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AN INTERPOLATION APPROACH TO $\zeta(2n)$

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Abstract. We present a method to find approximately the values $\zeta(2n)$ of the Riemann zeta-function.

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1. Starting point and main idea

We begin by recalling a well-known trigonometric identity, also called the *Dirichlet kernel formula*, which reads

$$\frac{1}{2}+\sum_{k=1}^{m}\cos(kx)=\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)},\qquad (m=1,2,\dots). \tag{1}$$

Formula (1) can be easily proved by first multiplying $\cos(kx)$ by $\sin(x/2)$ and then using that $\cos a\sin b=(\sin(a+b)-\sin(a-b))/2$; summing up from $k=1$ to $k=m$, the formula is obtained.

Now we come to the steps in which the computation of $\zeta(2n)$ will be developed:

- Multiplying (1) by an algebraic polynomial q, and then integrating over $[0,\pi]$, we obtain
$$\sum_{k=1}^{m}\int_0^\pi q(x)\cos(kx)\,dx=-\frac{1}{2}\int_0^\pi q(x)\,dx+\int_0^\pi q(x)\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)}\,dx.$$
- For each positive integer n, a polynomial q is chosen to get $\int_0^\pi q(x)\cos(kx)\,dx=1/k^{2n}$, which yields
$$\sum_{k=1}^{m}\frac{1}{k^{2n}}=-\frac{1}{2}\int_0^\pi q(x)\,dx+\int_0^\pi q(x)\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)}\,dx. \tag{2}$$
- Finally, after doing the computations (at the right-hand side of (2)) and taking limits as $m\to\infty$, our hope is to find a closed form for $\zeta(2n)$ by means of
$$\begin{aligned}\zeta(2n)&=\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\\&=-\frac{1}{2}\int_0^\pi q(x)\,dx+\lim_{m\to\infty}\int_0^\pi q(x)\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)}\,dx.\end{aligned} \tag{3}$$

Our plan works provided that we can find a polynomial q such that $\int_0^\pi q(x)\cos(kx)\,dx = 1/k^{2n}$, and provided also that we can obtain the above limit. These two issues constitute the target of the next two sections.

2. First obstacle: Does any polynomial matching our plan exist?

2.1. A notoriously simple formula

Let us first examine how $\int_0^\pi q(x)\cos(kx)\,dx$ can be calculated and its value expressed in terms of the derivatives $q^{(2j-1)}(\pi)$ and $q^{(2j-1)}(0)$.

PROPOSITION 1. *If k is a positive integer and q is a polynomial, then*

$$\int_0^\pi q(x)\cos kx\,dx = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(-1)^k q^{(2j-1)}(\pi) - q^{(2j-1)}(0)}{k^{2j}}. \tag{4}$$

(Notice that the above sum always terminates, since q is a polynomial and thus $q^{(2j-1)} = 0$ for all j such that $2j-1 > \deg(q)$.)

Proof. Integrating by parts we have

$$\begin{aligned}
\int_0^\pi q(x)\cos kx\,dx &= q(x)\frac{\sin kx}{k}\Big|_0^\pi - \frac{1}{k}\int_0^\pi q'(x)\sin kx\,dx \\
&= -\frac{1}{k}\int_0^\pi q'(x)\sin kx\,dx \\
&= \frac{1}{k}\left(q'(x)\frac{\cos kx}{k}\Big|_0^\pi - \frac{1}{k}\int_0^\pi q''(x)\cos kx\,dx\right) \\
&= \frac{(-1)^k q'(\pi) - q'(0)}{k^2} - \frac{1}{k^2}\int_0^\pi q''(x)\cos kx\,dx.
\end{aligned} \tag{5}$$

Hence, by iteration, with q replaced by q'' in the above formula, we get

$$\begin{aligned}
\int_0^\pi q(x)\cos kx\,dx &= \frac{(-1)^k q'(\pi) - q'(0)}{k^2} \\
&\quad - \frac{1}{k^2}\left(\frac{(-1)^k q^{(3)}(\pi) - q^{(3)}(0)}{k^2} - \frac{1}{k^2}\int_0^\pi q^{(4)}(x)\cos kx\,dx\right).
\end{aligned}$$

A new step in our deduction is obtained by following the same procedure, now replacing q by $q^{(4)}$ in (5) to obtain

$$\begin{aligned}
\int_0^\pi q(x)\cos kx\,dx &= \frac{(-1)^k q'(\pi) - q'(0)}{k^2} - \frac{(-1)^k q^{(3)}(\pi) - q^{(3)}(0)}{k^4} \\
&\quad + \frac{1}{k^4}\left(\frac{(-1)^k q^{(5)}(\pi) - q^{(5)}(0)}{k^2} - \frac{1}{k^2}\int_0^\pi q^{(6)}(x)\cos kx\,dx\right) \\
&= \frac{(-1)^k q'(\pi) - q'(0)}{k^2} - \frac{(-1)^k q^{(3)}(\pi) - q^{(3)}(0)}{k^4} \\
&\quad + \frac{(-1)^k q^{(5)}(\pi) - q^{(5)}(0)}{k^6} - \frac{1}{k^6}\int_0^\pi q^{(6)}(x)\cos kx\,dx.
\end{aligned}$$

Following in the same manner (with a formal inductive argument if desired), the proof is complete. ■

2.2. A pleasant consequence

In view of Proposition 1 we deduce that for each positive integer n, if a polynomial P_{2n} of exact degree $2n$ exists such that $P'_{2n}(0) = P'''_{2n}(0) = \cdots = P^{(2n-3)}_{2n}(0) = 0$, also $P^{(2n-1)}_{2n}(0) = (-1)^n$, and $P'_{2n}(\pi) = P'''_{2n}(\pi) = \cdots = P^{(2n-1)}_{2n}(\pi) = 0$, then (4) simplifies to

$$\int_0^\pi P_{2n}(x)\cos kx\,dx = \frac{1}{k^{2n}}, \qquad (k = 1, 2, \dots), \tag{6}$$

and this is, exactly, what we were looking for.

2.3. Intermezzo. A journey to Lidstoneland

Summarizing, for a fixed positive integer n, we are thus led to find a polynomial P_{2n} such that $P^{(2j-1)}(0) = (-1)^n \delta_{j\,n}$ and $P^{(2j-1)}(\pi) = 0$ for $1 \le j \le n$. (We have used the *Kronecker delta* $\delta_{j\,n}$ which equals 0 if $j \neq n$ and which satisfies that $\delta_{n\,n} = 1$.) This situation reminds us to the celebrated problem of polynomial interpolation of finding the unique algebraic polynomial Q_n, of degree at most n, for which $Q_n(x_i) = y_i$, where the x_is and the y_is are $n+1$ given values with $x_i \neq x_j$ for $i \neq j$. But we can find many more interpolation problems that admit unique solution. For example, the so-called *Taylor interpolation*, which consists in finding the unique polynomial Q_n with $\deg(Q_n) \le n$ for which $Q_n^{(j)}(x_0) = y_j$, where $0 \le j \le n$, and where the point x_0 and the y_js are given numbers. If conditions on the consecutive derivatives are given not just for a single point x_0, but for two points x_0, x_1, we will be facing the so-called *two point Taylor interpolation*. This is again a well posed interpolation problem with a single solution. We refer the reader to the very classic source (and still an excellent one) [3, Ch. I], in which the interpolation problem is fully and deeply analyzed, and where many more examples can be found.

A not so popular (but nevertheless very well studied) interpolation situation emerges when the conditions are given for two points but, instead of the consecutive derivatives, now the given derivatives are just the even ones. This problem was originally posed and solved by Lidstone in [5], and extended by many other authors (see, for example, [7]). In the rest of this intermezzo we show the very basic facts on Lidstone interpolation that we will need in order to overcome our first obstacle. The interested reader can verify each one of the claims without any further reading.

The unique polynomial p of degree at most $2n-1$ such that $p^{(2j)}(0) = a_j$ and $p^{(2j)}(1) = b_j$ for $0 \le j \le n-1$ is

$$p(x) = \sum_{j=0}^{n-1} \left(a_j \Lambda_j(1-x) + b_j \Lambda_j(x)\right), \tag{7}$$

where the so-called *Lidstone polynomials* Λ_k are recursively defined by

$$\begin{aligned}
&\Lambda_0(x) = x,\\
&\Lambda_k''(x) = \Lambda_{k-1}(x),\\
&\Lambda_k(0) = \Lambda_k(1) = 0, \qquad (k = 1, 2, \dots).
\end{aligned}$$

Define $e_k(x) = x^k$. Since $e_{2k+1}^{(2j)}(0) = 0$ and $e_{2k+1}^{(2j)}(1) = \dfrac{(2k+1)!}{(2k+1-2j)}$ for $0 \le j \le k$, then

$$x^{2k+1} = \sum_{j=0}^{k} \frac{(2k+1)!}{(2k+1-2j)!} \Lambda_j(x), \qquad (k = 0, 1, \dots),$$

which yields the recursion

$$\Lambda_k(x) = \frac{x^{2k+1}}{(2k+1)!} - \sum_{j=0}^{k-1} \frac{\Lambda_j(x)}{(2k+1-2j)!}. \tag{8}$$

Well, that is what we need. No hard machinery after all. Now it is time to meet the polynomials P_{2n} that fulfill (6).

2.4. Passing the first challenge

For each $n = 1, 2, \dots,$ our task is to find a polynomial P_{2n} such that

$$P_{2n}^{(2j-1)}(0) = (-1)^n \delta_{j\,n} \quad \text{and} \quad P_{2n}^{(2j-1)}(\pi) = 0, \quad \text{for} \quad j = 1, \dots, n.$$

But such a polynomial exists. Define $Q_n(x) = P_{2n}'(\pi x)$ and note that conditions above transform to

$$\begin{aligned}
Q_n^{(2j)}(0) &= \pi^{2j} P_{2n}^{(2j+1)}(0) = \pi^{2j} P_{2n}^{(2(j+1)-1)}(0) = (-1)^n \pi^{2j} \delta_{j+1\,n},\\
Q_n^{(2j)}(1) &= \pi^{2j} P_{2n}^{(2j+1)}(\pi) = \pi^{2j} P_{2n}^{(2(j+1)-1)}(\pi) = 0, \qquad (1 \le j+1 \le n).
\end{aligned}$$

Hence, by (7), we deduce

$$Q_n(x) = (-1)^n \pi^{2n-2} \Lambda_{n-1}(1-x).$$

Let us mention an important detail. Since $2n$ conditions are the ones for P_{2n} to satisfy, the degree of P_{2n} is one unit more than required. In other words, if we insist to look for a polynomial of precise degree $2n$, then it is still possible to add one more condition on P_{2n}. Just for the sake of simplicity, let us impose $P_{2n}(0) = 0$. This choice yields

$$\begin{aligned}
P_{2n}(x) &= \int_0^x Q_n\left(\frac{t}{\pi}\right) dt = (-1)^n \pi^{2n-2} \int_0^x \Lambda_{n-1}\left(1 - \frac{t}{\pi}\right) dt \\
&= (-1)^n \pi^{2n-1} \left(\Lambda_n'(1) - \Lambda_n'\left(1 - \frac{x}{\pi}\right)\right),
\end{aligned} \tag{9}$$

and thus

$$\int_0^\pi P_{2n}(x)\, dx = (-1)^n \pi^{2n} \Lambda_n'(1). \tag{10}$$

Finally notice that since Λ_n is a polynomial of precise degree $2n+1$ (see (8), then we deduce from (9) that $\deg P_{2n} = 2n$.

3. Second obstacle: How to compute the limit of an integral without computing the integral

As mentioned in Section 1, the success of our plan strongly depends on our ability to handle the sum

$$-\frac{1}{2}\int_0^\pi P_{2n}(x)\,dx + \lim_{m\to\infty}\int_0^\pi P_{2n}(x)\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)}\,dx. \tag{11}$$

(This is the right-hand side of formula (3), with q replaced by P_{2n}.) Notice that we have just calculated the first term in the above sum (it equals minus half the value in (10), so now we have to consider only the second term. But we must emphasize that our concern is not to compute

$$\int_0^\pi \frac{P_{2n}(x)}{2\sin(x/2)}\sin((m+1/2)x)\,dx,$$

but its limit as $m\to\infty$. To this end, we have at hand a classical result:

Lemma 1. [Riemann-Lebesgue] (see [3, p. 296] *If f is a function with continuous derivative on $[0,\pi]$, then*

$$\lim_{t\to\infty}\int_0^\pi f(x)\cos(tx)\,dx = \lim_{t\to\infty}\int_0^\pi f(x)\sin(tx)\,dx = 0.$$

The proof of the above "easy version" of the Riemann-Lebesgue lemma is nothing but integrating by parts. Let us show how this famous results works to compute the limit in (11). The trick consists in writing $P_{2n}(x) = xR_{2n-1}(x)$ (R_{2n-1} being a polynomial of degree $2n-1$), which is possible since by (9) we have that $P_{2n}(0)=0$. This factorization yields

$$\frac{P_{2n}(x)}{2\sin(x/2)} = \frac{x/2}{\sin(x/2)}R_{2n-1}(x),$$

and the continuity of $P_{2n}(x)/(2\sin(x/2))$ and its derivative on $[0,\pi]$ follows readily. Therefore,

$$\lim_{m\to\infty}\int_0^\pi P_{2n}(x)\frac{\sin\big((m+1/2)x\big)}{2\sin(x/2)}\,dx = 0.$$

4. Epilogue

The good news is that all the workings have already been done. Indeed, substituting (10) and (12) into (3), and taking into account (8), we conclude that

$$\zeta(2n) = \sum_{k=1}^\infty \frac{1}{k^{2n}} = (-1)^{n+1}\frac{\pi^{2n}\Lambda_n'(1)}{2}, \qquad (n=1,2,\dots), \tag{13}$$

where $\Lambda'_0(1) = 1$ and $\Lambda'_n(1) = \dfrac{1}{(2n)!} - \sum_{j=0}^{n-1} \dfrac{\Lambda'_j(1)}{(2n+1-2j)!}$. The first few values of $\Lambda'_n(1)$, $n = 1, \ldots, 5$ are $1/3$, $-1/45$, $2/945$, $-1/4725$, $2/93555$. One can go on computing them as long as patience will permit.

Remark. By changing the conditions on P_{2n}, the same problem of summing up $\sum_{k=1}^{\infty}(1/k^{2n})$ could have been solved in a different way. For example, the conditions $P_{2n}(0) = 0$, $P_{2n}^{(2j-1)}(0) = P_{2n}^{(2j-1)}(\pi) = 0$ for $1 \le j \le n-1$, $P_{2n}^{(2n-1)}(\pi) = 0$ and $P_{2n}^{(2n)}(0) = (2n)!$ define uniquely the polynomials P_{2n} by means of

$$P_{2n}(x) = \sum_{k=0}^{2n-1} \binom{2n}{k} (2\pi)^k B_k \, x^{2n-k},$$

which are closely related with Bernoulli polynomials (the B_k are Bernoulli numbers as described in [1]). In some sense, this resembles the approach in [2] to compute $\zeta(2n)$.

In much a similar way, Euler polynomials could have appeared. In fact, this is the approach in [4].

In closing, let us mention that formula (4.6) in [7] has some resemblance with our (13) (we came across with [7] after the submission of this note). However, it is worth mentioning that our method is much more elementary, and also that we give the key to recursively compute $\zeta(2n)$, by relating $\Lambda'_n(1)$ with $\Lambda'_j(1)$ for $j = 0, 1, \ldots, n-1$.

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