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AN EMPIRICAL STUDY IN THE NOTION OF AREA: A SOCRATIC EDUCATIONAL EXPERIENCE ANCHORED IN VAN HIELE'S MODEL

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'Can you do Addition?' the White Queen asked.
'What's one and one and one and one and one
and one and one and one and one and one?'
'I don't know,' said Alice. 'I lost count.'
'She can't do Addition,' the Red Queen interrupted.
Lewis Carroll, "Through the Looking Glass"

Abstract. In this article our goal is to design a suitable strategy to be implemented on High School students in order to prepare them for the formal study of approximate and exact integration via a Socratic semi-structured interview. Our dialog will be closely dependent on the use of a computer generated tool (applet) to encourage students participation, provide them with numerical and visual data and allow the linking of the processes of discovery, understanding and conceptualization in the frame of an educational model. What follows contains a description of the interview which is also our instrument for pointing out and detecting the levels postulated by van Hieles educational model.

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MSC Subject Classification: 97I50, 97U50

Key words and phrases: Area; approximation; error; limits; van Hiele's model.

1. Objective

Our purpose is to lay down the basic ideas which are behind the topic of the definite integral and it is aimed to High School students which have studied neither limiting processes nor integration yet, but are close to doing so. Since most students do not really understand the computation of area as a limiting process and since those limiting processes are neglected at school, we are not interested in what students can do naturally, but in what they can do accompanied by instruction. We want to guide them in a journey of discovery and inquiry, not random but purposeful, testing at every significant stage of the experience the foundations of their beliefs or their ways of reasoning, very much in the Socratic spirit, and to explore how far progress can be made in providing meaningful information by using colloquial language admitting a certain degree of ambiguity present (which hopefully allow students to invest meaning in the problem studied) while manipulating a computer generated visualization tool. The cognitive structure associated to a mathematical concept which includes all mental images, visual representations, experiences and impressions as well as associated properties and processes is called concept image, Tall and Vinner [7]. By means of a semi-structured clinical interview, we shall provide the means for the construction of a solid concept image of area which incorporates visual, numerical and algebraic connotations, for if one cannot handle a mathematical concept in more than one register, then understanding of it is limited (Duval). For this purpose and with the help of a mathematical assistant, we shall design a tool, technological in nature, covering all those aspects (see Section 4).

Our aim is not to develop a substitute for the concept definition (the conventional linguistic statement precisely delimiting the frontiers of application of the concept, Tall and Vinner [7]), but rather a somewhat narrower objective: to describe a battery of actions that should be implemented prior to formal mathematical instruction in the classroom, as done in Herceg and Herceg [1], with the purpose of, on one hand, constructing a suitable concept image which does not distort the desired concept definition allowing a smooth transition to it and, on the other, provoke the need for such a concept definition at a stage where the notion of limit has not been covered yet. The suitable place to implement such a strategy is High School.

2. Van Hiele's educational model

We shall anchor our experience in Van Hiele's educational model (van Hiele [8, 9]) which provides a description of the learning process, by postulating the existence of levels of reasoning (not identified with computational skills) classified as Level 0 (Pre-descriptive), Level 1 (Visual Recognition), Level 2 (Analysis), Level 3 (Classification and Relation) and Level 4 (Formal Deduction).

In Level 1 students are guided by a series of visual characteristics and lead by their intuition. In Level 2, individuals notice the existence of a network of relationships. This is the first level of reasoning that can be called "mathematical" because students are able to describe and generalize through observation and manipulation properties that they still do not know. Reasoning in Level 3 is related to the structure of the second level and conclusions are no longer based on the existence or non-existence of links in the network of relationships of the second level, but rather on existing connections between those links. Level 4 speaks for itself. Level 3 is our last port of call since we are interested mainly in the construction of a solid concept-image of a mathematical concept which makes easy the mental transition to its concept-definition, once the necessary logical-algebraic maturity is available.

Its success in dealing with concepts outside of the realm of Geometry (see Llorens and Pérez Carreras [2], Navarro and Pérez Carreras [3–6]) is explained because the model can be understood as more concerned on how students think about a specific topic than with the topic itself and also because the model mimics the genesis of some mathematical concepts: first, the discovery of isolated phenomena; second, the acknowledgement of certain characteristics common to all of them; third, the search for new objects, their study and classification and, fourth, through consideration of examples and counterexamples to proposed definitions, the emergence of definitive formulations. The application of this model to a specific subject requires the establishment of a series of descriptors for each level to enable their detection. To be considered within van Hiele's model, several conditions should be satisfied: (i) levels must be hierarchical, recursive, and sequential; (ii) levels must be formulated so that they include a progression in the level of reasoning as a result of a gradual process, resulting from learning experiences; (iii) tests designed for the detection of levels should take into account the existing relationships among levels and the language used by apprentices; and (iv) the fundamental objective of the design must be the detection of levels of reasoning, without confusing them with levels of computational skill or previous knowledge.

3. Our approach

According to the constructivist perspective which focuses in individual thinking and the continuing act of creating learning opportunities, all that can be accomplished via an appropriate Socratic discovery-oriented interview design that, within the context of the model and paying attention to many of the recurrent themes described in the research literature, allows the detection of students' levels of thinking with respect to the specific mathematical concept we are dealing with. This interview is not a wild-goose chase: it has to be carefully planned with a strategy in mind and orderly implemented, avoiding inappropriate tactics at the wrong times or places and allowing for feedback to planning and assessment.

Why an interview as a Socratic dialog? According to (iii) above, we need students to be able to talk profusely. Our questions should be sharp and short, their answers are expected to be tentative and long. Most students are very comfortable functioning without formal definitions in most cases and hence common language is a perfectly reasonable tool of communication to use: it is rich in implicit mathematical rules, meanings and conventions and its proper use entails a special competence. There is no such thing as a neutral communication system: ours allows the interviewer a sense of direction proposing those questions which concentrate in conceptual understanding and avoiding those more concerned in assessing performance in procedural items. We shall stress the importance of mathematical concepts and pattern matching. The dialog provides motivation and tests the solidity of those ideas and arguments which the student brings and that are based in prior educative experiences when challenged to dig out and verbalize his own beliefs; it also allows the student to form his own connections. Since students have been well trained to be passive in the classroom, the interview allows us to break this pattern and turn them in active learners: the downside is that causes discomfort, but it can be usually ameliorated by frank and open discussion. Since the beliefs and perceptions of students influence and jeopardize their approach to learning mathematics, the most delicate aspect of the dialog is how to fight the attitude of most students which are comfortable with inconsistencies, contradictions and competing meanings which are the main cause of failure in completing the interview as our questions grow in sophistication. Our only weapon is to be persuasive and gently reveal their ignorance and misinformation along with valid knowledge so that we can help replace one with the other. Summarizing, we try to approach our problem with an explorer mentality in a living environment where challenges occur allowing better focalization in mathematics as a process by encouraging the development of qualities as creativity, imagination, rationality, critical analysis and, last but not least, a bettering of their communication skills. This journey can be described as the plot of a good epic: be confronted by a challenge; leave the safety of your narrow previous knowledge; bond with new ideas; survive showdown and discover a truth.

Why do we use a computer generated tool with visual and computational capabilities? Regarding visualization, one has to consider that our evolutionary development has provided us with a remarkable capability to absorb information from instantaneous vision and we feel this capability should be used in the learning process, although some aspects of it are not learned automatically but have to be explicitly taught, not as separate topics, but throughout the experience in a parallel manner. The generation of images promotes the integration of the separate components of the item in question and accessing parts of the information encoded in memory prompts the retrieval of all other pieces of information contained in the image. The main concern is to create a learning environment which makes difficult the appearance of attitudes common in students which resign themselves to learning strategies in order to cope without understanding. In a more practical vein, when the student is asked questions about dynamics, such as a limiting process, it ought to be easier to extract dynamic information from a dynamic presentation, which is what our tool can provide, rather than from a static graph. There are other advantages using such a tool: its use somehow provokes less disappointment if the machine proves them wrong, favours the appearance of conjectures and frees students' thought processes. Last but not least, we believe that abstraction is facilitated by concretization via visual support. Regarding computation, and with an eye on condition (iv) in the former section, we leave computational skill to the machine.

Design and implementation. It took considerable time to design the interview and some failed attempts were needed in order to reach a satisfactory one. After that, twenty interviews on High School students of Valencia and Seville were carried out, each successful one consuming roughly one and a half hour time. Students were selected on the basis of their willingness to participate and every volunteer was accepted. They agreed to the audio recording of the interviews and also to the use of their corresponding anonymous transcriptions in our analysis. Their previous upbringing in mathematics was the usual consistent (but sadly declining) one provided by the Baccalaureate in Spain.

How we decided to present it to you. Hoping that the experience might offer some guidance to teachers wanting to introduce the topic to their students, we have unified our transcriptions and edited heavily the final text in order to present a narrative in mature language avoiding clumsiness, false starts, hesitations and fragmented and incoherent arguments hopefully preserving what we were able to experience in terms of clarity of thought and adaptability of our most successful students, if not in their exact actual words. Overcoming difficulties contribute to conceptual learning but in order to keep this article within reasonable bounds, dysfunctions do not show in the following transcription of our interview, since the responses belong to the more gifted students.

4. Description of the tool

The tool is an interactive screen and the user does not need to have previous knowledge of the program (MATLAB 6.1) used to design it. An introductory screen with zooming capability (Figures 1) shows an arc of circumference, allows us to choose one of its points, draws the tangent line to the curve on this point and activate two other different screens. The first one (Figure 2) is a combination of a graphical (GW1) and a computational (CW) window and the second (Figure 3) is a combination of another graphical window (GW2) and the same CW. In both combinations it is possible to enter a function, its domain, the number of trapezoids to be considered and the number of digits allowed in numerical expressions.

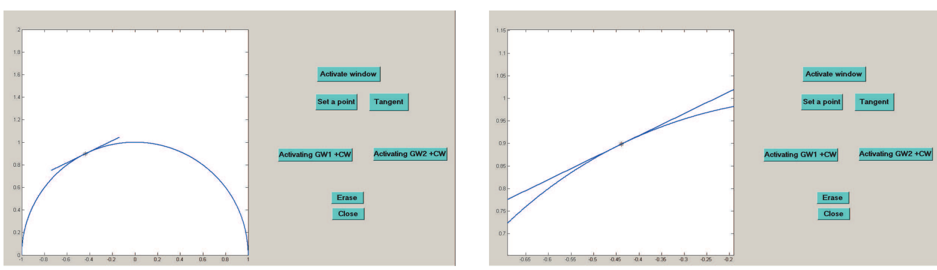


Fig. 1

GW1 deals with a region chopped in trapezoids and CW shows three columns depicting successive approximations to the actual area (calculated with the quad command of MATLAB) as a result of accumulating areas of trapezoids, error and percentage of error committed.

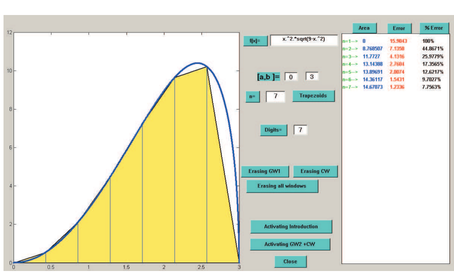


Fig. 2

Fig. 3

GW2 is a visual alter ego of CW. It depicts the former approximations $\{x_n\}$ as bi-dimensional points $(1/n, x_n)$ forming what we call a '*cloud of points*' and shows the evolution of those approximations; namely, whether they tend to stabilize around some value which will stand for the actual area.

In order to study proximity of the cloud to some conjectured value (chosen by the interviewee and seen as a red line in GW2), we need the capability of zooming and drawing bands of preselected semi-width (ε); a message window above will inform us of the height chosen, the number of points of the cloud lying outside the band and the ε chosen (Figure 4). Pressing 'Enlarge', the square contiguous to the ordinate axis will appear and more points of the cloud can be added to GW2 and CW (Figure 5), a new red line can be drawn (if a change a conjecture becomes mandatory) and further enlargements are possible.

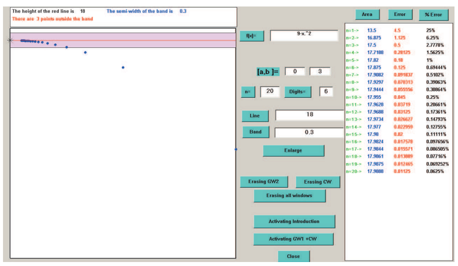


Fig. 4

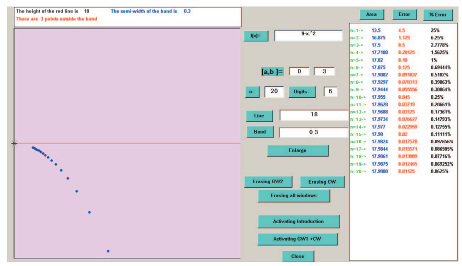


Fig. 5

5. Outline of the interview

Prerequisites deal with the naive sense of area (an undefined term), how to measure it in linear figures and the transition from exactness to approximation when dealing with curvilinear figures, leading to the idea that the term 'area' remains undefined. If a definition is going to be developed, we ask what kind of properties should reasonably be present in such a notion.

In Level 1 we deal with information inferred visually with the help of the tool. The zooming capability allows us to explore local straightness of curvilinear paths which is essential in how to proceed from known exact areas for linear figures to unknown areas for curvilinear figures. Local straightness allows the use trapezoids as building blocks. GW1, as amplifier of our representing capabilities, allows us to show dynamically how to proceed by exhausting progressively the surface under a curve using trapezoids. The activation of GW2 is followed by a discussion of what it represents, namely an abstraction of GW1 where only essential elements are kept in display and it acts as a predictive tool to where the partial areas are heading.

Level 2 adds numbers to our former analysis and CW shows the evolution of calculated partial areas and the ideas of numerical approximation and error come forward. The interviewee will see that GW2 and CW are two sides of the same coin and its combined use will allow the appearance of conjectures for the area under study: GW2 shows tendencies and hence provides a first conjecture; CW permits refinement in our choice by dealing with the error committed. Moreover, by playing with the number of digits allowed to appear in the window, we can advance the dynamical procedure needed to define what 'actual' value of the area means.

Once the idea of approximation has been developed, Level 3 deals with 'closeness' and how to check whether a candidate for area is a reasonable one. Both goals can be achieved by using the tool with the added bonus of introducing the experimental factor in our experience. Our first task is dealt with GW2 by visualizing our considerations over error into bands around a candidate; our second task, by reverting the dynamic process of taking smaller and smaller errors into a game to be played in GW2, the so-called *sceptic's game*. Since first conjectures produced will hardly be adequate, the experimental factor enters by trial and error.

A further abstraction process takes place as the interviewee notices that only a tiny part of GW2 is needed to play and reach conclusions, namely the companion message window above which contains the three key ingredients needed to produce the definition of area as a limiting process (integral of a positive continuous function). Students should be ready now for a formal treatment in the classroom.

6. The interview and students' responses

6.1. Level 0 (prerequisites)

Pr.: We all have some naive sense of 'area', but let's try to really understand what area is. In fact, the concept of area is not at all a simple one.

St.: We are taught certain formula as children, for example, the area of a triangle are half its base times its height.

Pr.: Yes, but what does this actually mean?

St.: A way of measuring area.

Pr.: Right, but the term 'area' remains undefined.

St.: Area is a quantity representing amount or extent of a surface.

Pr.: Mm, it seems that we cannot arrive to a true way to *define* area, but this does not mean that we have no way of properly studying area. Let us find a more humble goal than defining area – to know when two shapes have *the same* area or not. Imagine two squares.

St.: They have the same area if both have the same side length, that is, if both are the same square. Otherwise they have different areas.

Pr.: Could you imagine a movement in the plane where one square is moved to coincide with the other?

St.: Yes, a translation. Area doesn't change when a translation is performed.

Pr.: How to proceed if both are rectangles?

St.: (drawing) I use translation, and rotation if necessary. Again, area doesn't change when those movements are performed.

Pr.: What if one is square and the other is a rectangle?

St.: Translation and rotation won't do. We apply the formulas and we are done.

Pr.: In absence of formulas?

St.: Apart from the obvious cases where different areas might be appreciated, if I could construct a square with the same area of the rectangle, I could compare with the other square.

Pr.: Interesting idea to reduce the problem to squares. Let us take another route: if we take the area of a square as intuitive, does it suggest a way of defining area by using squares?

St.: The area of a figure is the number of squares required to cover it completely, like tiles on a floor, but it seems to me that it does not work when dealing with polygons other than rectangles.

Pr.: If p tiles are needed to cover the length of one side and q to cover the other side, how many tiles do you need to cover the surface of the rectangle?

St.: $p \times q$ tiles and hence the formula for the area of the rectangle.

Pr.: Thus, the formula allowing us to calculate the area of a rectangle is intuitive enough, as intuitive as the one leading to the area of a square.

We take a small detour by asking him over the difference between perimeter and area; can they be equal? He will state that perimeter is the length of the contour and it is measured in centimetres or whatever unit; that area is measured in square units and that area and perimeter can't really be equal, because they are measured in different units; technically, they might be "numerically equal", not actually "equal". Turning our attention away from rectangles, exactness (aka, formula) takes a step back and the idea of approximation comes forward.

Pr.: Let us go back to the area problem. When dealing with curvilinear plane figures intuition is too rough a guide and simple formulae fail. We may have to sacrifice exactness.

St.: I don't know. Anyway, that it is not always the case. What about the area of the circle? We have the usual formula πr^2, which is supposed to be exact, although I do not remember a justification for such a formula.

Pr.: π is the ratio of the circumference of a circle to its diameter. That is, if you have a string the length of the circumference, π is how many times it will cover the diameter. It's a little bit more than 3.

To avoid losing exactness, he speculates whether disconnecting the circle, measuring it in a straight line and comparing it with its diameter would provide π as a precise number and the formula would be exact. We express our doubts in getting anything better than 3 for π, since it is not determined by actually measuring physical circles; π is the ratio of circumference to diameter for any mathematical circle; but there are no mathematical circles in the real world, so we can't ever find π by measuring something. We advance that, in due time, he will learn that π has all sorts of other strange properties: it's a number that can never be written all of. That's why we just say "pi" instead of writing it down; if we start to write π, it looks like 3.141592653589 but that's only the beginning of it, it goes on forever. Moreover, π is not the exclusive property of circles: every closed contour can be related to π.

St.: Hmm ... , even in the case of the circle, is there no way to calculate this area exactly?

Pr.: Once we define the number π in a precise way, and it will come the time, the formula will be exact for the area of the circle. For the time being, that formula provides you with an approximation, the better the more digits of π you use.

St.: I see. And the same happens for other curvilinear figures?

Pr.: That's the point. When dealing with the rectangle, we defined area by the formula that calculates it, but this approach doesn't work when dealing with plane curvilinear figures, unless we had at our disposal formulae for all possible figures, which circles and other figures apart, we don't.

We need to define the term 'area', as well as try to find it. We shall attempt to find a (partial) solution for both problems along this interview.

Pr.: If trying to provide a definition to 'area', this definition has to comply with our intuitive ideas or make use of them.

St.: Such as?

Pr.: For starters, the product of length and breadth defines the area of a rectangle.

St.: That again! Sure.

Pr.: But also, that if you slide a figure rigidly along the plane or rotate it, that is, if you arrive to a congruent figure, it will continue to have the same area.

St.: (impatient) Right. Is there anything else?

Pr.: Yes, something important: the area of a figure composed of a finite number of non-overlapping parts is the sum of the areas of the parts.

St.: (pensive) Fair enough.

Pr.: Having those three conditions in mind, what can you say about the area of a triangle?

St.: I know that: complete the figure of the triangle to construct a rectangle, trace the altitude and the figure shows that the triangle covers half the area of the rectangle.

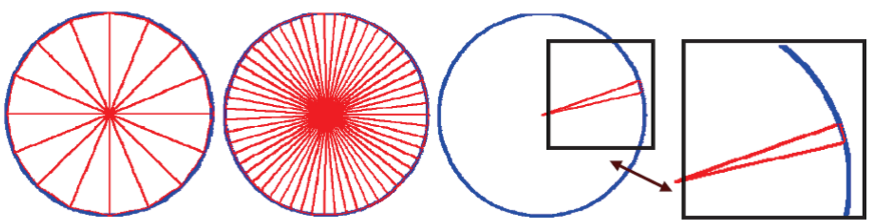
Pr.: Right. What about calculating areas of trapezoids? What about other polygons?

St.: The area of a trapezoid is obtained by adding the area of a rectangle and a triangle. For other polygons, cover them with non-overlapping triangles and add areas. And all those formulas are exact.

6.2. Level 1 (from verbal to visual)

Pr.: Returning to the circle and forgetting about the formula, could you proceed as in Figures 6?

St.: Yes, I understand the idea. The more triangles considered, the better the estimation. Is this the justification for the formula I was missing?



Figures 6

Pr.: Essentially yes, but it takes some time to arrive from the pictures to πr^2: we need to consider infinity of triangles to get all the way to the formula; using a finite number of them we are computing something other than the formula. Anyway, we are not interested now in justifying the formula, but in the method itself.

St.: Covering partially an extension with figures of known areas . . .

Pr.: And depleting the extension left by increasing progressively the number of them, following a certain procedure . . .

St.: Getting hopefully a close approximation to the area enclosed.

Pr.: That this procedure works is based in an idea we have to elaborate: first, a portion of continuous curve between two points may look very different from the portion of secant joining them and hence not very representative.

St.: Sure, but not so much if the points stay close together.

Pr.: The closer they are, the better the coincidence?

St.: Yes, obviously. Pr.: In other words, a continuous curve can be viewed as locally straight.

St.: What do you mean?

Pr.: (activating the tool's introductory screen where an arc of circumference is shown) Imagine that you direct your attention to a point in the curve (Figure 7) and perform a zooming process several times. What do you see?

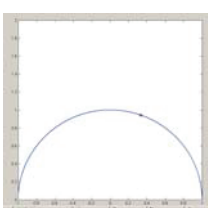


Fig. 7

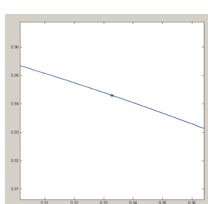


Fig. 8

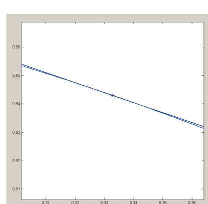


Fig. 9

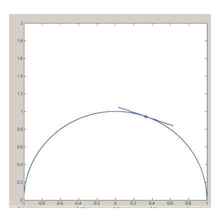


Fig. 10

St.: Zooming process means?

Pr.: Going local by applying an equal reduction of scale on both axes around a chosen point. The more you zoom, the smaller the scale used, like using a microscope with different magnifying lenses.

St.: After several zooms I see a straight line, well ... a segment (Figure 8).

Pr.: Imagine you maintain the segment and proceed to undo the zooming process recuperating the curve (Figure 9). What is the relation between the curve and the segment?

St.: The segment stays tangent to the curve (Figure 10).

Pr.: That curves are locally straight allows the bases of the triangles to almost confound themselves with portions of the circumference.

St.: What you've shown is that the graph over short intervals deviates very little from the tangent line. Hmm ... and this idea ... does it work for any curvilinear figure?

Pr.: You can see it for yourself by drawing a large number of blocks (as we did with triangles) to approximate the area under a complex curve, hopefully getting a better answer if you use more blocks.

St.: What kind of blocks? Squares? The triangles are especially suited for the circle, because there is a centre for the figure, but in absence of a centre, triangles seem hardly suitable.

Pr.: (referring to Figure 6) Right. Imagine we want to estimate the area enclosed. What kind of blocks can you suggest for covering and exhausting purposes?

St.: (pointing at the screen) I could use triangles again or I may pile up little squares into stripes, that is, I could use rectangles.

Pr.: (using paper and pencil) The area of such a little block under the curve can be thought of as the width of the strip weighted by (i.e., multiplied by) the height of the strip which has to be chosen, let us say at midpoints. In order to compute the enclosed area we chop up the region into lots and lots of little strips.

St.: But, doing so, we will not be able to succeed completely because there will always be regions with curved sides untouched.

Pr.: Sure. But the key idea is that the sum of the areas of the strips will be a very close approximation of the actual area and the more strips we cut, the closer our approximation will be.

St.: By adding all stripes areas ... Yes, it should work.

We inquire if, due to continuous curves being locally straight, we may choose strips looking as trapezoids. With paper and pencil and considering a small portion of curve he draws a trapezoid using a portion of tangent line and reflects on its area as being coincident with the corresponding midpoint rectangle. We object to the use of those trapezoids since tangent lines are required which seems to complicate our original problem. Can we settle for trapezoids defined by the secant lines? He will admit that those trapezoids stay also very close to the curve if there are plenty of them, although he insists that the other trapezoids seemed a better choice. We confirm that he is right: using tangent-trapezoids (equivalently, midpoint rectangles) is twice as accurate as using secant-trapezoids, but for our purposes those last trapezoids are good enough. We activate GW1 for the function $9-x^2$ in $[0,3]$ (see Figure 11) and we choose a uniform grid for the whole interval, see Figure 12.

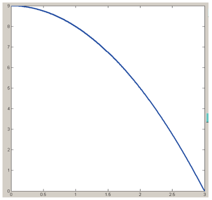
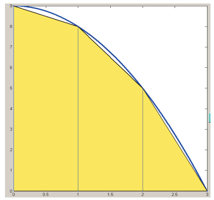
Pr.: Let us illustrate our point that the global approximation may be made better and better following a finite number of steps. It would look like this. Feel free to try different grids.

St.: (taking command of the tool and selecting different grids, Figure 13). The smaller the bases, the better the coincidence which means that I need to consider more trapezoids. Is that the idea?

Pr.: So it looks. Looking at GW1, our estimations are larger or smaller than the number we are searching?

St.: Smaller, indeed.

Pr.: (erasing GW1, activating GW2 and explaining the behaviour of the cloud from right to left, see Figure 13) The corresponding cloud of points seems to grow in height as the number of

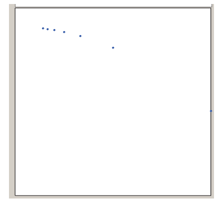


Fig. 11 Fig. 12 Fig. 13 Fig. 14

trapezoids is increased.

St.: Seen from right to left, the points in the cloud correspond to underestimations; they grow and should stay below the desired value, which is not to be seen in this window.

Pr.: (activating GW1 for a decreasing concave upwards path of $(x-3)^2$, $x \in [0,3]$ and GW2 thereafter, see Figure 14) What is the situation now?

St.: Now we are dealing with overestimations and the cloud of points should stay above the desired value.

Pr.: Could you affirm that the method seems to work for monotonic continuous paths?

St.: So it seems. Whether we are dealing with over or underestimations depends on the concavity of the curve.

Pr.: Let us summarize our findings: the trapezoid choice provides apparent good approximations for monotonic continuous curves because, selecting an appropriate uniform grid, the curve can be well approximated by lines and, moreover, that choice provides the exact value for linear paths.

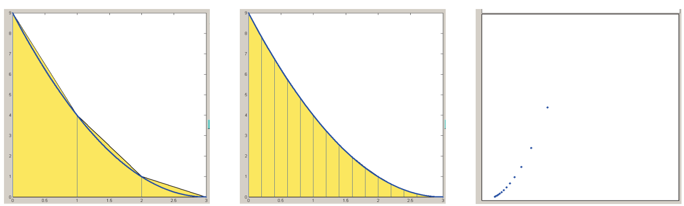


Fig. 15

Pr.: (producing a non-monotonic graph at the screen, Figure 16) What shall we do with this graph?

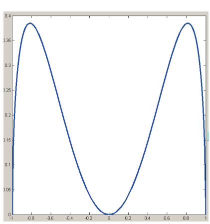
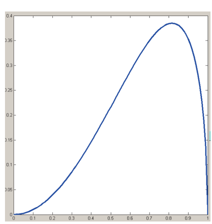
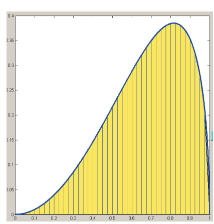
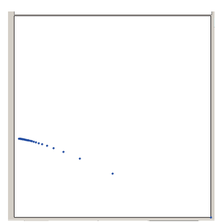





Fig. 16 Fig. 17 Fig. 18 Fig. 19

St.: Hmm ..., decompose it in monotonic paths, construct trapezoids to each path and add areas.

Pr.: Right. (We show the graph of $x^2(1-x^2)^{1/2}$, Figure 17) Here you have two monotonic paths. Is it easier to evaluate approximately the area beneath the slowly changing path than the one which changes fast?

St.: (pensive) Fast meaning steeper and slow meaning flatter. A good grid for the slow changing one may not be good enough for the fast changing path (see Figure 2). It is a nuisance to repeat procedure two times. Why not select a grid for the whole interval determined by our choice in the second path?

Pr.: Why don't you try using the tool?

St.: (proceeds with GW1, Figure 18 and GW2, Figure 19) OK, it works.

6.3. Level 2 (from visual to numerical)

Pr.: What do you understand under the term 'error'?

St.: The difference between a true value and an approximation obtained by any measurement means.

Pr.: Let us call it absolute error. Absolute error divided by true value, is called relative error, that is, a measure of accuracy; it tells us how close a particular measurement is to the correct value. Multiply the relative error by one hundred and you get the 'percentage of error'.

St.: I see. But we can only determine them when we know the true value.

Pr.: Right. We are going to put numbers to our analysis above and we are going to use our tool to do so. Be aware that an unavoidable error, called round-off error, occurs because only a finite number of digits appear in the display. Calculations are performed with approximate representations of the actual numbers.

St.: How does it affect the calculations?

Pr.: Fortunately this error is negligible for our computing purposes. Thus, consider them as actual values. (Activating CW for five digits to be shown, 8 trapezoids and our former function $9 - x^2$). The numerical translation of our former pictures and beyond can be seen in CW: you can see how the approximations to the hoped actual area are evolving (see Figure 20)

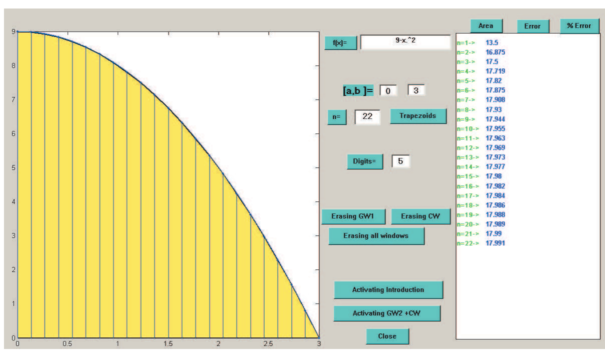


Fig. 20

St.: And I can guess the numerical value of the actual area to be 17.9.

Pr.: Why are you sure that these three digits are exact?

St.: The last two entries have the same first four digits, so is it reasonable certain that the first three digits represent the actual area. Isn't it?

Pr.: Given the smooth path of our graph, it is reasonable. But remember that what you are seeing is an approximate guess: be conscious of the inevitable error that accompanies the approximation. By the way, are those overestimations or underestimations of the supposed actual area?

St.: We covered that already in the visual screen. Since they are evolving upwards and each one is supposed to be better than the former one, I guess we are dealing with underestimations.

Now we deal with the delicate concept of 'actual' area under that curve. We point out that we are still lacking a definition of what area really is; he advances the idea that being able to measure the extension shown in Figure 20 could act as a definition. We ask him to elaborate this idea further; after some considerations he adventures a tentative definition. First, a method that works for our figure and all possible others should be available; he admits that the trapezoid method should be considered, even if his direct experience dealing with it is very limited. This method complies with his intuition over the area of the rectangle, but how does he deal with the fact that this method doesn't deliver an exact figure for the extension in Figure 20? He seems puzzled, but considers that the next best thing to exactness is that the method provides successively better and better estimations and that should be good enough. How does he know that those numbers are really related to the 'actual' area? Apart from intuition and what the screens GW1 and GW2 provide, he admits that it should be nice to have some warranty that those figures are heading inexorably somewhere and that place should correspond to the actual area.

Pr.: Even if we won't or can't get exactness, let us settle for a narrower objective for the time being. Is it reasonable at least to ask to be able to have some kind of control over the error committed?

St.: Meaning knowing how far I am from the actual value? But I do not know the actual value! Anyway, if there is any connection between the figures shown and the estimations calculated, those estimations using trapezoids look pretty good to me.

Pr.: One should be cautious. How good a figure looks depends on the scale used. Observe that some may think that our estimations are not brilliant: it took 7 trapezoids to get 3 exact digits and we had to go further to ensure it.

We point out that, in absence of figures or dealing with heavily oscillating curves where the trapezoids may go wildly (see Figures 21), just a list of values might be misleading since we might be very far from the actual area, supposing it exists.

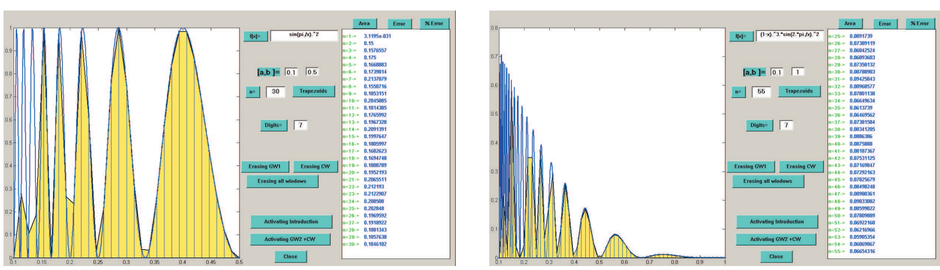


Fig. 21

St.: I am confused. Supposing there is an actual value, considering trapezoids and using the tool, the estimations obtained might not be approximations to the actual value. Is that what you mean?

Pr.: Yes. Some information on how the error behaves along the procedure of taking more and more trapezoids is necessary in order to know how far you are from the actual value. In a perfect world I would be happy providing you with a verifiable statement such as: if you want to have an error of magnitude ... in the estimation of the area provided by our procedure of using trapezoids, you need to take ... trapezoids.

St.: That would provide us with a clear rule and how far the approximation is from the actual value.

Pr.: Yes, but such a statement warrants a deeper insight on how this procedure works and its relation with the function considered and will come later in your mathematical life.

St.: No actual value and no way of knowing if my estimations are really approximations. What then?

Pr.: Let us see if our tool has predicting powers over the actual value and a glimpse on how fast the approximations run.

St.: I guess you are talking about, again, an approximate prediction.

We return to the function $9-x^2$, $x \in [0,3]$ and we activate GW2 and CW in our tool.

Pr.: What do you see?

St.: A cloud of points heading to the ordinate axis. (Looking at the area column) They seem to come close to the value 18.

Pr.: (directing his attention to the other two columns) Observe that the first one shows the difference between estimations and your predicted value and the second provides the percentage of error (see Figure 22) Observe that as $n = 5, 10, 20, 100$ increases, the error decreases.

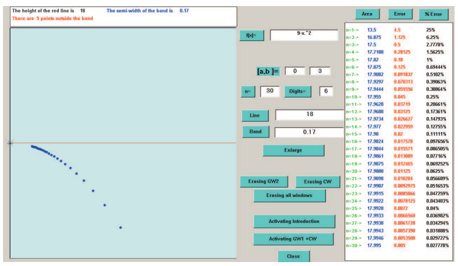


Fig. 22

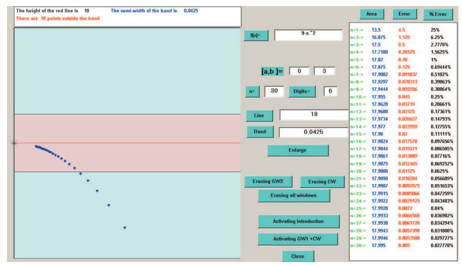


Fig. 23

St.: Is what it should happen if the method works, isn't it?

Pr.: Yes, but we are interested in how does this change occur. Have a look at the percentage window.

St.: From n jumping from 5 to 10 and (Figure 23), then, from 10 to 20 the percentages decrease to one fourth of the preceding number.

Pr.: Could you express what you just said but referring to GW2 in terms of band-width and location of the cloud?

St.: If the red line is drawn at height 18 and if we divide by one fourth the semi-width of the band around the red line, then the number of points of the cloud lying outside the band doubles (Figures 23 and 24).

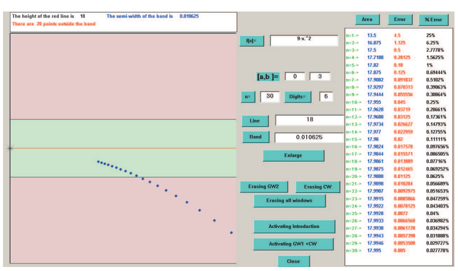


Fig. 24

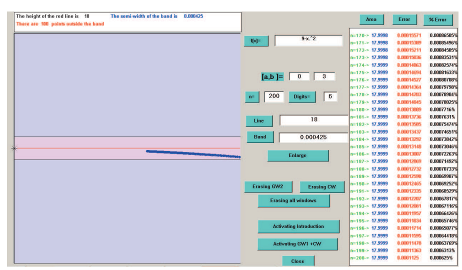


Fig. 25

Pr.: All the information you need is contained in the small window above which says that is, if I double n, the percentage shrinks one fourth. What about from 10 to 100 (Figure 25)?

St.: The percentage goes to 1/100 of the former value.

Pr.: What is your interpretation on GW2?

St.: If I take a band of semi-width 1/100 of the semi-width of the band which left ten points outside, now one hundred points are lying outside the new band.

Pr.: (adding more values for n) Squaring 10 you get 100. As you can see, this trend seems to continue. In this case, what can you conjecture about the existing relation between error and the value of n?

St.: The percentage of error is inversely proportional to the square of n. And this fact holds for other functions when choosing trapezoids?

Pr.: It does, but what we have done here is merely a hint; to prove it is a different matter.

St.: Then, the trapezoids procedure works fast, isn't it?

Pr.: Yes, it does although it is not necessarily the best available. Anyway, remember that our analysis was possible because we made a good guess for the actual area.

St.: Visually, the tool seems to provide a good guess about the actual area and, computationally, a reasonably powerful method for calculating areas in an approximate way, the larger n the better the approximation and the percentage error shrinking faster.

Pr.: Right. Could you explain how did you arrive to 18 as a guess for the actual area?

St.: Looking at GW2, the heights of the points in the cloud seem to stabilize around that height, although the area column in CW is enough to conjecture 18 as a good guess.

Pr.: Let us explore that window (erasing GW2 and leaving CW and reducing to five the number of digits shown Figure 26). What do you see?

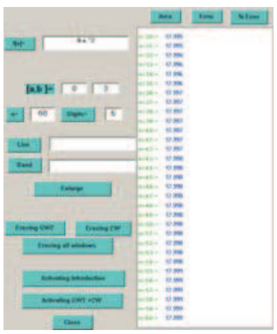


Fig. 26

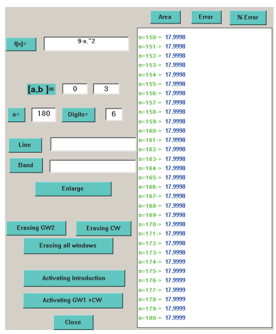


Fig. 27

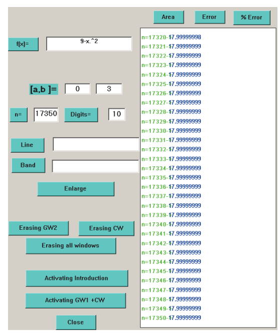


Fig. 28

St.: Are you cheating? The estimations tend to settle now to the value 17.999.

Pr.: (increasing the number of approximations) 17.999 being the difference between 18 and .001. What happens if I allow six digits to be shown?

St.: Stabilization to 17.9999, that is, 18 minus .0001 (Figure 27).

Pr.: What makes you think of 18 as a good guess instead of picking 17 followed by a certain number of nines?

St.: Because, theoretically, I could allow more and more digits present (Figure 28) and I can get as close to 18 as I want, since .00 . . . 01 gets as small as desired.

6.4 Level 3 (to the definition of area)

Concerning the needed algebraic operation of putting a dynamic process to rest,

Pr.: (we select the function $(9 - x^2)^{1/2}$, $x \in [0, 3]$ and we show GW1+CW to end up with GW2+CW, (Figures 29 and 30). Leaving outside everything which is not essential, you are left with a collection of numbers in CW which seem to approach a certain value that has to be determined precisely or a cloud of points in GW2 heading somewhere at the ordinate axis, which should be the actual area.

St.: You want me to consider numbers or points?

Pr.: We are on the road to abstraction and abstraction benefits from having the ability to deal with different representations of the same phenomenon, which is surely a good sign of understanding what is going on. Let us strip the problem to its bare essentials and forget about trapezoids,

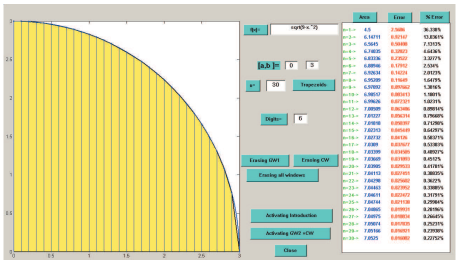


Fig. 29

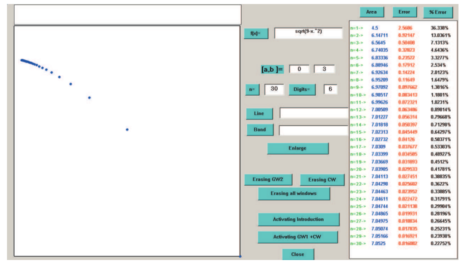


Fig. 30

approximations, areas and so on: (taking paper and pencil) algebraically, write a_1, a_2, ... , a_n, ... to denote those values shown in CW (that is, the approximations), the sub-indices standing for the first, the second, etc., and call it a *sequence* of numbers. If you look at GW2 you can also see the sequence, not as a table, but as a cloud of points which are ordered from right to left as the first, second, etc., their abscissas indicating position and their heights indicating numerical value.

St.: Right. And we have to determine where the sequence or the cloud of points is heading to, as we did above and that number will stand for the wanted area.

Pr.: Yes, but keep in mind that we are interested in three aspects: *If, where and how fast*. We have to determine whether the cloud approaches some value or not and, if it does, we should be precise about its concrete value, which should be guessed from the pool of candidates you may have, according to the information gathered at CW or GW2. The 'how fast' aspect refers to the efficiency of the method. In the event that such a number exists, we refer to it as the *limit* of the sequence.

St.: Shall we work in GW2?

Pr.: Yes and the limit (if existent) will be the height of the point in the ordinate axis where the cloud is heading to.

St.: It seems that the cloud approaches from below since the heights of their points increase; thus the limit has to be larger than any number in the sequence.

Now we ask him now if such a candidate for L is available. We suggest he uses the mouse to click over the ordinate axis at a chosen height. He chooses 7.06 as L and seems convinced of its suitability. We introduce him to the 'game of the sceptic' as follows: *I am the sceptic. I tell you: I need the cloud to get within a margin of error of .01 of L, or else I'm not convinced.* What I need is that you provide me with a position (that is an integer N) from which all points in the cloud are placed 'within' .01 of L. He demands more precision: What do you mean with 'within'? As visualization of margin of error, we invite him to draw a band of semi-width $\varepsilon = .01$ around his choice of L to visualize proximity. We explain that we use semi-widths instead of widths because, when other functions are considered, he may need to consider points at both sides of the line, that is, below and above from his chosen L and the tool should be useful in all instances. He produces the band (see Figure 31) and reads the message window above which reminds him of the height and error margin ε chosen as well as how many points of the cloud are placed outside it. He asks whether we are not interested or not on where the first N are placed. Well, we are not and we finish the explanation on the game: *suppose you are able to do a little computation or in our case (courtesy of the tool) a graphical manipulation, and come back and say, "No problem. As long as n is larger than a certain N (and you have to tell me to which N you are referring), you're guaranteed to be that close". Then, you win the game. If you are unable* *to produce N, I win*. That is the game: the tool manipulation is his proof; it's our responsibility to define 'close', but then it's his responsibility to show that he can get that close to L, provided that n is sufficiently large. Now we play with his candidate.

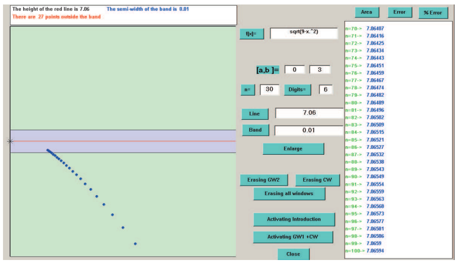


Fig. 31

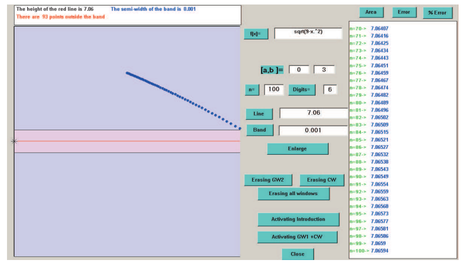


Fig. 32

St.: (manipulating the tool and looking at the window) Well, since I chose L as 7.06, the message tells me that there are 27 points outside; hence N equals 28, easy enough. And that's all?

Pr.: No. The point is we should be able to play this game as many times as I want, challenging you with smaller and smaller margins of error and you being able to produce the required integers N every time. If this happens, your guess L would be correct.

St.: Well, it reminds me on what you were doing before, allowing more and more digits to be seen.

Pr.: Indeed, it is the same idea, but we are performing the task now visually.

St.: Visually ... How to proceed?

Pr.: Use the tool again, and for your candidate L, draw a band around it of semi-width $\varepsilon = .001$ and use the zooming capability if needed. It may happen that you need to consider a larger value of n, that is, a larger cloud.

St.: (see Figure 32) Something isn't right: the points go over the line.

We ask him to elaborate why the game has collapsed. He changes the number of points taken, zooms around the line and so on, but ends up stating that probably his choice of L was not correct in the first place and that another candidate should be tried arguing that it should be close to the former election but larger. Since we are dealing with an arc of circumference, we suggest the choice 7.068583. He starts the game again.

St.: (manipulating the tool, considering first $\varepsilon = 0.001$ and then $\varepsilon = 0.0001$) Taking a semi-width of $\varepsilon = 0.001$, I find N as 192 (Figure 33). Now with $\varepsilon = 0.0001$, I need more points on the screen. Right, setting n as 1500, there are 885 points outside, hence N equals 886.

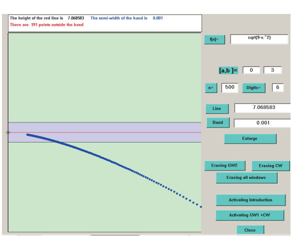


Fig. 33

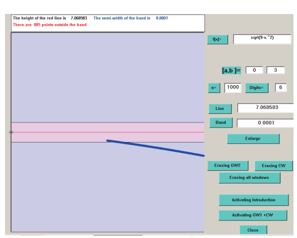


Fig. 34

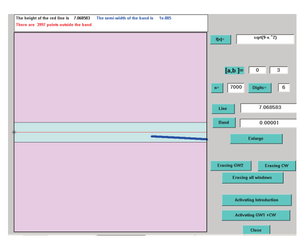


Fig. 35

Pr.: Where are the other points of the cloud?

St.: All of them inside the band. If I draw more points (manipulating the tool with a larger n, Figure 34), I still have the same number of points outside.

Pr.: Good. Now take $\varepsilon = 00001$.

St.: (using the tool again, see Figure 35) A larger n then, let us take $n = 7000$. Then $N = 3998$ can be selected.

Pr.: Now take $\varepsilon = .000001$.

St.: No need. I understand the game: you are taking smaller and smaller error margins ε and I am producing N's, so I win. Your chosen L seems to work if the trend continues.

Pr.: Right. Is there any relation between margins of error ε and positions N?

St.: The smaller ε is, the bigger is N. That is, the smaller the band, the more points of the cloud are lying outside and it goes on forever.

We ask him to synthesize what we are doing. What to do in absence of good guess on L? One could go on and on improving ones approximations, although none would produce the right answer, if there is actually a right answer. Even if there is one, it may happen that our estimations are far away from the actual value. So he has to go for a choice of L and, if correct, what can be checked by playing the game indefinitely, the limit L is the precisely the number which produces the actual area.

Pr.: Indeed. How could you know if your choice for L is not good enough?

St. If, at some step, the game collapses in the sense that there is no possibility of finding N and then, you win. If that happens, I should explore other choices by trial and error.

Pr.: But surely an educated guess.

St.: That's what the tool provides. Well, all that accounts for the *If* and *Where*. What about the *How fast*?

Pr.: For the same margin of error, the smaller the N detected, the faster the sequence to reach its end.

St.: Understood, but ...

Pr.: You don't seem to be comfortable.

What is the contribution of the tool? He comments that it gives him a chance to guess the limit with a certain confidence, although he seems bothered: How does one reach the limit? Is it by following an infinite number of steps? He rightly points out that he has just travelled a finite number of them in the game and hopes for the best in the remaining ones. He poses the inevitable question: How do we play such a game in an infinite time?

Pr.: Well, we are not entitled to such a luxury. Some compromise with experience is required, so we have to enter into an imaginary mental world. Since we need to play the game in a finite time, we should abandon the visual setting and enter the world of algebra.

St.: Why do we have to go the algebraic path?

Pr.: Well, unfortunately there is no shortcut eluding mathematical reasoning: once a basic grasp is achieved and I think we have done that, we need refinement and more systematic articulation through algebra which is supposed to be the lingua franca of mathematics courses; we use it precisely to avoid the vagueness of words and pictures.

St.: 'Mathematical' meaning 'algebraic'? Is there an algebraic way to reach the limit?

Pr.: Yes and to discover that way is the delicate part of this story: translating this infinite game into a few finite symbols with which we can manipulate algebraically and logically and do precisely as intended, that is, reaching the limit. You will study this stuff when you enter University.

St.: But, then, what does the visual game provide?

Pr.: Basically, two ideas: first, the possibility of discarding several wrong choices; when a choice works in the game, it might not give you certainty, but a hope that it will work for smaller and smaller errors. Even if you are not certain about the choice, the true value will be lying very close to it. Secondly, the algebraic formulation of the game is hidden in the instructions of the game; once, you are proficient with symbolic and algebraic manipulations, you will be able to define ‘limit’ without pain and ensure therefore the existence of the actual area.

St.: But, even if the limit is warranted, I still do not see how to finish the game because it does not free me from making calculations indefinitely.

Pr.: Because a good by-product of that promised translation, and you shall learn it in due time, is a formula relating margins of error ε with integers N (something analogous to what our message window does) and the formula will speak for itself, it shows that you can react to any to my choices; the formula encapsulates an infinity of choices and calculations and you win the game.

We end our experience with examples showing that not every approximation has to be better than the preceding one and that a suitable candidate for area is sometimes not easy to find, showing the limitations of our approach. For the function $f(x) = \sin^2(4\pi/x)/x$ in the interval $[0.01, 0.5]$, GW1 and CW (Figure 36) illustrate our first point whereas Figures 37 and 38 for GW2 and CW show that a choice between candidates 1.8, 1.9 and 2 cannot be made (1500 trapezoids) where enlarging the number of trapezoids to 1600 we still have doubts between 1.8 and 1.9. The tool is available on request.

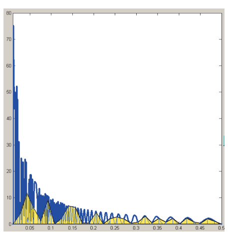


Fig. 36

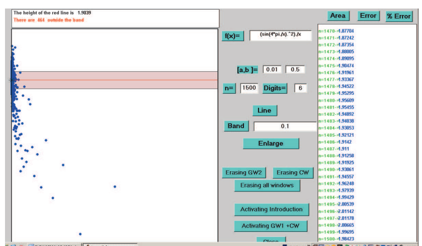


Fig. 37

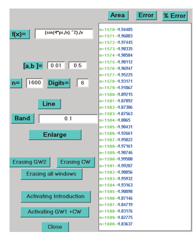


Fig. 38

Those last considerations show that a visual understanding of what a limiting process means is not enough to develop algorithms to calculate limits efficient and precisely. When we leave the safety of conjectural reasoning and intend to jump to exact reasoning through formalization, a different battery of actions is required. If what is needed is a precise algebraic definition of what a limit is and how to calculate it, we have to wait since it is not easy to make young students sensitive to such concerns which are not really part of their mathematical culture and establishing adequate criteria for precise definitions and the idea of proof lie beyond their present reach, showing that there are limits to what one hopes to achieve with High School students when confronted with delicate mathematical ideas without the benefit of a period of incubation. The transition towards more formal approaches, which usually takes place at the University, represents a tremendous jump, both conceptually and technically, but all ingredients are present in what we have achieved, except the algebraic formulation of an operation which treats an indefinite sequence of numbers (a dynamic process) and puts it to rest by showing a precise number which intuitively stands for the number the sequence approaches or gets close. In other words, what is needed is to define algebraically what “*approach*” or “*get* *close*" means. For a Socratic dialog leading to put all those considerations under the optic of a visualization tool, study how those images translate in mathematical statements favouring the use of logical quantifiers and framing the whole study in van Hiele's model allowing the study the cognitive obstacles that arise, we refer the interested reader to Navarro and Pérez Carreras [3].

7. Coda: Level descriptors

7.1 Level 0:

The interviewee

0.1 Identifies 'area' with the known formula for squares and rectangles. When pressed, is unable to produce a definition of the term, although this fact doesn't preclude us to explore this notion.

0.2 Advances the idea of calculating areas of plane figures by reducing the problem to the use of inserted squares and/or rectangles. Recognizes that this idea doesn't work for curvilinear figures.

0.3 Agrees that in order to produce a definition of area, certain intuitive characteristics have to be respected by the wanted definition and is able to develop area calculations for polygons.

0.4 Confronted with curvilinear figures, the idea of visual approximation comes to light.

7.2 Level 1:

1.1 Agrees in extending the previous method of inserted polygons to calculate areas of curvilinear figures, recognizing that the method provides approximations instead of the actual value.

1.2 As a first use of the tool and in the setting 'area under a curve', perceives the importance of a continuous curve being locally straight in order to get satisfactory visual approximations to the area which are translated verbally in selecting blocks with tiny bases.

1.3 Starts choosing rectangles as natural building blocks for the method, once an appropriate selection of grid and heights is made. But, due to local straightness, settles for trapezoids.

1.4 Recognizes visually that, in the context of graphs of monotonic functions, area estimations by defect or excess are both possible and recognizes concavity as the deciding factor. For more complex paths, adventures the idea of partition of the path in monotonic paths to apply the method.

1.5 Points out that 'the more trapezoids the better' as a verbal statement of what visual approximation means and extends that idea to the behaviour of a cloud of points in the plane heading to the ordinate axis, as shown in the graphical window.

1.6 Recognizes the importance of steepness of the path as a deciding factor on how to select the grid.

7.3 Level 2:

2.1 Understands the meaning of error.

2.2 Confronted with a table relating grids and estimations from our former curve deduces the defect/excess item and is able to settle for a certain number of digits which should be kept as 'true values of the actual area, as a statement of what numerical approximation means.

2.3 Presented with our tool, shows proficiency in its use and applies it to our former functions to check on previous visual conclusions.

2.4 Starts to grasp the idea of defining area by the method calculating it.

2.5 In absence of a way of deducing the value for the actual area, recognizes the need for conjecturing the actual value from the tool and also for a control of the error committed.

2.6 Using the tool, proceeds to conjecture the actual value and proceeds to relate grid and error to appreciate the convenience of dealing with a fast method.

2.7 Refines previous ideas on what numerical approximation means, by using the computational window, relating number of digits to be kept in view and the idea of getting close.

7.4 Level 3:

3.1 States that the product of an approximation process stands for a definition of area.

3.2 Translates the idea of numerical approximation from the computational one studied in Level 2 to a more refined use of the visual setting by identifying margins of error with two-sided bands to cover points approaching from above and below.

3.3 Establishes the relation between width of the band and number of points lying outside it.

3.4 Confidently plays the proposed game in a finite number of steps and verbalizes correctly the logical dependence of statements.

3.5 Confronted with the physical impossibility of playing the game indefinitely, understands the need to perform another change of setting, this time from the visual to the algebraic.

8. Addendum

In a forthcoming study we shall study all dysfunctions which do not show in our transcription of the interview. We shall point out the causes of failure and we shall propose remedial or tutorial actions for those students unable to reach the interview's conclusion, as well as validating our study in the frame of van Hiele's model by proving the existence of Levels as well as assigning each participant to the level reached according to our descriptors (Section 7). Let we advance some of our findings

(The interview) It was clear to us that verbal explanation presented serious challenges to many students, although the most gifted and curious ones were able to refine their language in terms of verbal accuracy along the interview even if a certain disdain regarding precision in verbal communication skills was noticeable: once an idea was understood, no much value was placed in expressing it unambiguously as if insight was only relevant to the individual and no communication abilities were deemed relevant.

(The tool) Curiously enough, lack of familiarity with a computer-generated tool presented no problems whatsoever and students quickly adapted to their use, some of them with remarkable proficiency. Visualization of limiting processes animated the experience triggering their natural curiosity and they enjoyed taking partial command of the interview. Executing commands in the tool and producing graphs and tables was an easy task for them and we sensed that they benefited from the open door to higher mathematics provided by the experience and from their first encounter with the tool. Not unexpectedly, it generated excitement and its manipulation became sometimes the unintended focus of their efforts, but we sensed also that efficient graphical capabilities without flexible power of interpretation could lead them to failure when interpretation was demanded.

(Success/Failure) From our pool of twenty interviewed students, three of them had to be discarded from the interview when the word 'unending' intruded in Level 0 (π as a never-ending-sequence of digits) and entering Level 1 (use of trapezoids in a never-ending procedure) as they were convinced 'finitists' and no appeal to their imagination was successful: for them infinity was metaphysical and '*one should avoid speaking about it*'. The rest accepted the process as legitimate: a vast majority of them trusted what they saw at the screen without further thought and made no objections whatsoever, accepting that the process can be repeated indefinitely, at least theoretically. *Three* more students were discarded when the 'cloud' made his appearance since they were unable to exhibit a clear understanding of the relationship between trapezoids and points of the cloud. Fourteen students entered Level 2 and *two* of them were uneasy with the idea of error and the need for a definition of the concept area and left due to their reluctance to admit that a never-ending process could result in a valid conclusion, which for them should be reached in a finite number of steps. Level 3 witnessed the failure of *three* more students due to their inability to provide sound verbal explanations on how convergence should be formulated in terms of epsilons and integers. Nine students completed successfully the interview exhibiting precise economical verbal explanations, showing that implicit logical quantification was part of their reasoning armoury, although sailing through Level 3 was a bumpy road for some of them and this part of the interview took more time than expected.

REFERENCES

[1] D. Herceg, Đ. Herceg, *The definite integral and computer*, The Teaching of Mathematics, **XII**, 1 (2009), 33–44.

[2] J. L. Llorens, P. Pérez Carreras, *An extension of van Hiele's model to the study of local*

approximation, International Journal of Mathematical Education in Science and Technology, **28** (1997), 713-726.

[3] A. Navarro, P. Pérez Carreras, *Constructing a concept image on convergence of sequences in the Van Hiele Framework*, F. Hitt, G. Harel and A.Selden, eds., Research in Collegiate Mathematics Education VI, CBMS Issues in Mathematics Education 13, AMS/MAA, pp. 61–98.

[4] A. Navarro, P. Pérez Carreras, *Local rate of change: a Socratic experience in van Hiele's model framework*, The Teaching of Mathematics, **XIII**, 2 (2010), 63–92.

[5] A. Navarro, P. Pérez Carreras, *The indefinite accumulation of finite amounts*, International Journal for Mathematics Teaching and Learning (2011-06-15).

[6] A. Navarro, P. Pérez Carreras, *An empirical study in convergence via iteration and its convergence*, The Teaching of Mathematics, **XV**, 2 (2012), 85–112.

[7] D. Tall, S. Vinner, *Concept image and concept definition in mathematics with particular reference to limits and continuity*, Educational Studies in Mathematics, **12** (1981), 151–169.

[8] P. M. Van Hiele, *Structure and Insight: A Theory of Mathematics Education*, New York: Academic Press, 1986. Inc. verbal material, J. Educational Psychology, **51** (1978), 267–272.

[9] P. M. Van Hiele, *La pense de l'enfant et la geometrie*, Bull. l'Association des Professeurs Mathematiques de l'Enseignement Public, 1959, pp. 198–205.

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