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A RESEARCH ON THE CREATION OF PROBLEMS FOR MATHEMATICAL COMPETITIONS

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Abstract. This paper describes the steps that a specialist in problem posing takes in order to create problems for mathematics competitions. We focus on a specialist's techniques and strategies on problem posing and we attempt to answer the following research questions: a) what characteristics make a mathematical problem interesting and suitable for competitions, b) how do techniques in problem posing differ from one expert to another, and what kind or level of creativity is required of problem posing for mathematics competitions.

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MSC Subject Classification: 97D50

Key words and phrases: mathematical problem posing; mathematics competitions; creativity.

Introduction

In the last twenty years, one of the main areas of interest of the International community of the Didactics of Mathematics has been research on the creation of mathematical problems. Several papers focus on how students and prospective teachers are trained in mathematical problem posing in order to improve their ability to understand mathematical concepts and problem solving. However, there is relatively limited research on how "difficult" problems, such as problems for mathematics competitions, are created. In particular, there is limited empirical research on how specialists in mathematical problem posing create, design and modify problems for competitions.

In this paper we examine the way that a specialist in problem posing worked when we asked him to create problems for mathematics competitions. We are also interested in comparing the techniques that problem posers use to create problems. We first present a brief overview of the relevant literature to determine the framework within which our research takes place.

Overview of the relevant literature

Specialists in problem posing (from now on we will refer to them simply as specialists) involved in mathematics competitions do not usually publish how they compose such problems. It is highly possible that, to some of them, the way in which problems are created is a "professional secret". Nevertheless, there have been occasional publications on this subject, for example papers appearing in the journal "Mathematics Competitions", the official journal of the International Mathematical Olympiads, (I.M.O.), Engel [2], Panaitopol & Stefanescu [9], Gardiner [3], Junda & Jianping [4], Soifer [14].

The most relevant and insightful paper to our research is the paper by Sharygin [11]. Igor Sharygin (1937–2004) was a coach of the I.M.O. Russian team, and his paper [11] is an overview of some of the techniques he used to create novel problems for mathematical contests, for the I.M.O., and for didactic reasons.

He presents many detailed examples as instances of the following six strategies:

1. Reformulating: disguise a known mathematical fact (e.g. a theorem) and formulate it in a radically different way. For example, Sharygin mentions that "*when a geometric problem is translated into an algebraic one, this translation may result into a striking and elegant problem. For example, if we consider the problem of constructing a triangle when given its three heights x, y, z and we call a, b, c its sides that correspond to these heights, then we can produce a fairly hard system of algebraic equations*".

2. Chaining: "*When the structure of a problem is complicated, its solution is often done in steps*".

3. Considering a special case: by considering special cases of fundamental theorems one can compose an interesting and elegant problem.

4. Generalizing: an essential method in mathematics of any level and form.

5. Varying the given data: "*A small change in the formulation of a problem can consequently lead to tremendous changes in its level of difficulty*".

6. Discovery: According to Sharygin, "*the main source of new problems is curiosity, our wish to discover the essence of a problem, the ability to observe a known fact from a new point of view. That is when the most interesting geometrical problems appear, problems we can characterize as discoveries*". Sharygin describes one of his "*best geometric discoveries*", yet he does mention, "*for example, Archimedes potentially knew it*".

Of course these techniques follow Polya's tradition. And although the first five techniques support the view that a new problem is usually inspired by problems the specialist is familiar with, Sharygin [11], the technique of Discovery supports exactly the point made by Kontorovich and Koichu [8], that "*even an extended pool of familiar problems is not sufficient for posing high-quality problems*".

We believe Sharygin's views are of paramount pedagogical value. In his paper Kontorovich [6], the researcher focuses on the goals that specialists try to achieve with the problems they set for mathematics competitions. Kontorovich studied 26 adult participants from the Competition Movement (i.e. coaches, experts in problem posing and organizers of mathematics competitions), "*The findings of the preliminary study suggested that the participants shared a pedagogical agenda consisting of four interrelated goals: to provide students with opportunities to learn meaningful mathematics, to strengthen their positive attitude towards mathematics, to create cognitive challenges for the students and to surprise them. Competition problems* *were perceived by the participants of the preliminary study as a means of achieving these goals*", Kontorovich [6, p. 2].

Another important paper on expert problem posing is [8] by Kontorovich and Koichu. This paper is not directly related to our study, but we briefly present their original idea for completion.

In [8], Kontorovich and Koichu focus on how specialists classify and group their "*personal pool of problems*", and how they use it to create new ones. More specifically, they examine the techniques used by a specialist ("Leo") to compose new problems, given as a starting point a list of 17 problems he had composed in the past. The key idea they introduce to study Leo's actions and to describe the abilities that characterize a specialist, is the concept of *nesting ideas*, "*familiar problems are described as 'eggs'*", therefore a "nest" would accommodate "eggs" that share similar attributes, and additionally, the nest would "*serve as a useful framework for 'laying' new (eggs)*". They note that "*similar attributes*" that classify problems are a personal choice, and they depend completely on the individual specialist. However, they do describe three kinds of nesting ideas, that is, three kinds of reasons to include problems in the same class:

1. Deep structure nesting ideas.
2. Surfaces structure nesting ideas.
3. Nesting ideas based on particularly rich mathematical concepts.

In our study we are not concerned with the concept of nesting ideas.

Similar papers on problem posing for mathematics competitions are scarce; even though there is a number of studies on how specialists solve problems in the community of the Didactics of Mathematics, "*additional research effort is needed in order to grasp the essence of expert problem poser-performance*" as Kontorovich & Koichu [8] emphasize.

The research questions

In this study we address two fundamental issues:

a) What principles does a specialist use to create problems, and what kind of creativity is required of problem posing for mathematics competitions?

b) What does it take for a problem to be interesting and appropriate for competitions? How do techniques in problem posing differ from one specialist to another? Are the techniques used by specialists in posing problems of high quality the same as the ones used by specialists in posing problems for other purposes?

The framework and the procedure of the present research

For our research purposes, we interviewed a specialist and asked him to describe in detail how he composes problems for competitions. We will refer to this expert as V. We chose V for the following reasons. V is one of the coaches of the Greek team for I.M.O. Not only did we have a personal opinion and insight into the problems composed by V, but more importantly, V was expressive, in the sense that he could develop fully and clearly his thoughts during the creation of a problem.

We had two interview sessions with V, each of which lasted two hours. The interview sessions took place during the Summer School for preparation of students for mathematics competitions, which was organized by the Greek Mathematical Society in July 2015. The purpose of the interview sessions was made absolutely clear to V, who answered each of the set questions with great willingness and interest. We explained to him that we would publish the conclusions of this interview in accordance with the standards of international literature.

During these interviews and throughout the entire procedure, we asked V questions, thus communicating with him directly. The first question posed to V was how does he determine the level of difficulty of a problem he is about to construct. V stated that "*it may not be known from the beginning*". He also said that "*I have some basic templates ('patterns' as he calls them), in which I rely on to compose a problem*". He said that he had constructed a personal pool of familiar problems. He has certain preferences with regard to the mathematical themes and he believes that these preferences are his basic starting point. He said that "*one should be attracted to this process*". He believes that one of his "secrets" in creating a problem is that the problem should lead to unexpected conclusions, even if it is initially based on "well-known" routes. This is his criterion of "good work".

In the case of geometrical problems, in particular, he tries to start from known theorems. As soon as he chooses a geometric shape to which he can associate well-known theorems, he examines how the theorems respond in accordance with "varying shapes". The term "varying shapes" stands for geometric shapes, certain elements of which have been modified, for example the measure of an angle, the length of a side, etc. He then adds new objects on these varying shapes, and examines whether the conclusions of the theorems, which were associated to the initial shape, have been altered or remain the same. In the end, the prospective solver is asked to identify and/or prove these variations of these theorems.

V gave us specific examples of problem posing during the interview sessions, in particular, he composed problems for students ages 15–18 years old, and his initial aim was to compose problems of the Balkan Mathematical Olympiad level. We point out that he was not prepared in advance for this task; the interview began as soon as he had agreed to participate in the research. This is crucial to our research, since this enabled us to witness composition-creation of mathematical problems "in vivo". After V finished formulating his problems, we discussed the process with him and asked him clarifying questions via emails. What follows is a detailed presentation of V's creations.

Outline of the construction of the first problem

Step 1. V drew the triangle ABC and its circumscribed circle, as in Figure 1. He mentioned that during the problem-posing process he would change certain elements of this shape, more specifically he said that "*I usually change the position of vertex A*".

STEP 2. He drew the height AD of the triangle ABC and he extended it to intercept the circumscribed circle at point E. Thus a new element appeared, namely the segment EC, which often appears in similar problems. He chose EC and not the segment BE, because in this particular shape the area on the right of the segment AE is larger than the area on the left of AE.

STEP 3. He added new elements on the shape, as shown in Figure 1. Point H is symmetric to point E with respect to BC. At this step, he mentioned that he tried to find relations that existed between the elements of the shape. He said that some relations could be known to the poser, perhaps to the solvers as well, yet other relations could turn out to be new and unexpected to both parties. As an example, he mentioned that H is the orthocenter of the triangle ABC, a relation that is a well-known theorem and hence, so it can be posed as a competition problem.

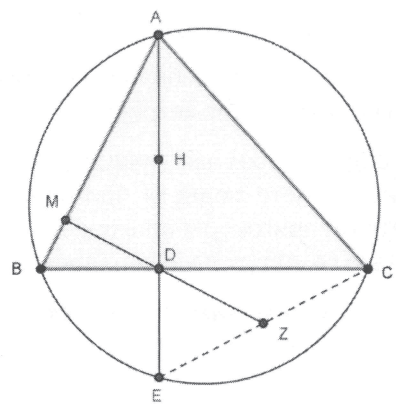

Fig. 1

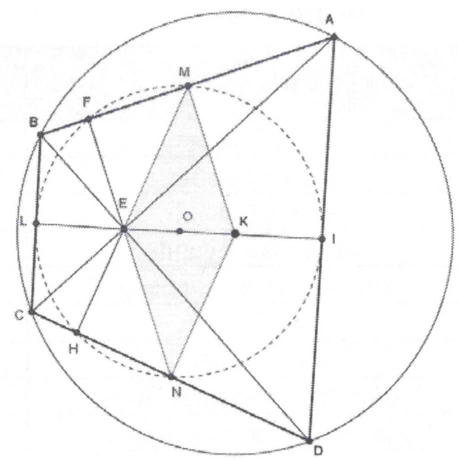

Fig. 2

STEP 4. He drew the segment DZ, where Z is the midpoint of the line segment EC. He justified this move by explaining that the triangle EHC is an isosceles triangle and the points D and Z are the midpoints of the sides EH and EC, hence $DZ \parallel HC$. He also mentioned that he would change the name of the orthocenter H since H is the letter commonly used to denote the orthocenter in Greek literature, and therefore its name could potentially "reveal" its property as an orthocenter. He then extended the segment DZ and named M the point of interception with AB. He knew that the line segment DM is perpendicular to AB, since $DZ \parallel HC$ and HC belongs to the height of ABC through the vertex C.

STEP 5. V decided that proving that DM is perpendicular to AB is an easy task. However he wanted to give a role to the point H. At this moment he recalled theorems related to this shape, i.e. a quadrilateral with perpendicular diagonals inscribed in a circle. At a second interview, V gave us a list with a few of these theorems, with illustrations as in Figure 2.

a) The points E, F and N are collinear points. E is the intersection of the diagonals of $ABCD$, F is the projection of E on one of the sides of the quadrilateral, and N is the midpoint of the opposite side.

b) The quadrilateral with vertices $KMEN$ is a parallelogram. M and N are the midpoints of AB and CD, respectively.

c) The midpoints M, L, N and I of the sides of $ABCD$, the projection of E on AB and the projection of E on CD, lie on the same circle.

d) The center O of this new circle is the midpoint of the line segment KE.

Based on the above theorems and attributes associated with these shapes, V posed the following problem:

> Take the quadrilateral $ABCD$ inscribed in a circle whose diagonals intersect perpendicularly at the point E. The circumscribed circle of the triangle EDC, intersects the extension of AD at point F. The straight line EF intersects the segment AB in L. Prove that the CL is perpendicular to AB, Figure 3.

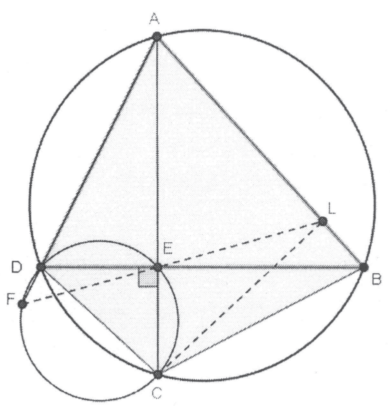

Fig. 3

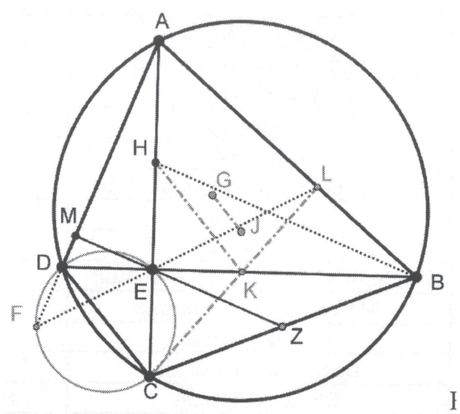

Fig. 4

At this point, we concluded the first interview. During the second interview the following day, V worked further on this problem.

The line segment CE is perpendicular to BD and the line segment CF is perpendicular to AD, because CD is the diameter of the circumcircle of the triangle CDE. As a result, EF is the Simson's line of the triangle ABD that corresponds to the point C. It is therefore concluded that the point L is the projection of the point C on the line segment AB.

If K denotes the point of intersection of the lines BD and CL, then the point K is the orthocenter of the triangle ABC. For this reason, potential solvers can be asked to prove that the point K is the orthocenter of the triangle ABC, hiding the fact that the line segments CL and AB are perpendicular.

Now, by adding some new elements, the problem can be made even more challenging. Observe that:

1. The point H is symmetric to point C with respect to the line BD and is the orthocenter of the triangle ADB.
2. The circumscribed circles of both triangles ACB and ABD have the same center.
3. If G is the barycenter of the triangle ABD and J is the barycenter of the triangle ABC, then the points O, G, H and O, J, K are on the Euler's lines of these triangles.

Hence, the problem becomes more difficult phrased as follows:

> Take the quadrilateral $ABCD$ inscribed in a circle whose diagonals intersect perpendicularly at point E. The circumscribed circle of the triangle EDC intersects the extension of AD in point F. The straight line EF intersects AB in L and the line CL intersects the line BD in point K. Point H is symmetric to point C with respect to the line BD, point G is the barycenter of the triangle ABD and point J is the center of the circumscribed circle of the triangle ABC. Prove that GL is parallel to the line HK and that $GL = \frac{1}{3} HK$.

V also noted the following. By adapting the facts one can prove that the quadrilateral $CDHK$ is a rhombus. One can also employ the following property: "*The vertical lines from the point M to the sides of the quadrilateral pass through the middles of its opposite sides*". He concluded his exposition by remarking that this will enable him to formulate additional new questions.

Outline of the construction of the second problem

V drew a triangle ABC and its Miquel point (the point from which the circumscribed circles of the triangles ADZ, BDE and CZE pass), where D, E and Z are points on the sides AB, BC and AC respectively, see Figure 5. He examined the cases where the Miquel point overlaps with other "characteristic" points of the triangle. His steps were the following:

Step 1. He chose the center of the circumscribed circle of the triangle, as the Miquel point O.

Step 2. Point E on the side BC was chosen not to be the midpoint of the side, since he considered this case trivial.

Step 3. He drew the circumscribed circle of the triangle BEO, that intercepts the side AB at the point D, and similarly, the circumscribed circle of the triangle EOC, intercepting the side AC at the point Z, as shown in Figure 5.

Step 4. V knew (by Miquel's theorem) that the points A, D, O and Z lie in the same circle, something that a prospective solver of competitions problems might know. For this reason, he decided not to make any reference to the circumscribed circle of the triangle AOZ. This circle appeared in Step 6, without mentioning that point D lies on it.

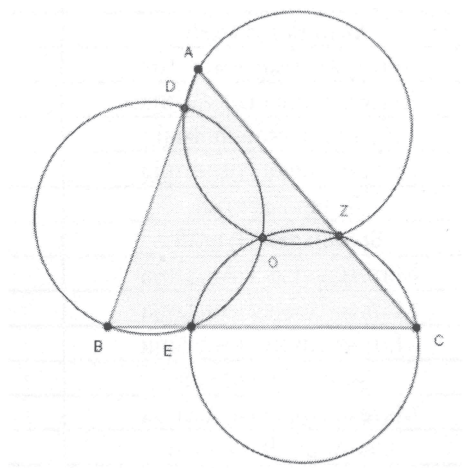

Fig. 5

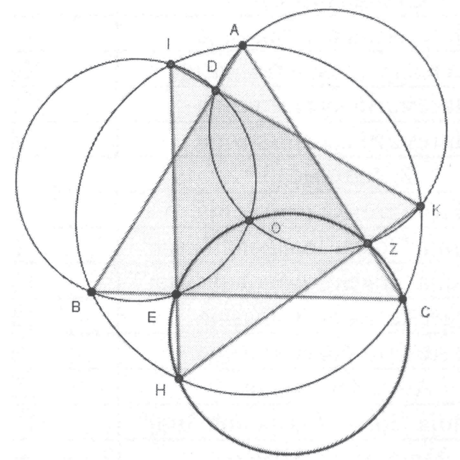

Fig. 6

STEP 5. V noted that it can be proved that the three circles of Figure 5 are equal. He felt that this was the first interesting attribute of this problem, which requires a proof. He said that he could request this proof from the solvers, in case he wanted to increase the problem's difficulty.

STEP 6. He drew the circumscribed circle of triangle ABC, as shown below in Figure 6, and he appointed as I, H, K the intersection points with the other three circles.

The new question that emerged was the relation of the triangle IHK with the initial triangle ABC. He conjectured that the two triangles were similar to each other, which was something he was sure about. He even speculated that the aforementioned triangles were congruent.

In this way V constructed a sequence of related problems instead of one single problem.

1. If he proved that triangles ABC and IHK were similar, he could ask the solver to "prove that triangles ABC and IHK are similar".

2. If he proved that triangles ABC and IHK were congruent, he could ask the solver to "prove that triangles ABC and IHK are congruent".

3. If he proved that triangles ABC and IHK were not similar, he could ask the solver to "prove that these triangles are not similar and find under which conditions they would become similar".

4. If he proved that triangles ABC and IHK were not congruent, he could ask the solver to "prove that these triangles are not congruent and find under which conditions they would become similar".

At this moment, V felt that he had completed the necessary steps for creating the second problem based on the properties of the Miquel point of a triangle. He said that the final step would be to write the problem in a form of one or two questions-tasks.

A short outline of the construction of the third problem

V declared that he would construct a problem based on Combinatorial Geometry. He revealed that he prefers problems of Combinatorial Geometry rather than problems of Combinatorial Enumeration because he likes Geometry more than other subjects and he feels more familiar with such problems. The steps of his creation were the following:

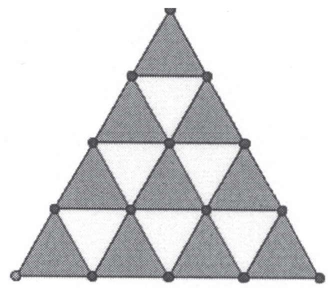

Fig. 7

STEP 1. He chose a familiar pattern. This was the equilateral triangle divided in smaller equilateral triangles, as shown in Figure 7.

STEP 2. From this pattern, he attempted to generate problems, such as the following:

a) Find the number of regular hexagons, when the original figure has n rows, consisting of small equilateral triangles.

b) Find the number of rhombuses consisting of two (or more) equilateral triangles.

c) Find the number of parallelograms of Figure 7 that are not rhombuses.

He revealed that he uses a personal database to establish whether the problem is already known or has been published. If this is the case, he makes modifications, improvements, and generalizations on the problem (at this point, we noticed that some of the six techniques described by Sharygin applied directly in this case). Of course he solves the problem to be certain of the result of his construction.

At a second interview he said to us that the basic idea to compose this problem came from a problem of the Balkan Juniors Mathematical Olympiad 2011. Another idea comes from a problem that was proposed at the Balkan Mathematical Olympiad 2014, "*Let n be a positive integer. A regular triangle with side length n is divided into equilateral triangles with side length 1 by lines parallel to its sides. Find the number of regular hexagons all of whose vertices are among the vertices of those equilateral triangles*".

New questions for V

Some of the following questions were raised after the interview, and were answered through emails.

1. We asked V if he had read Sharygin's article in the Greek edition of the journal "Quantum", "The art of posing novel problems", which is the translation of the Russian article of Sharygin [11]. The answer was negative. This question was raised because the first problem that Sharygin constructs in his article is related to Miquel's point. In addition, many points of Sharygin's presentation on composing that problem resemble closely V's spontaneous practice.

2. Our second question to V was how he finds out if his problems are received as original, difficult or “beautiful” from the wider mathematical community. He answered “through personal conversations with contestants and colleagues and through debates on social media and forums”.

3. The third question was whether he had a personal database, a “personal pool of familiar problems” according to Kontorovich and Koichu [8]. In general, we wondered how he organized statements and solutions of problems and how he recorded new ideas for problem posing. He answered that “*Since the problems I work on are usually geometrical, I ‘store’ my ideas into shapes, via the Geogebra software. It is possible to forget certain data of a specific problem, in which case I have to reconstruct it – a written text with its precise statement might not be available at that point in my database. This gives me the chance to add new questions or modify the queries I had originally thought*”.

We asked V these additional questions in order to form as clear a picture as possible of how specialists think and act when composing problems for competitions. We describe the conclusions we arrived at from V’s answers in the following section.

Discussion of the results and final remarks

In response to the question how do specialists create and pose problems for mathematics competitions, we must underline the observation that specialists take similar steps in composing challenging problems. While V was working, it was easy to recognize many of the techniques described by Sharygin [11]. According Sharygin a new problem usually originates from other problems that the poser is familiar with it. In our research, V demonstrated in vivo all these techniques.

During our research we observed an additional technique used by the specific specialist, namely that of creating a misleading wrapper to the problem. V uses wording in such a way, so as to supposedly create a stumbling block or unhelpful image of the problem, which would distract the solvers from the solution of the problem. By using this method, the specialists increase the cognitive difficulty of the problem and make it more novel and surprising to the solver. When V wanted to increase the difficulty of the problem, that is when he attempted to move the solver away from the solution, he tried to hide some of the characteristic faces of the problem.

An ever-present “challenge” of the procedure of problem posing is the possibility that a simple solution – different from the one intended by the poser – may exist. A poser – like V – tries to avoid this. Sharygin’s remark [8] that “*many problems are so constructed that they can be solved with the use of a particular idea. However, it often happens that a problem has a different (and several times a much simpler) solution*”, seem to answer this question.

An additional important issue is how the problems, which specialists create, will be accepted by the participants of the competition and related parties, such as students, coaches, fellow posers and the competition organizers. The specialists compare the posed problem with a particular set of existing problems. The comparison concerns cognitive complexity of the posed problem, its mathematical significance, its novelty and the element of surprise.

A very important question, which arises, but not answered in this paper, is the following: "*What is the role of software in the process of problem posing for competitions?*" We point out that V informed us that, when he composes problems, he often uses software like Geogebra. There is some research on this question, Santos-Trigo [10], Shimomura et al. [13], but the answer is not yet clear.

The road to understanding how experts and specialists in problem posing create problems is long and research in this field has still a long way to go.

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