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# DIVISION—A SYSTEMATIC SEARCH FOR TRUE DIGITS, II

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**Abstract.** This paper is an improvement and a continuation of the author's previous paper published under the same title (The Teaching of Mathematics, 2005, vol. VIII, 2, 89–101).

Let us call a division $A : B$ canonical when $\frac{A}{B} < 10$. Then, each long division splits into a number of canonical ones and we will consider mainly that case of division. Let us suppose that $A : B$ is a canonical division, where $A = a_1a_2\dots$ and $B = b_1b_2\dots$, ($a_i$ and $b_j$ being the digits of $A$ and $B$). For a real number $x$, let $[x]$ denote the greatest integer not exceeding $x$. Then, using this notation, the true digit of $A : B$ is the number $m = \left[\frac{A}{B}\right]$ and we consider a trial number $m'$ to be "good", when $m' \leqslant m$. When $A < B$, then $m = 0$ and when $A \geqslant B$ and $A$ and $B$ have the same number of digits, when $a_1 = b_1$ then $m = 1$. These two cases of division we will consider to be trivial. Thus, two possibilities remain (I): $A$ and $B$ have the same number of digits and $a_1 > b_1$ and (II): $A$ has one digit more than $B$, ($A = a_1a_2\dots a_na_{n+1}$, $B = b_1b_2\dots b_n$). Let us define the first (second) pair of guide numbers to be in the case (I): $a_1$ and $b_1$, ($a_1a_2$ and $b_1b_2$) and in the case (II): $a_1a_2$ and $b_1$, ($a_1a_2a_3$ and $b_1b_2$).

The first increase-by-one method is a successive calculation of the trial numbers $\left[\frac{A'}{B'+1}\right]$, where $A'$s are the first guide numbers of the dividend and of the numbers that remain and $B'$ is the guide number of the divisor. A number of tries (sometime even five of them) are needed until the true digit is attained.

The second increase-by-one method consists of just one calculation of the trial number $m' = \left[\frac{A'}{B'+1}\right]$, where $A', B'$ is the second pair of guide numbers and when $m'$ is equal to or just 1 less than the true digit. A disadvantage of this method is the difficulty to perform mentally canonical divisions of three-digit numbers by two-digit numbers.

A combined method is the application of the first method just once and then of the second method, when mental calculation reduces to the division of numbers less than 200.

At the end and at the top is the second method accompanied with the written performance following the combined method of canonical divisions of three-digit dividends by two-digit divisors. This way of performing long division is the culmination of our search for true digits.

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## 1. Introduction

This paper is a continuation of the author's previous paper Division—A Systematic Search for True Digits, The Teaching of Mathematics, 2005, vol. VIII, 2, 89–101 and its electronic form can be found at `htpp://elib.mi.sanu.ac.rs/journals/tm`. When referring to this paper, it will be denoted by [Div].

First we give a condensed presentation of [Div], exposing its content in somewhat more formal and precise way. Nevertheless, the reading of [Div] is suggested to the reader, before he/she proceeds with reading this text.

The procedure of division is the easiest to be understood as the continual subtraction of the divisor from the dividend. Each such subtraction contributes 1 to the quotient and therefore, such a procedure may be too much lengthy to be practically feasible. For example, when the quotient is a somewhat larger number, say 3752, we could only imagine that number of subtractions.

Let us recall that the procedure of division is based on Division Theorem which states: *For each pair of natural numbers $A$ and $B$, there exists a unique pair of non-negative integers $q$ and $r$ such that $A = Bq + r$, where $0 \leqslant r < B$.* Related to the division $A : B$, the number $q$ is called *quotient* and the number $r$, *remainder*.

For a real number $x$, $[x]$ denotes the greatest integer not exceeding $x$. In this context $q = \left[\dfrac{A}{B}\right]$ and we will write "$B$ *into* $A$: $q$" to be an abbreviation of the phrase "$B$ goes into $A$, $q$ times".

Starting to subtract a divisor from the number of dividend having the highest place value, a more efficient calculation follows. Let us consider an example.

$$
\begin{array}{lrl}
\text{(a)} & \mathbf{2025}63 : 951 = 111 & \qquad 202563 = 2025 \cdot 100 + 63 \\
& \underline{951} \qquad\quad 1\ \ 1 & \\
& 1074 \qquad\quad \underline{\ \ \ 1} & \\
& \underline{951} \qquad\ \ 213 & \\
& 1236 \qquad\qquad\quad & \qquad 123 \text{ hundreds and } 6 \text{ tens make } 1236 \text{ tens} \\
& \underline{951} \qquad\qquad\ \ & \\
& 2853 \qquad\qquad & \qquad 285 \text{ tens and } 3 \text{ units make } 2853 \text{ units} \\
& \underline{951} \qquad\qquad & \\
& 1902 \qquad\qquad & \\
& \underline{951} \qquad\qquad & \\
& 951 \qquad\qquad & \\
& \underline{951} \qquad\qquad & \\
& 0 \qquad\qquad &
\end{array}
$$

Instead of 213 subtractions, we have only $2 + 1 + 3$ $(= 6)$ of them.

On the other hand, application of Division Theorem also provides us with an explanation how the procedure of division runs.

$$
\begin{aligned}
\text{(b)}\quad 202563 &= \mathbf{2025} \cdot 100 + 63 = (951 \cdot \mathit{2} + 123) \cdot 100 + 63 \\
&= 951 \cdot (\mathit{2} \cdot 100) + 1236 \cdot 10 + 3 = 951 \cdot (\mathit{2} \cdot 100) + (951 \cdot \mathit{1} + 285) \cdot 10 + 3 \\
&= 951 \cdot (\mathit{2} \cdot 100 + \mathit{1} \cdot 10) + 951 \cdot \mathit{3} = 951 \cdot (\mathit{2} \cdot 100 + \mathit{1} \cdot 10 + \mathit{3}) = 951 \cdot \mathit{213}
\end{aligned}
$$

Now let us present this procedure of division in its usual form which exposes the place values:

$$
(c) \qquad \begin{array}{l}
\mathbf{2025}63 : 951 = \mathit{213} \\
\underline{1902} \\
\phantom{0}1236 \\
\phantom{0}\underline{\phantom{0}951} \\
\phantom{00}2853 \\
\phantom{00}\underline{2853} \\
\phantom{00000}0
\end{array}
$$

In the two last cases nothing indicates the way how the true digits 2, 1 and 3 are found. But the aim of this paper is a direct way of finding true digits which is free from trial and error, erasing and starting over again.

Inspecting last example, we see that this division splits into shorter ones: $2025 : 951$, $1236 : 951$ and $2853 : 951$. Let us call a division $A : B$ *canonical* (in [Div] we used the term "short division") when $\dfrac{A}{B} < 10$. Now we consider the general case $A : B$, where $A = a_1a_2 \dots a_k$, ($a_1, a_2, \dots, a_k$ being the digits of $A$), $B = b_1b_2 \dots b_n$, ($b_1, b_2, \dots, b_n$ being the digits of $B$). Excluding the trivial case when $A < B$, we also suppose that $k \geqslant n$. Let $A_0 = a_1a_2 \dots a_n$. When $A_0 \geqslant B$, the *leading number of the dividend* $A$ is $A_0$ and when $A_0 < B$, that number is $10 \cdot A_0 + a_{n+1}$. When $A_0 \geqslant B$, $\dfrac{A_0}{B} < \dfrac{10^n}{10^{n-1}} = 10$ and when $A_0 < B$, ($A_0 + 1 \leqslant B$), then $\dfrac{10 \cdot A_0 + a_{n+1}}{B} < \dfrac{10 \cdot A_0 + 10}{B} \leqslant 10$. In both cases this division is canonical. When $r$ is the remainder of a canonical division $A : B$, ($r < B$, i.e. $r + 1 \leqslant B$) and $a_j$ is the digit which is taken down, then

$$\frac{10 \cdot r + a_j}{B} < 10 \cdot \frac{r+1}{B} \leqslant 10.$$

Therefore, we see that all divisions which follow the first one are canonical. Hence, each algorithm of long division splits into a number of canonical divisions. Thus, we will be considering mainly the cases of canonical divisions $A : B$ and we will be searching for true digits, proceeding directly. Let $m$ denote the true digit. When $A < B$, then $m = 0$ and when $A = a_1a_2 \dots a_n > B$, $a_1 = b_1$, then $m = 1$.

Indeed,

$$
\begin{aligned}
1 \leqslant \frac{A}{B} &= \frac{a_1 \cdot 10^{n-1} + A''}{b_1 \cdot 10^{n-1} + B''} = \frac{a_1 + \frac{A''}{10^{n-1}}}{b_1 + \frac{B''}{10^{n-1}}} \\
&< \frac{a_1 + 1}{b_1} = 1 + \frac{1}{b_1} \leqslant 2.
\end{aligned}
$$

Hence, m = 1.

These two cases of division will be treated as trivial in our further considerations.

Two possible cases remain:

(i) $A = a_1a_2\dots a_n$, $a_1 > b_1$

and

(ii) $A = a_1a_2\dots a_na_{n+1}$.

Now we define the *first (second) pair of guide numbers* of $A$ and $B$ to be in the case (i): $a_1$ and $b_1$, ($a_1a_2$ and $b_1b_2$) and in the case (ii): $a_1a_2$ and $b_1$, ($a_1a_2a_3$ and $b_1b_2$), respectively.

For example, in the case of division 654 : 305, the first (second) guide numbers are 6 and 3, (65 and 30), while in the case of division 2743 : 659 these numbers are 27 and 6, (274 and 65).

Let $A : B$ be a canonical division and $A'$ and $B'$ be the guide numbers of $A$ and $B$ respectively. Then

$$A = A' \cdot 10^k + A'', \quad B = B' \cdot 10^k + B''$$

where $A''$ ($B''$) is the number obtained from $A$, ($B$) omitting the digits belonging to $A'$, ($B'$), while $k$ is the number of digits of $A''$ (i.e. $B''$). Let us notice that

$$A' \cdot 10^k \leqslant A, \quad B' \cdot 10^k \leqslant B, \quad A'' < 10^k, \quad B'' < 10^k,$$
$$(B'+1) \cdot 10^k > B, \quad \frac{A'}{B'+1} < \frac{A}{B}.$$

Indeed,

$$B = B' \cdot 10^k + B'' < B' \cdot 10^k + 10^k = 10^k(B'+1),$$

and

$$10 > \frac{A}{B} = \frac{A' + \frac{A''}{10^k}}{B' + \frac{B''}{10^k}} > \frac{A'}{B'+1}.$$

We take $m' = \left[\frac{A'}{B'+1}\right]$ to be the *trial number* and since $m = \left[\frac{A}{B}\right]$ is the true digit, $m' \leqslant m$ *always holds*. Dependently on the case when $A'$ and $B'$ are the first (second) guide numbers of $A$ and $B$, the way how $m'$ is calculated is called the *first (second) increase-by-one method*.

Now we will be estimating the difference $\Delta = \frac{A}{B} - \frac{A'}{B'+1}$ as being the measure of the accuracy of an increase-by-one method. We have

$$\begin{aligned}
\Delta &= \frac{A' \cdot 10^k + A''}{B' \cdot 10^k + B''} - \frac{A'}{B'+1} = \frac{A' + \frac{A''}{10^k}}{B' + \frac{B''}{10^k}} - \frac{A'}{B'+1} \\
&= \frac{A'B' + B' \cdot \frac{A''}{10^k} + A' + \frac{A''}{10^k} - A'B' - A' \cdot \frac{B''}{10^k}}{(B' + \frac{B''}{10^k})(B'+1)} \\
&= \frac{(B'+1) \cdot \frac{A''}{10^k} + A' \cdot (1 - \frac{B''}{10^k})}{(B' + \frac{B''}{10^k})(B'+1)}.
\end{aligned}$$

Being $10^k > A''$ and $10^k > B''$, we have the following

Estimation. *If* $\Delta$ *is the difference* $\dfrac{A}{B}-\dfrac{A'}{B'+1}$ *where* $A'$ *and* $B'$ *are the guide numbers of dividend* $A$ *and divisor* $B$*, then*

$$\Delta<\frac{B'+1+A'}{B'(B'+1)}=\frac{1}{B'}\left(1+\frac{A'}{B'+1}\right).$$

## 2. First increase-by-one method

We will interpret this method using a number of examples. We will also use the following abbreviations: "l. n." for leading number, "g. n." for guide numbers.

| | | | |
|---|---|---|---|
| (d) | $47458:691=68$ | | l. n.: 4745; g. n.: 47 and 6; 7 *into* 47: 6 |
| | $\underline{4146}$ | | |
| | 5998 | | g. n.: 59 and 6; 7 *into* 59: 8 |
| | $\underline{5528}$ | | |
| | 470 | | |

| | | | |
|---|---|---|---|
| (e) | $1974:201=6$ | | l. n.: 1974; g. n.: 19 and 2; 3 *into* 19: 6 |
| | $\underline{1206}$ | 2 | |
| | 768 | $\underline{1}$ | g. n.: 7 and 2; 3 *into* 7: 2 |
| | $\underline{402}$ | 9 | |
| | 366 | | g. n.: 3 and 2; 3 *into* 3: 1 |
| | $\underline{201}$ | | |
| | 165 | | |

| | | | |
|---|---|---|---|
| (f) | $987:101=4$ | | l. n.: 987; g. n.: 9 and 1; 2 *into* 9: 4 |
| | $\underline{404}$ | 2 | |
| | 583 | 1 | g. n.: 5 and 1; 2 *into* 5: 2 |
| | $\underline{202}$ | 1 | |
| | 381 | $\underline{1}$ | g. n.: 3 and 1; 2 *into* 3: 1 |
| | $\underline{101}$ | 9 | |
| | 280 | | g. n.: 2 and 1; 2 *into* 2: 1 |
| | $\underline{101}$ | | |
| | 179 | | trivial case: 101 *into* 179: 1 |

| | | | |
|---|---|---|---|
| (g) | $1085:199=5$ | | l. n,: 1085; g. n.: 10 and 1; 2 *into* 10: 5 |
| | $\underline{995}$ | | |
| | 90 | | |

As we see, in some cases this method may be hardly better than continual subtraction, particularly when the first digit of a divisor is 1 or 2 (but it also depends on its second digit). The advantage of this method is a reduction of the algorithm of long division to the mentally easily performed divisions by one digit numbers. Its disadvantage may be the number of steps when the method applies, as the examples (e) and (f) reveal it clearly.

### 3. Second increase-by-one method

The greatest advantage of this method is the fact that the trial number is equal to or just one less than the true digit. First we prove this fact.

Proposition 1. *Let $A : B$ be a case of canonical division. Let $m$ be the true digit and let $m'$ is obtained as the trial digit when the second increase-by-one method is applied, then $m - m' \leqslant 1$.*

*Proof.* We use the Estimation from Section 1. When $B' = 10$, and $A' \leqslant 99$, than $\Delta < \frac{1}{10}\left(1 + \frac{99}{11}\right) = 1$, what implies that $m - m' \leqslant 1$. When $B' = 10$ and $A' = 100, \dots, 109$, then $\dfrac{A'}{B'+1} \geqslant \dfrac{100}{11} > 9$ what implies that $m' = 9$ and hence $m - m' = 0$. When $B' \geqslant 11$, then $\Delta < \frac{1}{11}(1 + 10) = 1$, hence $m - m' \leqslant 1$ always holds. ■

Now we use a number of examples to interpret this very efficient method, leaving aside the question of division of at most three-digit numbers by two-digit numbers.

(h) $987 : 101 = 8$ l. n.: 987; g. n.: 98 and 10; 11 *into* 98: 8

$\underline{808}$ $\quad\quad \underline{1}$

$179$ $\quad\quad 9$ $\quad 179 > 101$ and according to Proposition 1., 1 is added to the trial number 8

(i) $2641 : 387 = 6$ l. n.: 2641; g. n.: 264 and 38; 39 *into* 264: 6

$\underline{2322}$

$319$

(j) $7485 : 964 = 7$ l. n.: 7485; g. n.: 748 and 96; 97 *into* 748: 7

$\underline{6748}$

$737$

Noticeable disadvantages of this method are canonical divisions with the two-digit divisors, when they have to be carried out mentally.

### 4. Combining two methods

The main point of combining two methods is the fact that we accept as admissible mental divisions when dividends are less than 200. Combination of two methods relies on the following

Proposition 2. *Let $A : B$ be canonical division and let $A = a_0a_1a_2\dots a_n$, $B = b_1b_2\dots b_n$. Let us suppose that the first increase-by-one method has been* *applied, when the trial number $k$ is equal to* $\left[\dfrac{a_0a_1}{b_1+1}\right]$. *Then,* $A-kB=c_0c_1c_2\dots c_n$ *and* $c_0c_1c_2<200$, *(including also the trivial cases when* $a_0=0$*).*

*Proof.* Being $k$ the trial number, we have

$$(k+1)(b_1+1)>10a_0+a_1\geqslant k(b_1+1).$$

Let us write $d=A-kB$ and $A=a_0a_1a_2\cdot 10^{n-2}+A''$, $B=b_1b_2\cdot 10^{n-2}+B''$. Then

$$\begin{aligned} d&=(a_0a_1a_2-k\cdot b_1b_2)\cdot 10^{n-2}+A''-B''\\ &\leqslant[10(10a_0+a_1)+a_2-k\cdot(10\cdot b_1+b_2)]\cdot 10^{n-2}+A''\\ &\leqslant[10\cdot(k+1)(b_1+1)+a_2-10\cdot k\cdot b_1-k\cdot b_2]\cdot 10^{n-2}+A''\\ &=[10\cdot(k+b_1+1)+a_2]\cdot 10^{n-2}+a_3\cdot 10^{n-3}+\dots+a_n. \end{aligned}$$

Hence,

$$c_0c_1c_2\leqslant 10\cdot(k+b_1+1)+a_2\leqslant 10\cdot(9+9+1)+9\leqslant 199.$$

The sketch of a proof of this proposition is found in [Div]. ∎

The combined method consists of the application of the second increase-by-one method, when $a_0a_1a_2\leqslant 200$. When $a_0a_1a_2>200$, the first method applies, reducing that case of division to the one when $c_0c_1c_2\leqslant 200$ and when it is easy to carry out a division mentally.

Now concrete examples of this type of division follow.

(k) $$\begin{array}{r} 1743 : 355 = 4\\ \underline{1420}\phantom{ : 355 = 4}\\ 323\phantom{ : 355 = 4} \end{array}$$ 174 < 200 and the second method applies: 36 *into* 174: 4

(l) $$\begin{array}{rr} 49548 : 5327 = & 8\\ \underline{42616}\phantom{ : 5327 =} & \underline{1}\\ 6932\phantom{ : 5327 =} & 9\\ \underline{5327}\phantom{ : 5327 =} & \\ 1605\phantom{ : 5327 =} & \end{array}$$ First method applies: 6 *into* 49: 8

Second method applies: 54 *into* 69: 1

(m) $$\begin{array}{rr} 7982 : 913 = & 7\\ \underline{6391}\phantom{ : 913 =} & \underline{1}\\ 1591\phantom{ : 913 =} & 8\\ \underline{913}\phantom{ : 913 =} & \\ 678\phantom{ : 913 =} & \end{array}$$ First method applies: 10 *into* 79: 7

Second method applies: 92 *into* 159: 1

## 5. At the end and at the top

When the second increase-by-one method is applied the trial number is obtained performing a small canonical division, where the divisor is a two-digit number. The second guide numbers determine uniquely the trial number which is the same no matter what are all other digits and how many of them exist. When mental calculation of the trial number is not very easy, its written form has to be done aside (and these small divisions are usually done combining two methods). Now we use some examples to demonstrate that proceeding in this way, we have a more efficient performance of long division than when two methods are combined.

(n) Combining two methods:

$$\begin{array}{lll}
2713352 : 301914 = 6 & & 4 \textit{ into } 27\text{: } 6 \\
\underline{1811484} & \underline{2} & \\
\phantom{0}901868 & 8 & 31 \textit{ into } 90\text{: } 2 \\
\phantom{0}\underline{603828} & & \\
\phantom{0}298040 & &
\end{array}$$

Second method (plus a small division):

$$\begin{array}{llll}
2713352 : 301914 = 8 & 271 : 31 = 6, & & 4 \textit{ into } 27\text{: } 6 \\
\underline{2415312} & \underline{186} & \underline{2} & \\
\phantom{0}298040 & \phantom{0}85 & 8 & 31 \textit{ into } 85\text{: } 2
\end{array}$$

(o) Second method (plus a small division):

$$\begin{array}{lllll}
2415313 : 301914 = 7 & & 241 : 31 = 6, & & 4 \textit{ into } 24\text{: } 6 \\
\underline{2113398} & \underline{1} & \underline{186} & \underline{1} & \\
\phantom{0}301915 & 8 & \phantom{0}55 & 7 & 31 \textit{ into } 55\text{: } 1
\end{array}$$

Let us notice that according to Proposition 1., when the remaining number (here: 301915) is larger than the divisor (here: 301914), 1 is added to the trial number to obtain the true digit.

(p)
$$\begin{array}{llll}
79901 : 8291 = 9 & 799 : 83 = 8, & & 9 \textit{ into } 79\text{: } 8 \\
\underline{74619} & \underline{664} & \underline{1} & \\
\phantom{0}5282 & 135 & 9 & 83 \textit{ into } 135\text{: } 1
\end{array}$$

(q)
$$\begin{array}{llll}
59911 : 6132 = 9 & 599 : 62 = 8 & & 7 \textit{ into } 59\text{: } 8 \\
\underline{55188} & \underline{496} & \underline{1} & \\
\phantom{0}4723 & 103 & 9 & 62 \textit{ into } 103\text{: } 1
\end{array}$$

(r)
$$\begin{array}{lllll}
289451 : 41329 = 6 & & 289 : 42 = 5 & & 5 \textit{ into } 28\text{: } 5 \\
\underline{247974} & \underline{1} & \underline{210} & \underline{1} & \\
\phantom{0}41477 & 7 & \phantom{0}79 & 6 & 41477 > 41329 \text{ and 1 is added} \\
 & & & & \text{to the trial number 6}
\end{array}$$

This way of performing long division is the culmination of our search for true digits.

Remark. All divisions $a_1a_2a_3 : b_1b_2$ that are done aside are performed first applying the first increase-by-one method. Let $m'$ be the trial number and $m$ the true digit. Then, using Estimation (Section 1):

$$\Delta < \frac{1}{b_1}\left(1 + \frac{a_1a_2}{b_1+1}\right),$$

it is easily checked that: $m - m' \leqslant 2$, for $b_1 = 9, 8, 7, 6$, $( \dots \frac{1}{6}\left(1 + \frac{69}{7}\right) = \frac{76}{42} < 2)$; $m - m' \leqslant 3$, for $b_1 = 5, 4$, $( \dots \frac{1}{4}\left(1 + \frac{49}{5}\right) = \frac{54}{20} < 3)$; $m - m' \leqslant 4$, for $b_1 = 3$. When $b_1 = 2$ and $a_1a_2 = 15, 16, \dots, 29$ then $m' \geqslant 5$ and when $b_1 = 2$ and $a_1a_2 < 15$, $\Delta < \frac{1}{2}\left(1 + \frac{14}{3}\right) = \frac{17}{6} < 3$, therefore in both cases $m - m' \leqslant 4$. When $b_1 = 1$ and $a_1a_2 = 10, 11, \dots, 19$, then $m' \geqslant 5$, hence $m - m' \leqslant 4$. We see that in all these cases, mental divisions produce at most 4, what is much easier than dividing, in general, numbers less than 200 by two-digit divisors.

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