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A POWER SERIES APPROACH TO AN INEQUALITY AND ITS GENERALIZATIONS

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Abstract. In this paper, we present and prove an inequality and its several generalizations by using power series and Muirhead inequality.

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Power series provide a very useful tool for proving several inequalities. Thus, many difficult problems can be easily solved and often even extended. For example, C. Mortici in [3] and M. Jeong in [1], proved Nesbith's inequality using power series and modified it to obtain several other inequalities.

In this paper, we present and prove an olympiad-type inequality and several of its modifications by using power series and Muirhead inequality.

Problem (19th Moscow Mathematical Olympiad). *If $a, b \in \mathbb{R}$ and $|a| < 1$, $|b| < 1$, then*

$$\frac{1}{1-a^2} + \frac{1}{1-b^2} \geqslant \frac{2}{1-ab}.$$

Proof. By using power series, since $|a| < 1$, $|b| < 1$, we can write

$$\frac{1}{1-a^2} = \sum_{k=0}^{\infty} (a^2)^k, \quad \frac{1}{1-b^2} = \sum_{k=0}^{\infty} (b^2)^k, \quad \frac{1}{1-ab} = \sum_{k=0}^{\infty} (ab)^k. \tag{1}$$

Thus, it is enough to prove that

$$\sum_{k=0}^{\infty} a^{2k} + \sum_{k=0}^{\infty} b^{2k} \geqslant 2\sum_{k=0}^{\infty} (ab)^k,$$

that is, $\sum\limits_{k=0}^{\infty} (a^{2k} + b^{2k} - 2a^k b^k) \geqslant 0$. Hence, we need just to prove that

$$\sum_{k=0}^{\infty} (a^k - b^k)^2 \geqslant 0,$$

which is obvious. The proof is complete. ∎

In order to prove several generalizations of the previous inequality, we first recall the following well-known notions and assertion (see, e.g., [2]).

Definition 1. Let $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_n)$ be two arrays from $\mathbb{R}^n$. We say that a majorizes b (denoted as $a \succ b$) if: (1) $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$; (2) $a_1 + a_2 + \cdots + a_k \geqslant b_1 + b_2 + \cdots + b_k$ for all k, $1 \leqslant k \leqslant n-1$; (3) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$.

Definition 2. Let $a = (a_1, a_2, \dots, a_n)$ be an array of nonnegative real numbers and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Denote

$$T[a_1, a_2, \dots, a_n](x_1, x_2, \dots, x_n) = \sum\nolimits^{!} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $\sum^{!}$ is the sum of $n!$ summands, taken over all possible permutations of the sequence $x = (x_i)_{i=1}^n$. We shall write just $T[a_1, a_2, \dots, a_n]$ if it is clear which sequence x is used.

Theorem. [Muirhead inequality] *The expression $T[a]$ is comparable with the expression $T[b]$ for all positive sequences x, if and only if one of the sequences a and b majorizes the other one in the sense of relation $\prec$. If $a \prec b$ then $T[a] \leqslant T[b]$. The equality holds if and only if the sequences a and b are identical, or all the x_i's are equal.*

In other words, if $x_1, x_2, \dots, x_n$ are positive reals, and $a = (a_i)_{i=1}^n$ majorizes $b = (b_i)_{i=1}^n$, then we have the inequality

$$\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geqslant \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}.$$

For example, since $(5,0,0) \succ (3,1,1) \succ (2,2,1)$, we obtain

$$\begin{aligned} a^5 + a^5 + b^5 + b^5 + c^5 + c^5 &\geqslant a^3bc + a^3bc + b^3ca + b^3ca + c^3ab + c^3ab \\ &\geqslant a^2b^2c + a^2b^2c + b^2c^2a + b^2c^2a + c^2a^2b + c^2a^2b. \end{aligned}$$

From this we derive $a^5 + b^5 + c^5 \geqslant a^3bc + b^3ca + c^3ab \geqslant abc(ab + bc + ca)$.

Notice that Muirhead inequality is symmetric, not cyclic. For example, even though $(3,0,0) \succ (2,1,0)$, it gives only

$$2(a^3 + b^3 + c^3) \geqslant a^2b + a^2c + b^2c + b^2a + c^2a + c^2b,$$

and in particular this does not imply $a^3 + b^3 + c^3 \geqslant a^2b + b^2c + c^2a$.

Proposition 1. *Let $a_i \in \mathbb{R}$ and $|a_i| < 1$, $i = 1, 2, \dots, n$, and let $a_{n+1} = a_1$. Then*

$$\sum_{i=1}^{n} \frac{1}{1 - a_i^2} \geqslant \sum_{i=1}^{n} \frac{1}{1 - a_i a_{i+1}}.$$

Proof.

$$\begin{aligned}
&\frac{1}{1-a_1^2}+\frac{1}{1-a_2^2}+\cdots+\frac{1}{1-a_n^2}\\
&\qquad=\sum_{k=0}^{\infty}(a_1^2)^k+\sum_{k=0}^{\infty}(a_2^2)^k+\cdots+\sum_{k=0}^{\infty}(a_n^2)^k\\
&\qquad=\frac{1}{2}\left(\sum_{k=0}^{\infty}(a_1^{2k}+a_2^{2k})\right)+\frac{1}{2}\left(\sum_{k=0}^{\infty}(a_2^{2k}+a_3^{2k})\right)+\cdots+\frac{1}{2}\left(\sum_{k=0}^{\infty}(a_n^{2k}+a_1^{2k})\right)\\
&\qquad\geqslant\sum_{k=0}^{\infty}a_1^ka_2^k+\sum_{k=0}^{\infty}a_2^ka_3^k+\cdots+\sum_{k=0}^{\infty}a_n^ka_1^k\\
&\qquad=\frac{1}{1-a_1a_2}+\frac{1}{1-a_2a_3}+\cdots+\frac{1}{1-a_na_1}.
\end{aligned}$$

The proof is complete. ■

Proposition 2. *If $a,b\in(0,1)$ and $n\in\mathbb{N}$, then*

$$\frac{a^n}{1-a^2}+\frac{b^n}{1-b^2}\geqslant\frac{a^n+b^n}{1-ab}.$$

Proof. By using power series, we can write the relations (1). Thus, we need to prove

$$a^n\sum_{k=0}^{\infty}a^{2k}+b^n\sum_{k=0}^{\infty}b^{2k}\geqslant(a^n+b^n)\sum_{k=0}^{\infty}(ab)^k,$$

i.e., $\sum\limits_{k=0}^{\infty}(a^{n+2k}+b^{n+2k}-a^{n+k}b^k-a^kb^{n+k})\geqslant 0$. This follows from $(a^k-b^k)(a^{n+k}-b^{n+k})\geqslant 0$ for each k, which is obvious. The proof is complete. ■

Generalization. If $a_i\in(0,1)$, $i=1,2,\ldots,n$, then

$$\sum_{i=1}^{n}\frac{a_i^n}{1-a_i^n}\geqslant\frac{\sum\limits_{i=1}^{n}a_i^n}{1-\prod\limits_{i=1}^{n}a_i}.$$

Proof. Similarly as in Proposition 2, we have just to prove that

$$\begin{aligned}
a_1^{n+nk}+a_2^{n+nk}+\cdots+a_n^{n+nk}&\geqslant a_1^ka_2^k\cdots a_n^k(a_1^n+a_2^n+\cdots+a_n^n)\\
&=a_1^{n+k}a_2^k\cdots a_n^k+a_1^ka_2^{n+k}\cdots a_n^k+\cdots+a_1^ka_2^k\cdots a_n^{n+k}.
\end{aligned}$$

Since $(n+nk,\underbrace{0,\ldots,0}_{k-1})\succ(n+k,\underbrace{k,\ldots,k}_{k-1})$, using Muirhead inequality we have that

$$\begin{aligned}
&(n-1)!\left(a_1^{n+nk}+a_2^{n+nk}+\cdots+a_n^{n+nk}\right)\geqslant\\
&\qquad(n-1)!\left(a_1^{n+k}a_2^k\cdots a_n^k+a_1^ka_2^{n+k}\cdots a_n^k+\cdots+a_1^ka_2^k\cdots a_n^{n+k}\right),
\end{aligned}$$

which completes the proof. ■

Proposition 3. *If $a, b \in (0,1)$, $m, n \in \mathbb{N}$, $n \geqslant m$, then*

$$\frac{a^n}{1-a^{2m}} + \frac{b^n}{1-b^{2m}} \geqslant \frac{a^n+b^n}{1-(ab)^m}.$$

Proof is similar as for Proposition 2. ∎

Generalization. *If $a_i \in (0,1)$, $i = 1, 2, \dots, n$, $m \in \mathbb{N}$, $n \geqslant m$, then*

$$\sum_{i=1}^{n} \frac{a_i^n}{1-a_i^{mn}} \geqslant \frac{\sum\limits_{i=1}^{n} a_i^n}{1-\big(\prod\limits_{i=1}^{n} a_i\big)^m}.$$

Proof is similar as for Generalization of Proposition 2. ∎

Proposition 4. *If $a, b \in (0,1)$, $n \in \mathbb{N}$, then*

$$\frac{a^n}{\sqrt{1-a^2}} + \frac{b^n}{\sqrt{1-a^2}} \geqslant \frac{a^n+b^n}{\sqrt{1-ab}}.$$

Proof. By squaring both sides, we obtain the equivalent inequality

$$\frac{a^{2n}}{1-a^2} + \frac{b^{2n}}{1-b^2} + \frac{2a^nb^n}{\sqrt{1-a^2}\,\sqrt{1-b^2}} \geqslant \frac{a^{2n}+b^{2n}+2a^nb^n}{1-ab}. \tag{2}$$

By Proposition 2 we know that $\dfrac{a^{2n}}{1-a^2} + \dfrac{b^{2n}}{1-b^2} \geqslant \dfrac{a^{2n}+b^{2n}}{1-ab}$. Furthermore, we have

$$\frac{1}{\sqrt{1-a^2}\,\sqrt{1-b^2}} = \frac{1}{\sqrt{1+a^2b^2-(a^2+b^2)}} \geqslant \frac{1}{\sqrt{1+a^2b^2-2ab}} = \frac{1}{1-ab}.$$

The previous two inequalities imply that (2) holds. The proof is complete. ∎

Proposition 5. *If $a, b \in (0,1)$, then* $\dfrac{a}{1-b^2} + \dfrac{b}{1-a^2} \geqslant \dfrac{a+b}{1-ab}$.

Proof. By using power series, we can write

$$\frac{a}{1-b^2} + \frac{b}{1-a^2} = a\sum_{k=0}^{\infty} b^{2k} + b\sum_{k=0}^{\infty} a^{2k} = \sum_{k=0}^{\infty} (ab^{2k} + a^{2k}b).$$

and

$$\frac{a+b}{1-ab} = (a+b)\sum_{k=0}^{\infty} a^kb^k = \sum_{k=0}^{\infty} (a^{k+1}b^k + a^kb^{k+1}).$$

Thus, we just need to prove that $ab^{2k} + a^{2k}b \geqslant a^{k+1}b^k + a^kb^{k+1}$. But this follows from $ab^{2k} + a^{2k}b - a^{k+1}b^k - a^kb^{k+1} = ab(a^k-b^k)(a^{k-1}-b^{k-1}) \geqslant 0$. This completes the proof. ∎

GENERALIZATION. If $a_i \in (0,1)$, $i = 1, 2, \ldots, n$, $a_{n+1} = a_1$, then

$$\sum_{i=1}^{n} \frac{a_i}{1 - a_{i+1}^n} \geqslant \frac{\sum\limits_{i=1}^{n} a_i}{1 - \prod\limits_{i=1}^{n} a_i}.$$

Proof is similar to that for Generalization of Proposition 2. ■

PROPOSITION 6. *If* $a, b \in (0,1)$ *and* $n \in \mathbb{N}$, *then* $\dfrac{a^{2n}}{1-b^2} + \dfrac{b^{2n}}{1-a^2} \geqslant \dfrac{a^{2n}+b^{2n}}{1-ab}$.

Proof is similar to that of Proposition 5. ■

PROPOSITION 7. *If* $a, b \in (0,1)$, *then* $\dfrac{a}{\sqrt{1-b}} + \dfrac{b}{\sqrt{1-a}} \geqslant \dfrac{a+b}{\sqrt{1-\sqrt{ab}}}$.

Proof. By squaring both sides, we obtain the equivalent inequality

$$\frac{a^2}{1-b} + \frac{b^2}{1-a} + \frac{2ab}{\sqrt{1-a}\,\sqrt{1-b}} \geqslant \frac{a^2+b^2+2ab}{1-\sqrt{ab}}. \tag{4}$$

Firstly, we will prove that

$$\frac{a^2}{1-b} + \frac{b^2}{1-a} \geqslant \frac{a^2+b^2}{1-\sqrt{ab}}. \tag{5}$$

By using power series, we can write

$$\frac{a^2}{1-b} + \frac{b^2}{1-a} = a^2 \sum_{k=0}^{\infty} b^k + b^2 \sum_{k=0}^{\infty} a^k = \sum_{k=0}^{\infty} (a^2 b^k + b^2 a^k).$$

Without loss of generality, assume that $a \geqslant b \geqslant 0$. Then we have $b^{k/2} - a^{k/2} \leqslant 0$, $a^2 b^{k/2} \leqslant b^2 a^{k/2}$ and hence $(a^2 b^{k/2} - b^2 a^{k/2})(b^{k/2} - a^{k/2}) \geqslant 0$, wherefrom $a^2 b^k + b^2 a^k \geqslant a^{2+\frac{k}{2}} b^{\frac{k}{2}} + b^{2+\frac{k}{2}} a^{\frac{k}{2}}$. It follows that

$$\frac{a^2}{1-b} + \frac{b^2}{1-a} \geqslant \sum_{k=0}^{\infty} \left(a^{2+\frac{k}{2}} b^{\frac{k}{2}} + b^{2+\frac{k}{2}} a^{\frac{k}{2}}\right) = (a^2+b^2) \sum_{k=0}^{\infty} (\sqrt{ab})^k = \frac{a^2+b^2}{1-\sqrt{ab}}.$$

Furthermore, we have

$$\frac{2ab}{\sqrt{1-a}\,\sqrt{1-b}} = \frac{2ab}{\sqrt{1+ab-(a+b)}} \geqslant \frac{2ab}{\sqrt{1+ab-2\sqrt{ab}}} = \frac{2ab}{\sqrt{1-ab}}. \tag{6}$$

Adding up relations (5) and (6), we obtain inequality (4). The proof is complete. ■

Proposition 8. *If $a, b \in (0,1)$, then* $\dfrac{a}{\sqrt{1-b^2}} + \dfrac{b}{\sqrt{1-a^2}} \geqslant \dfrac{a+b}{\sqrt{1-ab}}$.

Proof. By squaring both sides, we obtain the equivalent inequality

$$\frac{a^2}{1-b^2} + \frac{b^2}{1-a^2} + \frac{2ab}{\sqrt{1-a^2}\sqrt{1-b^2}} \geqslant \frac{a^2+b^2+2ab}{1-ab}. \tag{7}$$

Firstly, we will prove that

$$\frac{a^2}{1-b^2} + \frac{b^2}{1-a^2} \geqslant \frac{a^2+b^2}{1-ab}. \tag{8}$$

By using power series, we can write

$$\frac{a^2}{1-b^2} + \frac{b^2}{1-a^2} = a^2 \sum_{k=0}^{\infty} b^{2k} + b^2 \sum_{k=0}^{\infty} a^{2k} = \sum_{k=0}^{\infty} (a^2 b^{2k} + b^2 a^{2k}).$$

Without loss of generality, assume that $a \geqslant b \geqslant 0$. Then we have $b^k - a^k \leqslant 0$, $a^2 b^k \leqslant b^2 a^k$ and hence $(a^2 b^k - b^2 a^k)(b^k - a^k) \geqslant 0$, wherefrom $a^2 b^{2k} + b^2 a^{2k} \geqslant a^{2+k} b^k + b^{2+k} a^k$. It follows that

$$\frac{a^2}{1-b^2} + \frac{b^2}{1-a^2} \geqslant \sum_{k=0}^{\infty} \left(a^{2+k} b^k + b^{2+k} a^k\right) = (a^2+b^2) \sum_{k=0}^{\infty} (ab)^k = \frac{a^2+b^2}{1-ab}.$$

Furthermore, we have

$$\frac{2ab}{\sqrt{1-a^2}\sqrt{1-b^2}} = \frac{2ab}{\sqrt{1+a^2b^2-(a^2+b^2)}} \geqslant \frac{2ab}{\sqrt{1+a^2b^2-2ab}} = \frac{2ab}{1-ab}. \tag{9}$$

Adding up relations (8) and (9), we obtain inequality (7). The proof is complete. ∎

Proposition 9. *If $a, b, c \in (0,1)$ then*

$$\frac{a}{1-a^3} + \frac{b}{1-b^3} + \frac{c}{1-c^3} \geqslant \frac{a+b+c}{1-abc}.$$

Proof. By using power series, we can write

$$\begin{aligned} \frac{a}{1-a^3} + \frac{b}{1-b^3} + \frac{c}{1-c^3} &= a \sum_{k=0}^{\infty} a^{3k} + b \sum_{k=0}^{\infty} b^{3k} + c \sum_{k=0}^{\infty} c^{3k} \\ &= \sum_{k=0}^{\infty} (a^{3k+1} + b^{3k+1} + c^{3k+1}). \end{aligned}$$

and

$$\frac{a+b+c}{1-abc} = (a+b+c) \sum_{k=0}^{\infty} (abc)^k = \sum_{k=0}^{\infty} (a^{k+1} b^k c^k + a^k b^{k+1} c^k + a^k b^k c^{k+1}).$$

Thus, we just need to prove

$$a^{3k+1}+b^{3k+1}+c^{3k+1} \geqslant a^{k+1}b^kc^k+a^kb^{k+1}c^k+a^kb^kc^{k+1} \tag{10}$$

for $k \in \mathbb{N}\cup\{0\}$. Since $(3k+1,0,0) \succ (k+1,k,k)$, by using Muirhead inequality, we have that

$$\begin{aligned}
&a^{3k+1}+a^{3k+1}+b^{3k+1}+b^{3k+1}+c^{3k+1}+c^{3k+1}\\
&\qquad \geqslant a^{k+1}b^kc^k+a^{k+1}b^kc^k+a^kb^{k+1}c^k+a^kb^{k+1}c^k+a^kb^kc^{k+1}+a^kb^kc^{k+1},
\end{aligned}$$

wherefrom the inequality (10) follows. This completes the proof. ∎

Generalization. *If $a_i \in (0,1)$, $i=1,2,\dots,n$, then*

$$\sum_{i=1}^{n} \frac{a_i}{1-a_i^n} \geqslant \frac{\sum\limits_{i=1}^{n} a_i}{1-\prod\limits_{i=1}^{n} a_i}.$$

Proof is similar as for Generalization of Proposition 2. ∎

Proposition 10. *If $a,b \in (0,1)$ then* $\dfrac{a^3}{(1-a^2)^2}+\dfrac{b^3}{(1-b^2)^2} \geqslant \dfrac{(a+b)ab}{(1-ab)^2}$.

Proof. By using $\sum\limits_{k=1}^{\infty} kx^k = \dfrac{x}{(1-x)^2}$, $x \in (0,1)$, we can write

$$\begin{aligned}
\frac{a^3}{(1-a^2)^2}+\frac{b^3}{(1-b^2)^2} &= a\cdot\frac{a^2}{(1-a^2)^2}+b\cdot\frac{b^2}{(1-b^2)^2}\\
&= a\sum_{k=1}^{\infty} ka^{2k}+b\sum_{k=1}^{\infty} kb^{2k}=\sum_{k=1}^{\infty} k(a^{2k+1}+b^{2k+1})\\
&\geqslant \sum_{k=1}^{\infty} k(a^{k+1}b^k+a^kb^{k+1})=(a+b)\sum_{k=1}^{\infty} k(ab)^k\\
&= \frac{(a+b)ab}{(1-ab)^2}
\end{aligned}$$

(we have used Muirhead inequality with $(2k+1,0) \succ (k+1,k)$). The proof is completed. ∎

Generalization. *If $a_i \in (0,1)$, $i=1,2,\dots,n$, then*

$$\sum_{i=1}^{n} \frac{a_i^3}{(1-a_i^n)^2} \geqslant \frac{\Bigl(\sum\limits_{i=1}^{n} a_i\Bigr)\Bigl(\prod\limits_{i=1}^{n} a_i\Bigr)}{\Bigl(1-\prod\limits_{i=1}^{n} a_i\Bigr)^2}.$$

Proof is similar as for generalization of Proposition 2. ∎

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