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EXPLORING QUADRATIC EQUATIONS WITH DIGITAL TOOLS IN MATHEMATICS TEACHER EDUCATION

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Abstract. The paper offers teaching ideas for a technology-enhanced mathematics teacher education course related to an advanced inquiry into one-variable quadratic equations with parameters in place of coefficients. Mathematical activities behind the ideas have been interpreted using the notions of the technology immune/technology enabled (TITE) problem, digital fabrication, collateral learning, and hidden mathematics curriculum. The activities, leading to explorations that can be considered as rudiments of real problems arising in science and engineering, are in support of recommendations by the Conference Board of the Mathematical Sciences [The Mathematical Education of Teachers II, Washington, DC: Mathematical Association of America, 2012] for the preparation of secondary teacher candidates.

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1. Introduction

Many problems of science and engineering require knowledge of the roots of a one-variable polynomial $y = P(x)$ and their location in the (x, y)-plane. For example, one important property of $P(x)$ for applications is for its roots to be located to the left of the y-axis or inside the unit disk (e.g., [15]). Different methods exist to decide such properties of $P(x)$ without finding the roots explicitly (e.g., [14, 17]). When only real roots are considered, the problem of deciding the number of different (not multiple) roots of $P(x)$ that belong to a given interval was solved by a French mathematician of the 19th century Jacques Charles François Sturm [8, 23] who developed an algorithm (reminiscent of the Euclidean algorithm of finding the greatest common divisor of two integers) of exploring the change of signs of a recursively constructed sequence of auxiliary polynomials called Sturm's functions (starting from $P(x)$ and its first derivative) at the end points of the interval. However, those general methods, while have to be studied by the future extenders of the frontiers of STEM (science, technology, engineering, mathematics) knowledge, are outside the boundary of the mainstream mathematics curriculum, including that of a secondary mathematics teacher preparation program.

This paper is written to offer teaching ideas for such a program that include certain rudiments of STEM problems using grade appropriate mathematical machinery and content. Polynomials of the second degree and the use of their roots in different algebraic explorations are included into the secondary mathematics curriculum. Taking pedagogical advantage of the curriculum, instead of dealing with the two-dimensional plane, one can use the one-dimensional x-axis and study the roots' location about an interval on the number line. This study may be considered as a pre-requisite for real-life STEM investigations. In the classroom setting, even when there exists a formula for explicitly finding the roots in terms of the coefficients of a polynomial, it may be too complex to allow for an effective localization of the roots. This is true even in the simple case of quadratic equations. Although the quadratic formula is not difficult to deal with, determining the location of roots about an interval on the number line using this formula requires solving inequalities involving radical expressions. The use of such inequalities can be avoided due to qualitative methods which allow one not to deal with the roots directly. Those qualitative methods can be taught already at the secondary level under the conceptual umbrella of STEM education especially when the tools of technology are commonly available.

An effective approach to the localization of roots of polynomials on the number line is to use what may be called a geometrization of analytic situations through constructing the graphs of the polynomials. It goes back to the 17th century when René Descartes, a French mathematician and philosopher, realized that one-to-one correspondence between an algebraic equation in two variables and a set of points in the coordinate plane can be established. Due to this wondrous insight, algebra and geometry had become connected and the analytic geometry was born. The use of analytic geometry in solving algebraic problems can be demonstrated by using quadratic equations without explicitly expressing their roots through the quadratic formula. Another conceptual alternative to the quadratic formula is to use Vieta's Theorem (named after a French mathematician of the 16th century Franciscus Vieta, credited with the introduction of algebraic notation into mathematics). These methods and their description in terms of the modern educational constructs are the main focus of this paper. Briefly, the equivalence of the geometric/graphic method to the construction of Sturm's functions to be explored in terms of the change of signs at the endpoints of an interval will be demonstrated (Section 8) in the case of a quadratic equation. The teaching ideas presented in this paper stem from a capstone secondary mathematics education course that the author has taught to prospective teachers over the years.

2. The concept of a TITE problem

The concept of technology immune/technology enabled (TITE) problem was introduced elsewhere [2] as an extension of Maddux's Type II vs. Type I technology applications framework who referred to the former type as "new and *better* way of teaching" [16, p. 38, italics in the original]. This dichotomy between the two types made it possible to justify the need for moving away from drill and practice (Type I) towards using technology as a conceptual tool (Type II). In mathematics education, the suggested extension is necessary due to what may be considered as a negative affordance of technology even when its Type II applications are the main foci of instruction. The ever-increasing sophistication of mathematical software tools, enabling one either by the design of a tool or by his/her technological skills to solve various multistep problems at the push of a button, can significantly reduce the traditional complexity of problem solving. This unintentional consequence of otherwise positive affordances of technology blurs the distinction between the two types. In order to sustain educational gains from the Type II versus Type I framework in the context of mathematics, a teacher candidate has to be proficient in designing tasks that are still cognitively challenging despite (or perhaps because of) the available power of symbolic computations and graphic constructions.

In that way, the context of the TITE problem solving may be considered as Type II application of technology of the second order [1]. A TITE problem cannot be automatically solved by software; nonetheless, the role of technology in support of solving such a problem is significant. Nečesal and Pospíšil [20] made a similar point in the context of teaching engineering calculus when applying *Wolfram Alpha* (computational knowledge engine available free on-line at `www.wolframalpha.com`) to non-algorithmically solvable problems so that whereas a student is cognitively responsible for the whole problem-solving process, some part of this process can be outsourced to technology. In mathematics teacher education, an ability to pose (and solve) a TITE problem in the context of a digital tool (including *Wolfram Alpha*) can also be put in context of the technological pedagogical content knowledge (TPCK) framework [18, 21] for it constitutes an important skill allowing teacher candidates to "advance from novice to expert thinking about designing instruction with technology" [6, p. 162].

3. Location of roots of quadratics about an interval on the number line

In what follows, various TITE problems associated with the use of the geometric method in algebra (i.e., analytic geometry) supported by computer graphing and symbolic computation software tools will be considered. More specifically, the immediate goal of the paper is to determine conditions in terms of the coefficients of the quadratic equation

$$x^2 + bx + c = 0 \tag{1}$$

responsible for a specific type of location of its different real roots about an interval on the x-axis. The first, technology immune (TI) part of this problem is to determine the number of ways two roots can be located on the number line about an interval. To this end, the concept of combinations with repetitions [24] has to be considered. In general terms, one has to find the number of k-combinations selected out of m different types of objects, allowing a combination to include same object types. In the case of two roots and three intervals into which a given interval divides the number line, the roots have to form a 2-combination provided they may or may not belong to the same interval out of the total of three intervals; that is, $k = 2$ and $m = 3$.

The situation can be described in terms of two letters R (the roots) and two letters E (the interval's endpoints) being mutually arranged. There are six such arrangements which are not difficult to be listed: RREE, RERE, REER, ERER, ERRE, and EERR. In the general case of k and m, the number of such arrangements is equal to $(k+m-1)!/(k!\,(m-1)!)$. In the case $k=2$ and $m=3$ we have $4!/(2!\,2!)=6$.

Note that in finding the above six permutations, technology was not used and in that sense this part of the problem is technology immune. It prepares one to use a digital graphing tool in constructing all possible cases defined by the six permutations of letters in the string RREE. Technologically, this construction is not automatic and it requires one more pre-technological cognitive engagement to be considered.

4. Digital fabrication

Digital fabrication as an educational paradigm has been a way of introducing a pedagogy of student-computer interaction into a classroom in order to improve the teaching and learning of STEM disciplines [12, 25]. This interaction occurs in a problem-solving space where abstract mathematical relations and concrete images (described by those relations) meet [19]. One can see a connection between the concept of TITE problem and the paradigm of digital fabrication. In order to deal with a mathematical concept as a tool in turning the abstract (symbolic) into the concrete (visual) by using technology, one has to consider separately the TI and TE components of a problem in hand. That is, digital fabrication *is* a TITE problem solving with two distinct parts. A simple example of digital fabrication is the use of two-variable inequalities in constructing an interval "penetrated" by the x-axis. To this end, a system of the inequalities $|y|<\varepsilon$, $|x|<n$ can be entered into the input box of the program *Graphing Calculator* [7] capable of graphing images defined by two-variable inequalities. As a result, the program digitally fabricates the $(-n,n)$ interval of thickness 2ε (see Figs. 1–6 below where $n=1$).

Now, consider equation (1) the coefficients of which satisfy the inequality $b^2-4c>0$ to allow for two real roots, p and q, $p>q$. For simplicity, a symmetric about the origin interval $(-n,n)$ will be considered. The task is to explore the location of the roots about this interval and partition the plane (c,b) of parameters into the regions corresponding to six different types of the roots' location about the interval $(-n,n)$. Let $f(x,b,c)=x^2+bx+c$ be a function of variable x with parameters b and c.

How can a parabola – the graph of the quadratic function $y=f(x,b,c)$ – be constructed (alternatively, fabricated) using computer graphing software in some systematic way in order to establish its location about an interval? The graph of a quadratic function with two real roots can be defined as the graph of the relation $y=(x-p)(x-q)$ where, in the context of the *Graphing Calculator*, the roots p and q can become slider-controlled parameters. To this end, one can define the parameters $p=\text{slider}(-3,3)$ and $q=\text{slider}(-2,2)$ and use them as tools that can be altered in order for the parabola and the interval $(-1,1)$ to match each of the six permutations of letters in the string RREE. It is in that way that the graphs pictured below in Figs. 1–6 have been constructed. Such graphing activity is a digital fabrication which represents a technology enabled (TE) part of the problem. It is followed by a TI component which requires one to be able to interpret each of the six cases analytically.

5. Connecting the coordinate plane with the plane of coefficients

This section will explore the above six permutations of letters in the string RREE and, using the results of digital fabrication, express each case in the form of a system of two-variable rational inequalities among b, c, and n. As was mentioned above, the geometric/graphic method makes it possible to avoid the straightforward use of irrational inequalities. As will be shown in Section 9, the latter type of inequalities is hidden among the former type. Furthermore, the use of Vieta's Theorem (formulas) will be shown (Section 6) as a purely TI alternative to digital fabrication through which another kind of hidden inequalities can be recognized.

1. The case RREE: both roots of equation (1) are smaller than $-n$. As shown in Fig. 1, the inequality $f(-n, b, c) > 0$ should hold true and the vertex, $x_0 = -b/2$, of the parabola $y = x^2 + bx + c$ should be located to the left of the line $x = -n$. Therefore, the following system of two inequalities should be satisfied[1]:

(2) $$n^2 - bn + c > 0, \quad -\frac{b}{2} < -n.$$

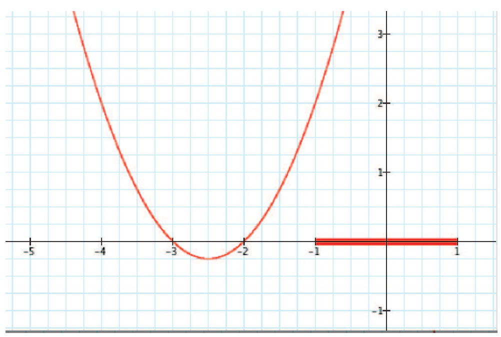

Fig. 1. Both roots are smaller than -1

2. The case RERE: one root of equation (1) is smaller than $-n$, another root belongs to $(-n, n)$. As shown in Fig. 2, the inequalities $f(-n, b, c) < 0$ and $f(n, b, c) > 0$ have to be satisfied. Hence

(3) $$n^2 - bn + c < 0, \quad n^2 + bn + c > 0.$$

[1]One may wonder as to why the inequality $f(n, b, c) > 0$ is not included in the description of the case RREE. Noting that the function $f(x, b, c)$, as shown in Fig. 1, strictly increases to the right of the point $x_0 = -b/2$, one can conclude that $f(n, b, c) > f(-n, b, c) > 0$ and, thereby, the inequality $f(n, b, c) > 0$ is an extraneous one. Similarly, in other cases, such extraneous inequalities will be intentionally omitted.

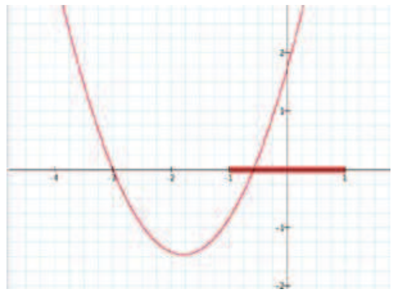

Fig. 2. One root is smaller that -1, another root belongs to $(-1, 1)$.

3. The case REER: one root of equation (1) is greater than n, another root is smaller than $-n$. As shown in Fig. 3, this case implies $f(-n, b, c) < 0$ and $f(n, b, c) < 0$. Hence

$$n^2 - bn + c < 0, \quad n^2 + bn + c < 0. \tag{4}$$

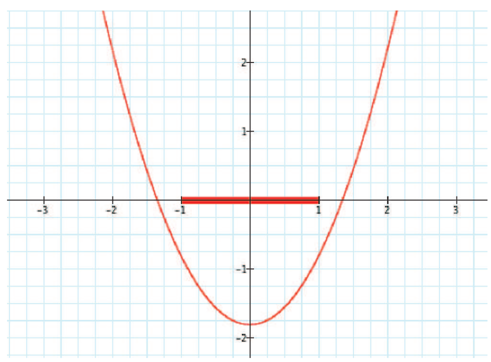

Fig. 3. The interval $(-1, 1)$ is inside the "inter-rootal" interval [26, p. 40].

4. The case ERER: one root of equation (1) is greater than n, another root belongs to $(-n, n)$. As shown in Fig. 4, this implies $f(-n, b, c) > 0$ and $f(n, b, c) < 0$. Hence

$$n^2 - bn + c > 0, \quad n^2 + bn + c < 0. \tag{5}$$

5. The case EERR: both roots of equation (1) are greater than n. As shown in Fig. 5, in addition to the inequality $f(n, b, c) > 0$, the vertex of the parabola $y = x^2 + bx + c$ should be located to the right of the line $x = n$. Hence

$$n^2 + bn + c > 0, \quad -\frac{b}{2} > n. \tag{6}$$

6. The case ERRE: both roots of equation (1) belong to $(-n, n)$. As shown in Fig. 6, this implies $f(-n, b, c) > 0$, $f(n, b, c) > 0$, and, to avoid the cases RREE

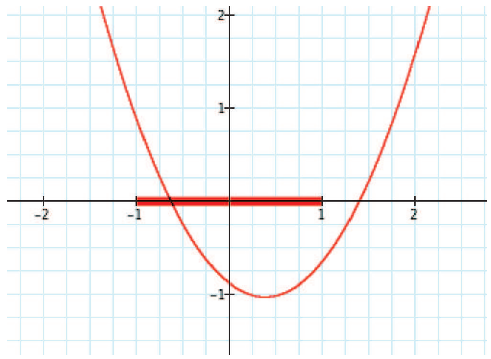

Fig. 4. One root is greater than 1, another root belongs to $(-1, 1)$.

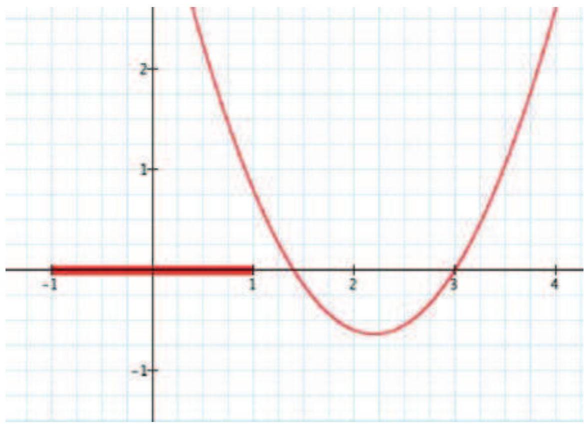

Fig. 5. The interval $(-1, 1)$ does not belong to the "inter-rootal" interval.

and EERR shown in Fig. 1 and Fig. 5, respectively, the line of symmetry of the parabola must be crossing the interval $(-n, n)$. Hence

$$n^2 - bn + c > 0, \quad n^2 + bn + c > 0, \quad |b| < 2n. \tag{7}$$

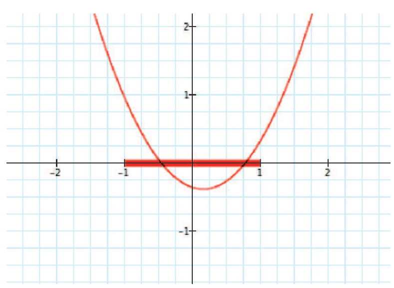

Fig. 6. The "inter-rootal" interval is within $(-1, 1)$.

6. Using Vieta's Theorem

Alternatively, as a purely TI component of the activities, inequalities (2)–(7) can be derived using Vieta's Theorem for equation (1), connecting its roots p and q with the coefficients b and c through the formulas

$$p + q = -b, \quad pq = c. \tag{8}$$

To this end, three distinct cases of the mutual arrangement of three points on the number line need to be considered.

If both p and q are smaller than the number t, then the (obvious) inequalities $p-t < 0$, $q-t < 0$ are equivalent to $(p-t)(q-t) > 0$, $p+q < 2t$, or $t^2-(p+q)t+pq > 0$, $\frac{p+q}{2} < t$, whence, due to formulas (8),

$$t^2 + bt + c > 0, \quad \frac{-b}{2} < t. \tag{9}$$

If both p and q are greater than the number t, then the (obvious) inequalities $p-t > 0$, $q-t > 0$ are equivalent to $(p-t)(q-t) > 0$, $p+q > 2t$, or $t^2-(p+q)t+pq > 0$, $(p+q)/2 > t$, whence, due to formulas (8),

$$t^2 + bt + c > 0, \quad \frac{-b}{2} > t. \tag{10}$$

Finally, if $q < t < p$, then the (obvious) inequalities $q - t < 0$, $p - t > 0$ are equivalent to $(p-t)(q-t) < 0$, or $t^2-(p+q)t+pq < 0$, whence, due to formulas (8),

$$t^2 + bt + c < 0. \tag{11}$$

Note that that the inequality $(p - t)(q - t) < 0$, due to the assumption $p > q$, implies the relations $q-t < 0$ and $p-t > 0$, so that, unlike (9) and (10), inequality (11) does not need to be augmented by a condition on b.

Now, the above six cases of the roots' location about the interval $(-n, n)$ can be revisited using formulas (9)–(11).

1. The case RREE can be described through inequalities (9) by setting $t = -n$, whence inequalities (2).

2. The case RERE can be described by setting $t = n$ in (9) and (11). This yields the inequalities $n^2+bn+c > 0$, $b > -2n$, $n^2-bn+c < 0$, which, in comparison with inequalities (3) require an additional condition for b. However, inequalities (3) imply the inequality $b > -2n$ and in that sense one can say that the last inequality is *hidden* in inequalities (3). Indeed, subtracting the inequality $n^2 - bn + c < 0$ from the inequality $n^2 + bn + c > 0$ yields $2bn > 0$, whence $b > 0 > -2n$.

3. The case REER can be described by using inequality (11) twice: setting $t = -n$ and $t = n$. This immediately yields inequalities (4).

4. The case ERER can be described by setting $t = -n$ in (10) and $t = n$ in (11). This yields the inequalities $n^2 - bn + c > 0$, $b < 2n$, $n^2 + bn + c < 0$. Once again, the inequality $b < 2n$ is not included, but rather *hidden*, in inequalities (5).

Indeed, subtracting the inequality $n^2+bn+c<0$ from the inequality $n^2-bn+c>0$ yields $-2bn>0$, whence $b<0<2n$.

5. The case EERR can be described by setting $t=n$ in inequalities (10). This immediately yields inequalities (6).

6. Finally, the case ERRE can be descried by setting $t=n$ in (9) and $t=-n$ in (10). This yields the inequalities $n^2+bn+c>0$, $b>-2n$, $n^2-bn+c>0$, $b<2n$ which coincide with inequalities (7).

Remark 1. Note that the cases of hidden inequalities were revealed in the context of using formulas (9)–(11) when one and only one root belongs to the interval $(-n,n)$. As shown in Fig. 2 and Fig. 4, the relationship between the vertex of the corresponding parabola and the right/left end point of the interval, something that a hidden inequality specifies, is automatically embedded into the corresponding visual illustration. This observation underscores the importance of formal reasoning being mediated by an interplay between visual and analytic representations of a mathematical concept. Whereas, generally speaking, the former representation may not be considered rigorous, the latter representation, in the absence of its situational referent in the form of a graph, might bring about symbolic information that has no effect on the outcome of a digital fabrication. By the same token, an analytic interpretation of the behavior of a graph in terms of relations among the three parameters is not straightforward and its outcome depends on the level of mathematical competence of an interpreter. All things considered, it appears that using jointly both representations ensures computational efficiency of the corresponding algorithm which connects the TI and TE components of the problem in question.

7. TITE activities in the plane of parameters

The next step (a purely TE one) of the activities is to partition the plane of parameters (c,b) into the regions corresponding to the above six types of roots' location about the interval $(-n,n)$ defined by inequalities (2)–(7), respectively. Note that the inequality $b^2-4c>0$ should also be taken into consideration when constructing the regions defined by the systems of inequalities (2), (6), and (7) as those systems would still be satisfied if the corresponding parabolas in Fig. 1, Fig. 5, and Fig. 6, respectively, would be located entirely above the x-axis. In the other three cases, as shown in Figs. 2–4, at least one of the inequalities $y(\pm1)<0$ (the case $n=1$) would imply that the corresponding parabola has points in common with the x-axis. This generalized partitioning diagram (where n is a slider-controlled parameter) is shown in Fig. 7.

Several TITE activities can be proposed using the partitioning diagram as a learning environment. A TE component of those activities may go beyond constructing the diagram to include some additional (digital) fabrication of points[2]

[2] In the context of the *Graphing Calculator* the point (c_0,b_0) can be either constructed as the intersection of the lines $x=c_0$ and $y=b_0$, or digitally fabricated through the inequalities $|x-c_0|<\varepsilon$, $|y-b_0|<\varepsilon$ for a sufficiently small $\varepsilon>0$.

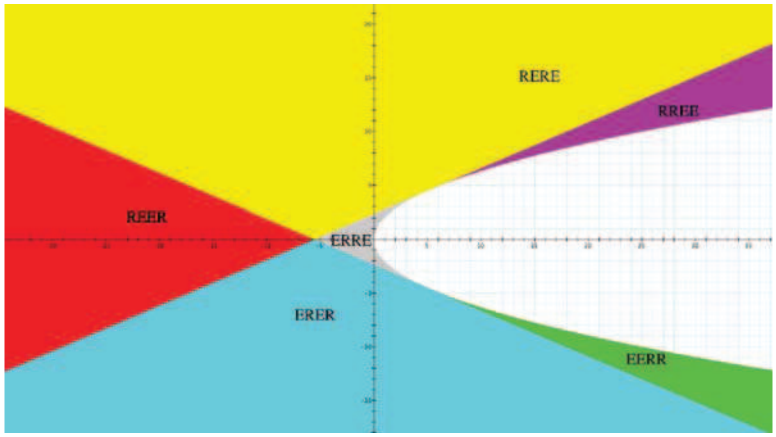

Fig. 7. Regions in the plane (c, b) defined by inequalities (2)–(7).

and lines. As will be demonstrated below, once the diagram is constructed, it serves as support system for a TI component of an activity.

Activity 1. One can select a point from the (c, b)-plane of parameters, notice to which of the six regions it belongs along with the value of n, use its coordinates as the coefficients of the quadratics, find the roots of the latter (perhaps, using technology), and check to see that their location about the interval $(-n, n)$ for the chosen n corresponds to the region to which the selected point belongs. This shows the advantage of theory over experiment when selecting parameters of a system with desired behavior.

Activity 2. Using the *Graphing Calculator*, one can check to see that adding to inequalities (3) the inequality $b > -2n$ has no effect on the digital fabrication of the region RERE. Likewise, adding to inequalities (5) the inequality $b < 2n$ has no effect on the region ERER. This TE-based observation can then be followed by a TI explanation as described in Section 6.

Activity 3. The apparent symmetry about the c-axis of the regions RERE and ERER, as well as the regions RREE and EERR can motivate another TI activity. To this end, let the point $(c, b) \in \{RERE\}$ for a certain value of n. This means that equation (1) has real roots p and q such that $q < -n < p < n$. It follows from the last chain of inequalities that $-n < -p < n < -q$; in other words, the status of the pair $(-p, -q)$ about the segment $(-n, n)$ is of the ERER type. Furthermore, the equalities $p^2 + bp + c = 0$ and $q^2 + bq + c = 0$ are equivalent to, respectively, $(-p)^2 - b(-p) + c = 0$ and $(-q)^2 - b(-q) + c = 0$. In other words, $-p$ and $-q$, for which $-n < -p < n < -q$ are the roots of the equation $x^2 - bx + c = 0$ into which equation (1) turns when the point (c, b) is replaced by the point $(c, -b)$. This completes the demonstration of the symmetry of the regions RERE and ERER about the c-axis. Similarly, other visually apparent symmetries about the c-axis in the (c, b)-plane can be demonstrated.

Activity 4. One can be asked to investigate how the roots behave when the case ERRE turns into that of RREE. The diagram of Fig. 7 shows that the two regions converge into the point of tangency of the straight line $n^2 - bn + c = 0$ and the parabola $b^2 - 4c = 0$. Solving the last two equations simultaneously yields $(c, b) = (n^2, 2n)$. At that point, equation (1) assumes the form $x^2 + 2nx + n^2 = 0$, whence $x = -n$ (i.e., $p = q = -n$). That is, the case ERRE bifurcates into the case RREE when both roots simultaneously pass through the left end point of the interval $(-n, n)$. Similarly, the case ERRE bifurcates into the case EERR when both root of equation (1) simultaneously pass through the right end point of the interval $(-n, n)$.

Activity 5. One can explore the behavior of the roots of equation (1) on the borders of the regions shown in Fig. 7. For example, the regions RREE and RERE share the border described by the line $b = n + \frac{c}{n}$ where equation (1) turns into $x^2 + (n + \frac{c}{n})x + n^2 = 0$. The roots of the last equation can be found using formulas (8): $p = -n$, $q = -\frac{c}{n}$. When $p = q$ we have $c = n^2$ and when $p > q$ we have $c > n^2$. That is, on the border between the regions RREE and RERE, while one of the roots remains outside the interval $(-n, n)$, another root becomes equal to $-n$. This (perhaps, intuitively clear) conclusion is reflected in the partitioning diagram of Fig. 7.

Similarly, in order to see what happens when the case ERRE bifurcates into the case REER at the point where the lines $n^2 + bn + c = 0$ and $n^2 - bn + c = 0$ intersect, one can find out (both analytically and graphically) that such point has the coordinates $(-n^2, 0)$ where equation (1) turns into $x^2 - n^2 = 0$, whence $x = \pm n$.

Remark 2. Note that the points located on the borders of the six regions provide equation (1) either with a double root, or, in some cases, with at least one root coinciding with an end point of the corresponding interval.

Activity 6. By changing n, one can observe in the (c, b)-plane that as $n > 0$ decreases, the region ERRE shrinks. Requesting an explanation of this phenomenon leads to another TITE activity. To this end, one can be asked to describe analytically the region ERRE in terms of inequalities among b, c, and n. Such a description can be done as a combination of analytic reasoning and symbolic computations using, once again, *Wolfram Alpha*. In order to find the smallest value of the c-range for that region, one has to solve simultaneously the equations $n^2 + bn + c = 0$ and $n^2 - bn + c = 0$ to get $c = -n^2$. The largest value of the c-range results from solving another pair of equations, $n^2 + bn + c = 0$ and $b^2 = 4c$, to get $c = n^2$. That is, in the region ERRE we have $-n^2 < c < n^2$. When $-n^2 < c < 0$ we have $-\frac{c}{n} - n < b < \frac{c}{n} + n$; when $0 < c < n^2$ we have $2\sqrt{c} < b < \frac{c}{n} + n$. This analytic description suggests that the region ERRE shrinks as n decreases, and it is attracted by the origin when $n \to 0$.

To conclude this section, note that the use of the partitioning diagram of Fig. 7 can be extended to include problems on geometric probability. One such problem may be as follows: if the coefficients of equation (1) are chosen at random from the rectangle $\{ (c, b) : |b| < 2n,\ |c| < n^2 \}$, find the probability that both roots belong to the interval $(-n, n)$. Many other like problems can be formulated using the diagram.

8. Making mathematical connections

As was mentioned in the introduction, a classic method of deciding the number of roots that belong to a given interval is due to Sturm and the following description of the method is borrowed from [8]. Let $f(x) = 0$ be an equation which does not have equal roots in the interval (a, b). Let us divide $f(x)$ by its first derivative, $f'(x) = f_1(x)$ and find the negative of the remainder, $-r_1(x) = f_2(x)$. This division process continues (changing the sign of each new remainder before using it as a divisor) until a remainder not depending on x is reached and its sign is changed as well to have $f_k(x) = \text{const} \neq 0$. This recursive process is reminiscent of the Euclidean algorithm when division stops once the zero remainder is reached. The functions $f(x)$, $f_1(x)$, $f_2(x)$, ... , $f_k(x)$ are called Sturm's functions. Sturm's Theorem states that the difference between the number of sign changes in the sequence of Sturm's functions when $x = a$ and $x = b$ is equal to the number of real roots of the equation $f(x) = 0$ in the interval (a, b).

As an example, consider the equation $x^2 - 5x + 6 = 0$ and the interval $(0, 4)$. Sturm's functions in this case are $f(x) = x^2 - 5x + 6$, $f_1(x) = 2x - 5$, and $f_2(x) = \frac{1}{4}$ as $x^2 - 5x + 6 = (2x - 5)(\frac{x}{2} - \frac{5}{4}) - \frac{1}{4}$. When $x = 0$ the sequence $f(0) = 6$, $f_1(0) = -5$, $f_2(0) = \frac{1}{4}$ has two sign changes (from + to − and then back to +); when $x = 4$ the sequence $f(4) = 2$, $f_1(4) = 3$, $f_2(4) = \frac{1}{4}$ does not change sign. According to Sturm's Theorem, the equation $x^2 - 5x + 6 = 0$ has $2 - 0 = 2$ real roots in the interval $(0, 4)$.

In the general case of quadratic equation (1), assuming $b^2 - 4c > 0$, one can recognize the equivalence of the geometric/graphic and Sturm's methods, both free from the explicit use of the quadratic formula (or Vieta's formulas). In that case, the sequence consists of three polynomials: the quadratic trinomial $x^2 + bx + c$, its derivative $2x + b$, and the negative of the remainder obtained by dividing $x^2 + bx + c$ by $2x + b$; that is, $\frac{1}{4}(b^2 - 4c)$. Indeed, one can check to see that within the relation among the dividend, the divisor, the quotient, and the remainder: $x^2 + bx + c = (2x + b)(\frac{x}{2} + \frac{b}{4}) - \frac{1}{4}(b^2 - 4c)$, the negative of the remainder turns out to be the $\frac{1}{4}$ multiple of the discriminant, $b^2 - 4c$, of the quadratic trinomial[3]. Alternatively, one can enter into the input box of *Wolfram Alpha* the command "solve $x^2 + bx + c = (2x + b)(px + q) + r$ in p, q, r" to get the solution $p = 1/2$, $q = b/4$, and $r = \frac{1}{4}(4c - b^2)$. Note that the inequality $b^2 - 4c > 0$, by making negative r positive, does guarantee the existence of two real roots of the equation $x^2 + bx + c = 0$. According to Sturm's Theorem, the number of real roots in the interval $(-n, n)$ is equal to the difference between the number of sign changes of the three polynomials at the points $x = -n$ and $x = n$. Due to the discriminant inequality (which does not depend on x), only the first two polynomials may be subject to sign changes.

[3] As an aside, note that the (apparently not commonly used) identity $x^2 + bx + c = \frac{1}{4}[(2x + b)^2 - (b^2 - 4c)]$ not only connects a quadratic trinomial, its first derivative, and the discriminant, but it also shows that the positive discriminant does provide the trinomial with two different real roots.

Consider the case ERRE, when such difference has to be equal to the number 2. Because $b^2 - 4c > 0$, for $x = -n$ we should have the inequalities $n^2 - bn + c > 0$ (the inequality of the opposite sign would not be consistent with two sign changes allowing either $- + +$ or $- - +$ combinations) and $-2n+b < 0$ (in other words, the negative value of the first derivative implies that the quadratic function decreases at the left end point of the interval); when $x = n$, no change of sign is required implying the inequalities $n^2 + bn + c > 0$ (the positive value of the quadratic function at the right end point of the interval implies exactly two intersections of the corresponding parabola with the interval) and $2n + b > 0$ (the positive value of the first derivative at the point $x = n$ confirms the increase of the quadratic function). That is, the case ERRE, according to Sturm's Theorem, requires that inequalities (7), augmented by the inequality $b^2 - 4c > 0$, hold true. This shows the equivalence of the geometric/graphics and Sturm's methods, each of which does not require either the explicit or implicit knowledge of the roots. Furthermore, interpreting the behavior of Sturm's functions in terms of the behavior of a quadratic function enhances the TI component of the activity. Similarly, other cases of the roots' location about the interval $(-n, n)$ can be established using Sturm's method.

Remark 3. As an aside note, that technological availability of symbolic computations makes it possible, without much difficulty, to develop Sturm's functions proceeding from the cubic polynomial $x^3 + bx^2 + cx + d$. Just as in the above case of the quadratic trinomial, using *Wolfram Alpha* one can first find the remainder $r_1 = \frac{2}{9}(3c - b^2)x + \frac{1}{9}(9d - bc)$ when dividing the cubic polynomial by its first derivative, and then divide the derivative, $3x^2 + 2bx$+c, by $-r_1$ to get the negation of the second remainder r_2 in the form

$$-r_2 = \frac{9}{4(b^2 - 3c)^2}(b^2c^2 - 4b^3d - 4c^3 + 18bcd - 27d^2).$$

Arriving at the remainder which does not depend on x terminates the recursive process of division. The expression $b^2c^2 - 4b^3d - 4c^3 + 18bcd - 27d^2$ is the discriminant of the cubic polynomial (information provided, among other sources, by *Wolfram Alpha*), which is equal to zero in the case of three equal roots (e.g., for the equation $x^3 + 3x^2 + 3x + 1 = 0$). Conditioning the discriminant to be positive enables two things: 1) the cubic polynomial has three real roots; 2) the inequality $-r_2 > 0$ holds true whatever the interval on the x-axis as r_2 does not depend on x.

9. Revealing hidden concepts through collateral learning

The use of graphs in defining the six cases of the roots location about an interval through rational inequalities (2)–(7) or formulas (9)–(11) made it possible to avoid solving inequalities involving radical expressions. In the case of quadratic equations, with an easy algorithm of finding roots (i.e., the quadratic formula), the computational power of a digital tool does not differentiate between dealing with radical or rational relations. Yet, an unwarranted use of this power might lead to significant difficulties (including just the typing errors) already in the case of polynomial equations of the second order. Furthermore, fostering the culture of computational efficiency of a mathematical investigation is an important skill that has to be emphasized as appropriate through the teaching of STEM-related disciplines. The issue of reducing computing complexity becomes meaningful in the context of STEM education if a computational environment used offers an improvement over the traditional one in terms of how calculations could be performed. By using an efficient computational algorithm, one can gain better conceptual understanding of mathematical relationships that underpin the algorithm. This also gives new meaning to the concepts involved. In particular, in the case of a quadratic equation, by comparing two computational algorithms, one based on irrational inequalities and another based on rational inequalities (whatever the method), the notion of hidden inequalities can be discussed. An interesting aspect of using digitally fabricated graphs as well as formulas (9)–(11) versus the use of the quadratic formula is that one type of inequalities may be found being hidden within another type (see also [4]). Similar examples of one procedure/concept being hidden within another procedure/concept are the repeated addition being hidden within the multiplication, or Sturm's Theorem in the case of a quadratic equation being hidden behind the geometric/graphic method. That is, a concept which provides a less cumbersome (or a less cognitively demanding) computational procedure hides a more complicated (or a more general) procedure within an efficient (or easy to use) one.

To clarify (in addition to two illustrations of Section 6 where the notion of hidden inequalities was discussed), consider the case RREE (Fig. 1) as an example. The recourse to the roots of equation (1) leads to the inequality $\frac{-b+\sqrt{b^2-4c}}{2} < -n$ whence

$$\sqrt{b^2 - 4c} < b - 2n. \tag{12}$$

Since $b^2 - 4c > 0$ implies $\sqrt{b^2 - 4c} > 0$, inequality (12) is equivalent to the following system of rational inequalities

$$b^2 - 4c > 0, \quad b - 2n > 0, \quad b^2 - 4c < (b - 2n)^2.$$

Simple transformations of the last two inequalities yield inequalities (2) for which the condition $b^2 - 4c > 0$ (enabling two real roots) was assumed. It is in that sense that one can say that irrational inequality (12) is *hidden* in rational inequalities (2). Similar relations of equivalence between irrational and rational inequalities can be demonstrated in other five cases of the roots of equation (1) location about the interval $(-n, n)$.

In such a way, under the umbrella of the educational construct of collateral learning [10], revealing hidden inequalities (or any hidden procedure/concept for that matter, like Sturm's Theorem) serves the purpose of educational efficiency "not only in reaching the projected end of the activity [e.g., digital fabrication] immediately at hand, but even more in securing from the activity the learning which it potentially contains" [13, p. 334]. This also brings to mind the notion of the didactical phenomenology of mathematics as "a way to show the teacher the places where the learner might step into the learning process of mankind" [11,

p. ix]. The notion of collateral learning encourages teachers to make connections among seemingly disconnected ideas by revealing to students hidden mathematics curriculum [3] – a didactic approach to the teaching of mathematics that motivates learning in a larger context and rejects "the greatest of all pedagogical fallacies ... that a person learns only the particular thing he is studying at the time" [10, p. 49]. The recurrent procedure, reminiscent of the Euclidean algorithm, of constructing the sequence of Sturm's functions designed to decide the number of roots within a given interval does belong to a hidden mathematics curriculum of secondary teacher education.

10. Conclusion

The goal of this paper was to share teaching ideas regarding the use of traditional (and mostly procedural) content of quadratic equations as a background for grade-appropriate mathematical investigations rudimental to real problems of science and engineering. Those rudiments of STEM explorations are pertinent to a secondary mathematics teacher education program. Towards this end, a single-variable quadratic equation with two real parameters was considered as an object of an analytic inquiry aimed at constructing regions in the plane of parameters responsible for a specified location of two real roots about a given interval. In the digital era, explorations of that kind can lead to conceptually rich mathematical activities that are both technology immune (TI) and technology enabled (TE). The importance of a TITE problem-solving mathematics curriculum is due to the availability of sophisticated computer programs capable of the variety of symbolic computations and graphic constructions. While the omnipresent use of digital tools in the modern classroom makes procedural skills less demanding, the conceptual component of mathematics teaching and learning may not be neglected. A TITE problem-solving mathematics curriculum has the potential to maintain the right balance between the procedural and the conceptual.

Why is it important to know how to construct regions in the plane of parameters responsible for a particular type of behavior of a system described by an equation the solution of which depends on the parameters? In the context of STEM education, when exploring mathematical models, a student must have experience with understanding a phenomenon that goes beyond pure intuition [5]. For example, it may not be intuitively clear that with the decrease of the interval's length, the region ERRE (when the interval contains both roots of a quadratic equation) shrinks and becomes attracted by the origin in the plane of parameters. Nowadays, the exploration of non-intuitive situations can be greatly enhanced by the use of technology. At the same time, their formal demonstration requires analytic reasoning skills which could be enhanced but not supplanted by the use of symbolic computations. Pedagogically speaking, this provides an opportunity using TITE problems in a mathematics classroom, including that of prospective secondary teachers. Having a mechanism for the appropriate selection of parameters of a mathematical model that provide required properties of the behavior of the model, demonstrates the value of TITE problem-solving activities in advancing STEM education at different levels.

The paper illustrated the use of computer graphing software as a tool of digital fabrication. Procedural skills of plotting graphs have to be informed by conceptual understanding of the properties of functions that the graphs represent. This interplay between the procedural and the conceptual is the core of digital fabrication. However, there is a difference between plotting parabolas $y = (x - p)(x - q)$ with different types of intersections of the x-axis and constructing regions, defined by inequalities, in the plane (c, b), where $c = pq$ and $b = -(p + q)$, corresponding to those different types. In plotting the graph of a quadratic function with a specific property, the visual guides the symbolic. On the contrary, in constructing a region in the plane of parameters, the symbolic enables the visual. This shows the real complexity of ideas that comprise technological pedagogical content knowledge [6, 18, 21] necessary for teaching mathematical problems with applied flavor.

A rather mundane (and perhaps even boring) activity of solving quadratic equations can be significantly enriched by considering same equations but with parameters in place of coefficients. A didactic value of this extension is at least twofold. It enriches secondary mathematics education with explorations redolent of real research experience in a STEM field that teacher candidates need to acquire. Conference Board of the Mathematical Sciences, an umbrella organization consisting of sixteen professional societies in the United States concerned with the mathematical preparation of schoolteachers, supports this position by noting that those teachers "who have engaged in a research-like experience for a sustained period of time frequently report that it greatly affects what they teach, how they teach, what they deem important, and even their ability to make sense of standard mathematics courses" [9, p. 65]. Also, the extension into equations with parameters makes it possible, by introducing historical perspectives into a mathematics education course, to open entries into a hidden mathematics curriculum comprised of many ideas developed by great mathematical minds of the past. Such ideas can be mapped to the traditional mathematics curriculum enabling mathematical connections to be developed through technologically-motivated collateral learning as "the proper use of technology make[s] complex ideas tractable, [and] it can also help one understand subtle mathematical concepts" [ibid, p. 57]. Finally, the blend of digital tools, research-like experience, historical perspectives, and appeal of hidden mathematics curriculum allow for a stronger collaboration between mathematicians and mathematics educators towards the dual goal for teacher candidates' seeing "the larger body of mathematics ... arising from the ideas under discussion ... [and] appreciating the nature and role of meaning in students' mathematical learning" [22, p. 320].

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