of problem posing (and, consequently, problem solving) by asking (and then answering) questions. Within this kind of classroom discourse, one can begin appreciating how a slight modification of a simple question or a search for an alternative procedure quite unexpectedly may become a source of conceptual developments in mathematics.

In his Oberwolfach talk, McCallum noted, among other things, "Reformulation is useful in learning" [22, p. 49]. The process of reformulation of a mathematical problem can be seen as a reflection on a solved problem in search of new ones, something that enriches both procedural skills and conceptual understanding. That is, the usefulness of reformulation (alternatively, problem posing) for the learning of mathematics is in a possibility of linking two types of knowledge. Furthermore, educational problem posing in mathematics might lead to significant conceptual outcomes. So, in 1901, an outstanding Russian mathematician Alexandr Lyapunov formulated and proved the Central Limit Theorem in the most general form (allowing random variables to exhibit different distributions) as he was preparing a new course for the students of the University of St. Petersburg [35]. Recently, while preparing technology-enhanced course materials for secondary mathematics teacher candidates, the author, proceeding from an iterative algorithm of constructing a certain class of polynomials associated with Fibonacci numbers, has come across an interesting property of those polynomials—the absence of complex roots. Verified computationally for the polynomials of degree $n \leqslant 100$, the formal proof of this property remains an open problem in mathematics by the time of writing this paper (for more details, see [4]). Whereas examples of that kind are rare, notwithstanding, mathematical problem posing in the context of education has great potential to serve as a link between algorithmic thinking and conceptual knowledge.

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