THE TEACHING OF MATHEMATICS
2017, Vol. XX, 1, pp. 13–19

CRAMER'S RULE FOR NONSINGULAR $m \times n$ MATRICES

**Azamat Akhtyamov, Meirav Amram,
Miriam Dagan and Artour Mouftahkov**

Abstract. In linear algebra, Cramer's rule is an explicit formula for the solution of a system of linear equations with as many equations as unknowns, that is, for the solution of a system with a square matrix. In this paper we want to generalize this method for an $m \times n$ system of linear equations, such that $m < n$. We offer a simple and convenient formula for systems with rectangular matrices using only the minors of the augmented matrix, as well as the usual method of Cramer. We also generalize the results in order to solve a matrix equation.

MathEduc Subject Classification: H65

MSC Subject Classification: 97H60

Key words and phrases: Cramer's rule; matrix; minors; linear algebra; augmented matrix

1. Introduction

A solution of systems of linear algebraic equations is one of the main topics of the linear algebra course. Usually, Gaussian elimination and Cramer's rule are studied in the course. Gabriel Cramer (1704–1752) published the rule for an arbitrary number of unknowns in 1750. Gaussian elimination was published in 1849 by Carl Friedrich Gauss (1777–1855).

In general, a preference is given to the method of Gauss. There are two main advantages of this method – first, the method involves a simple calculation for systems with a large number of unknowns, and second, we are able to solve systems with an infinite number of solutions.

However, the method of Gauss gives us an algorithm for the solution, but does not give the final formulas for finding the unknowns of the system. Moreover, the method of Gauss is inconvenient and cumbersome if the coefficients of the system are functions. For example, in the case of solutions of a linear differential nonhomogeneous equation, by the method of variation of variables, we obtain a system of linear equations whose coefficients are functions.

On the other hand, the advantage of Cramer's rule is that the method gives us a final formula for the solutions, when calculating the unknowns by the minors of the augmented matrix of the system. Cramer's rule can be used for various proofs of theorems in linear algebra, for calculating solutions of systems with parameters, and, in particular, for finding solutions of the above-mentioned differential equation.

Before 2010 most mathematicians were certain that using Cramer's rule was not practical in computational mathematics. However, it was shown in [5] that Cramer's rule may be implemented with a complexity comparable to the complexity of the Gaussian elimination.

In this article we show that Cramer's rule can be generalized, and, in the case of systems with an infinite number of solutions, get the final formula for calculating the unknowns by the minors of the augmented matrix of the system.

There are several interesting papers about the generalization of Cramer's rule [1–4, 6], but the authors of these papers use different formulations of the problem, or receive other final formulas and prove them by other methods.

The paper is divided as follows. In Section 2 we recall the ordinary Cramer's rule. In Section 3 we generalize Cramer's rule for a system with a rectangular matrix of the coefficients. In Section 4 we generalize the results of Section 3 matrix equation.

2. The ordinary Cramer's rule

In this section we set forth the ordinary Cramer's rule, formulate it by minors of an augmented matrix, and give a proof that is useful also in the proof of the generalization of the rule in Section 3.

Let us consider the following system of n linear equations with n unknowns:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = a_{1,n+1} \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = a_{2,n+1} \\ \qquad\qquad\qquad\qquad\qquad \dots\dots \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n = a_{n,n+1}. \end{cases}$$

Let A be a nonsingular $n \times n$-matrix of the coefficients of this system. Let $\bar{A}$ be an $n \times (n+1)$-augmented matrix of the system, i.e., $A \in \mathbb{K}^{n\times n}$, $\bar{A} \in \mathbb{K}^{n\times(n+1)}$, $\operatorname{rank} A = n$, where $\mathbb{K}$ is some field. The augmented matrix $\bar{A}$ has the following minors of order n for some chosen columns $j_1, j_2, \dots, j_n$:

$$M_{j_1,j_2,j_3,\dots,j_n} = \begin{vmatrix} a_{1,j_1} & a_{1,j_2} & \dots & a_{1,j_n} \\ a_{2,j_1} & a_{2,j_2} & \dots & a_{2,j_n} \\ \dots & \dots & \dots & \dots \\ a_{n,j_1} & a_{n,j_2} & \dots & a_{n,j_n} \end{vmatrix}.$$

We formulate Cramer's rule by minors of an augmented matrix.

Theorem 1. [Cramer's rule] *Let $\bar{A}$ be an $n \times (n+1)$-augmented matrix of the system of linear equations. If its minor $M_{1,2,\dots,n} \neq 0$, then the solution of the system is*

$$x_i = \frac{M_{1,2,\dots,i-1,n+1,i+1,\dots,n-1,n}}{M_{1,2,\dots,n}}, \tag{1}$$

where $i = 1, 2, \dots, n$.

To prove this theorem we need the following lemma.

LEMMA 2. *Let $A, \widetilde{A}$ be two $m \times n$-matrices. Let $M_{i_1,i_2,\dots,i_m}$ and $\widetilde{M}_{i_1,i_2,\dots,i_m}$ be the minors of matrices A and $\widetilde{A}$ respectively. If A and $\widetilde{A}$ are row-equivalent, then*

$$M_{i_1,i_2,\dots,i_m} = t \cdot \widetilde{M}_{i_1,i_2,\dots,i_m}, \qquad t \neq 0,$$

and t is independent of $(i_1, i_2, \dots, i_m)$.

Proof. Let A and $\widetilde{A}$ be two row-equivalent matrices, i.e., there is a nonsingular $m \times m$-matrix S such that $\widetilde{A} = S \cdot A$. Let $\det S = t$ and $t \neq 0$ (as S is nonsingular). Then $\widetilde{A}_{i_1,i_2,\dots,i_m} = S \cdot A_{i_1,i_2,\dots,i_m}$, where $A_{i_1,i_2,\dots,i_m}$ and $\widetilde{A}_{i_1,i_2,\dots,i_m}$ are sub-matrices of A and $\widetilde{A}$ respectively consisting of the columns $i_1, i_2, \dots, i_m$. Then, $\widetilde{M}_{i_1,i_2,\dots,i_m} = \det(S \cdot A_{i_1,i_2,\dots,i_m}) = \det S \cdot \det A_{i_1,i_2,\dots,i_m} = t \cdot M_{i_1,i_2,\dots,i_m}$. ■

Proof of Cramer's rule. Let $\bar{A}$ be an $n \times (n+1)$-augmented matrix of the system of linear equations

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & a_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & a_{n,n+1} \end{bmatrix}$$

Since $M_{1,2,\dots,n} \neq 0$, by elementary actions on the rows of $\bar{A}$, we can obtain a matrix row-equivalent to $\bar{A}$:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \widetilde{a}_{1,n+1} \\ 0 & 1 & \dots & 0 & 0 & \widetilde{a}_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & \widetilde{a}_{n-1,n+1} \\ 0 & 0 & \dots & 0 & 1 & \widetilde{a}_{n,n+1} \end{bmatrix}$$

By Lemma 2, $M_{i_1,i_2,\dots,i_n} = t \cdot \widetilde{M}_{i_1,i_2,\dots,i_n}$, where $\widetilde{M}_{i_1,i_2,\dots,i_n}$ is the corresponding minor of the latter matrix. If $(i_1, i_2, \dots, i_n) = (1, 2, \dots, n)$, then $t = \frac{M_{1,2,\dots,n}}{\widetilde{M}_{1,2,\dots,n}} = \frac{M_{1,2,\dots,n}}{1} = M_{1,2,\dots,n}$, and $\widetilde{M}_{i_1,i_2,\dots,i_n} = \frac{M_{i_1,i_2,\dots,i_n}}{t} = \frac{M_{i_1,i_2,\dots,i_n}}{M_{1,2,\dots,n}}$.

The solution of the system is then

$$x_i = \widetilde{a}_{i,n+1} = \widetilde{M}_{1,2,\dots,i-1,n+1,i+1,\dots,n-1,n} = \frac{M_{1,2,\dots,i-1,n+1,i+1,\dots,n-1,n}}{M_{1,2,\dots,n}},$$

where $i = 1, 2, \dots, n$, as in (1). ■

In the next section we formulate and prove our generalized Cramer's rule by the same method, i.e., by minors of an augmented matrix.

3. Cramer's rule for a system of linear equations with a nonsingular rectangular matrix of the coefficients

In this section we generalize Cramer's rule to a system with a nonsingular rectangular matrix of the coefficients.

Let us consider the following system of m linear equations with n unknowns (for $m < n$):

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = a_{1,n+1} \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = a_{2,n+1} \\ \qquad\qquad \cdots\cdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = a_{m,n+1}. \end{cases}$$

Let A be a nonsingular $m \times n$-matrix of the coefficients of this system ($m < n$). Let $\bar{A}$ be an $m \times (n+1)$-augmented matrix of the system, i.e., $A \in \mathbb{K}^{m\times n}$, $\bar{A} \in \mathbb{K}^{m\times(n+1)}$, $\operatorname{rank} A = m$, where $\mathbb{K}$ is some field.

In the following theorem we generalize Cramer's rule to a system with a nonsingular rectangular matrix $A \in \mathbb{K}^{m\times n}$ of the coefficients in a case that $M_{1,2,\dots,m} \neq 0$.

Theorem 3. [Generalized Cramer's rule 1] *Let $\bar{A}$ be an $m \times (n+1)$-augmented matrix of a system of linear equations ($m < n$), and let $M_{1,2,\dots,m} \neq 0$. Then the solution of the system is the set of all n-tuples $(x_1, x_2, \dots, x_n)$ such that $x_{m+1}, \dots, x_n$ are arbitrary elements of the field $\mathbb{K}$ and*

$$x_i = \frac{M_{1,2,\dots,i-1,n+1,i+1,\dots,m-1,m}}{M_{1,2,\dots,m}} - \sum_{j=m+1}^{n} \frac{M_{1,2,\dots,i-1,j,i+1,\dots,m-1,m}}{M_{1,2,\dots,m}} x_j, \tag{2}$$

where $i = 1, 2, \dots, m$.

Proof. Let $\bar{A}$ be an $m \times (n+1)$-augmented matrix of the system of linear equations:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m-1} & a_{2,m} & a_{2,m+1} & a_{2,m+2} & \cdots & a_{2,n} & a_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,m-1} & a_{m-1,m} & a_{m-1,m+1} & a_{m-1,m+2} & \cdots & a_{m-1,n} & a_{m-1,n+1} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m-1} & a_{m,m} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{m,n} & a_{m,n+1} \end{bmatrix}.$$

Since $M_{1,2,\dots,m} \neq 0$, by elementary actions on the rows of $\bar{A}$, we can obtain a matrix row-equivalent to $\bar{A}$:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \widetilde{a}_{1,m+1} & \widetilde{a}_{1,m+2} & \dots & \widetilde{a}_{1,n} & \widetilde{a}_{1,n+1} \\ 0 & 1 & \dots & 0 & 0 & \widetilde{a}_{2,m+1} & \widetilde{a}_{2,m+2} & \dots & \widetilde{a}_{2,n} & \widetilde{a}_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & \widetilde{a}_{m-1,m+1} & \widetilde{a}_{m-1,m+2} & \dots & \widetilde{a}_{m-1,n} & \widetilde{a}_{m-1,n+1} \\ 0 & 0 & \dots & 0 & 1 & \widetilde{a}_{m,m+1} & \widetilde{a}_{m,m+2} & \dots & \widetilde{a}_{m,n} & \widetilde{a}_{m,n+1} \end{bmatrix}.$$

By Lemma 2, $M_{i_1,i_2,\dots,i_m} = t \cdot \widetilde{M}_{i_1,i_2,\dots,i_m}$, where $\widetilde{M}_{i_1,i_2,\dots,i_m}$ is the corresponding minor of the latter matrix. If $(i_1, i_2, \dots, i_m) = (1, 2, \dots, m)$, then $t = \frac{M_{1,2,\dots,m}}{\widetilde{M}_{1,2,\dots,m}} = \frac{M_{1,2,\dots,m}}{1} = M_{1,2,\dots,m}$. This means that $\widetilde{M}_{i_1,i_2,\dots,i_m} = \frac{M_{i_1,i_2,\dots,i_m}}{t} = \frac{M_{i_1,i_2,\dots,i_m}}{M_{1,2,\dots,m}}$.

We can see that

$$\widetilde{a}_{i,j} = \widetilde{M}_{1,2,\dots,i-1,j,i+1,\dots,m-1,m} = \frac{M_{1,2,\dots,i-1,j,i+1,\dots,m-1,m}}{M_{1,2,\dots,m}},$$

where $i = 1, 2, \dots, m,\ \ j = m+1, m+2, \dots, n, n+1$.

Then $x_i = \widetilde{a}_{i,n+1} - \sum_{j=m+1}^{n} \widetilde{a}_{i,j} x_j$, i.e.,

$$x_i = \frac{M_{1,2,\dots,i-1,n+1,i+1,\dots,m-1,m}}{M_{1,2,\dots,m}} - \sum_{j=m+1}^{n} \frac{M_{1,2,\dots,i-1,j,i+1,\dots,m-1,m}}{M_{1,2,\dots,m}} x_j,$$

where $x_j \in \mathbb{K}, \quad i = 1, 2, \dots, m,\ j = m+1, m+2, \dots, n$, as in (2). ∎

Remark 4. If we substitute $m = n$ in Equation (2), we get the Cramer's rule, as in Equation (1).

In Theorem 3 we assume that $M_{1,2,\dots,m} \neq 0$. But it is enough to assume that at least one minor of order m of A is not 0. Therefore, we get the following theorem.

Theorem 5. [Generalized Cramer's rule 2] *Let $\bar{A}$ be an $m \times (n+1)$-augmented matrix of a system of linear equations ($m < n$). Let $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$ be a permutation, such that $M_{\sigma(1),\sigma(2),\dots,\sigma(m)} \neq 0$. Then the solution of the system is the set of all n-tuples $(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ such that $x_{\sigma(m+1)}, \dots, x_{\sigma(n)}$ are arbitrary elements of the field $\mathbb{K}$ and*

$$x_{\sigma(i)} = \frac{M_{\sigma(1),\sigma(2),\dots,\sigma(i-1),n+1,\sigma(i+1),\dots,\sigma(m-1),\sigma(m)}}{M_{\sigma(1),\sigma(2),\dots,\sigma(m)}} - \\ - \sum_{j=m+1}^{n} \frac{M_{\sigma(1),\sigma(2),\dots,\sigma(i-1),\sigma(j),\sigma(i+1),\dots,\sigma(m-1),\sigma(m)}}{M_{\sigma(1),\sigma(2),\dots,\sigma(m)}} x_{\sigma(j)},$$

where $i = 1, 2, \dots, m$.

We follow Theorem 5 and solve in the following example a system of linear equations.

Example 6. Let us consider the following system of two linear equations with three unknowns:

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 2x_1 + 2x_2 + 3x_3 = 7. \end{cases}$$

Each equation of the system describes the equation of a plane in a 3-dimensional space. The solution of the system determines a line in a 3-dimensional space, which is the intersection line of the two given planes.

$M_{1,2} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0,\ \ M_{1,3} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0$, then $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, and by Theorem 5, we can find the parametric equation of the above-named line:

$$\begin{cases} x_1 = x_{\sigma(1)} = \dfrac{M_{4,\sigma(2)}}{M_{\sigma(1),\sigma(2)}} - \dfrac{M_{\sigma(3),\sigma(2)}}{M_{\sigma(1),\sigma(2)}}\ x_{\sigma(3)} = \dfrac{M_{4,3}}{M_{1,3}} - \dfrac{M_{2,3}}{M_{1,3}}\ x_2 \\ x_3 = x_{\sigma(2)} = \dfrac{M_{\sigma(1),4}}{M_{\sigma(1),\sigma(2)}} - \dfrac{M_{\sigma(1),\sigma(3)}}{M_{\sigma(1),\sigma(2)}}\ x_{\sigma(3)} = \dfrac{M_{1,4}}{M_{1,3}} - \dfrac{M_{1,2}}{M_{1,3}}\ x_2. \end{cases}$$

$M_{2,3} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \quad M_{1,4} = \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 1, \quad M_{4,3} = \begin{vmatrix} 3 & 1 \\ 7 & 3 \end{vmatrix} = 2$, then

$$\begin{cases} x_1 = 2 - t \\ x_2 = t, \qquad t \in \mathbb{R}. \\ x_3 = 1 \end{cases}$$

4. Cramer's rule for matrix equations

In Sections 2 and 3 we solved the system $AX = B$ where $X \in \mathbb{K}^n$, $B \in \mathbb{K}^m$, and $\mathbb{K}$ is some field. Here we solve a more general question as follows: Let us consider the following matrix equation:

$$AX = B, \tag{3}$$

where $A \in \mathbb{K}^{m\times n}$, $X \in \mathbb{K}^{n\times k}$, $B \in \mathbb{K}^{m\times k}$, $\operatorname{rank} A = m$, $m \le n$.

The matrix equation $AX = B$ is equivalent to k systems $AX_l = B_l$, where $X_l \in \mathbb{K}^n$ is the column number l of matrix X, $B_l \in \mathbb{K}^m$ is the column number l of matrix B, $l = 1, 2, \dots , k$.

If $m = n$, the following theorem is a consequence of Theorem 1.

Theorem 7. *Let $\bar{A}$ be an $n \times (n + k)$-augmented matrix of the matrix equation* (3) $[m = n]$. *If its minor $M_{1,2,\dots,n} \neq 0$, then the solution of the matrix equation is*

$$x_{i,j} = \frac{M_{1,2,\dots,i-1,n+j,i+1,\dots,n-1,n}}{M_{1,2,\dots,n}},$$

where $i = 1, 2, \dots , n$, and $j = 1, 2, \dots , k$.

If $m < n$, the following theorem is a consequence of Theorem 5.

Theorem 8. *Let $\bar{A}$ be an $m \times (n + k)$-augmented matrix of the matrix equation* (3), *and $m < n$. Let $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$ be a permutation, such that $M_{\sigma(1),\sigma(2),\dots,\sigma(m)} \neq 0$. Then the solution of the matrix equation is the set of all $n\times k$-matrices with columns $(x_{\sigma(1),j}, x_{\sigma(2),j}, \dots , x_{\sigma(n),j})^T$ such that $x_{\sigma(m+1),j}, \dots ,$ $x_{\sigma(n),j}$ are arbitrary elements of the field $\mathbb{K}$ and*

$$x_{\sigma(i),j} = \frac{M_{\sigma(1),\sigma(2),\dots,\sigma(i-1),n+j,\sigma(i+1),\dots,\sigma(m-1),\sigma(m)}}{M_{\sigma(1),\sigma(2),\dots,\sigma(m)}} - \sum_{s=m+1}^{n} \frac{M_{\sigma(1),\sigma(2),\dots,\sigma(i-1),\sigma(s),\sigma(i+1),\dots,\sigma(m-1),\sigma(m)}}{M_{\sigma(1),\sigma(2),\dots,\sigma(m)}} x_{\sigma(s),j},$$

where $i = 1, 2, \dots , m$, and $j = 1, 2, \dots , k$.

We follow Theorem 8 and solve in the following example a matrix equation.

EXAMPLE 9. Let $AX = B$ be a matrix equation where $A \in \mathbb{K}^{2\times 3}$, $X \in \mathbb{K}^{3\times 2}$, $B \in \mathbb{K}^{2\times 2}$, and $M_{1,2} \neq 0$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{pmatrix} = \begin{pmatrix} a_{1,4} & a_{1,5} \\ a_{2,4} & a_{2,5} \end{pmatrix}.$$

By Theorem 8, we obtain the following solution of the equation:

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{pmatrix} = \begin{pmatrix} \frac{M_{4,2}}{M_{1,2}} - \frac{M_{3,2}}{M_{1,2}} t_1 & \frac{M_{5,2}}{M_{1,2}} - \frac{M_{3,2}}{M_{1,2}} t_2 \\ \frac{M_{1,4}}{M_{1,2}} - \frac{M_{1,3}}{M_{1,2}} t_1 & \frac{M_{1,5}}{M_{1,2}} - \frac{M_{1,3}}{M_{1,2}} t_2 \\ t_1 & t_2 \end{pmatrix}, \ t_1, t_2 \in \mathbb{K},$$

where $M_{i,j}$ are the minors of matrix

$$\bar{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \end{pmatrix}.$$

REFERENCES

[1] S. Burgstahler, *A generalization of Cramer's rule*, College Math. J., **14** (3) (1983), 203–205.

[2] G.M. Diaz-Toca, L. Gonzalez-Vega, H. Lombardi, *Generalizing Cramer's rule: solving uniformly linear systems of equations*, SIAM. J. Matrix Anal. Appl., **27** (3) (2005), 621–637.

[3] Z. Gong, M. Aldeen, *A generalized Cramer's rule and characterization of fixed modes*, Control Theory Adv. Technol., **10** (1995), 1833–1841.

[4] Z. Gong, M. Aldeen, L. Elsner, *A note on a generalized Cramer's rule*, Linear Algebra Appl., **340** (13) (2002), 253–254.

[5] K. Habgood, I. Arel, *Revisiting Cramer's rule for solving dense linear systems*, In: *Proceedings of the 2010 Spring Simulation Multiconference*, 2010.

[6] H. Leiva, *A generalization of Cramer's rule*, Adv. Linear Algebra Matrix Theory, **5** (2015), 156–166.

A.A.: Bashkir State University, Ufa, Russia & Institute of Mechanics of Ufa Branch, RAS, Ufa, Russia

E-mail: `akhtyamovam@mail.ru`

M.A.: Shamoon College of Engineering, Ashdod, Israel

E-mail: `meiravt@sce.ac.il`

M.D.: Shamoon College of Engineering, Beer-Sheva, Israel

E-mail: `dagan@sce.ac.il`

A.M.: Shamoon College of Engineering, Ashdod, Israel

E-mail: `artourm@sce.ac.il`