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LIMITS OF SEQUENCES AND FUNCTIONS FROM CERTAIN CLASSES

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Abstract. We present some general methods to calculate limits of functions and sequences belonging to certain classes.

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1. Main results

In many cases, calculating limits of functions or sequences is quite difficult. Therefore the knowledge of additional properties of functions or sequences can lead us to easier determination of these limits.

Throughout the paper we denote $\mathbb{R}_+^* = (0, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$.

DEFINITION 1. A function $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is called a B-function if there exists $r \in \mathbb{R}_+$ such that

$$\lim_{x\to+\infty} \frac{f(x+1)}{x^{r+1} f(x)} = a \in \mathbb{R}_+^* \quad \text{and there exists} \quad \lim_{x\to+\infty} \frac{\big(f(x)\big)^{1/x}}{x}.$$

DEFINITION 2. A positive real sequence $(a_n)_{n\geqslant 1}$ is called a B-sequence if there exists $r \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^{r+1} a_n} = a \in \mathbb{R}_+^* \quad \text{and there exists} \quad \lim_{n\to\infty} \frac{(a_n)^{1/n}}{n}.$$

DEFINITION 3. A function $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is called a G-function if there exists $s \in \mathbb{R}_+$ such that

$$\lim_{x\to+\infty} \frac{g(x+1) - g(x)}{x^s} = b \in \mathbb{R}_+^* \quad \text{and there exists} \quad \lim_{x\to+\infty} \frac{g(x)}{x^{s+1}}.$$

DEFINITION 4. A positive real sequence $(b_n)_{n\geqslant 1}$ is called a G-sequence if there exists $s \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{n^s} = b \in \mathbb{R}_+^* \quad \text{and there exists} \quad \lim_{n\to\infty} \frac{b_n}{n^{s+1}}.$$

Example 1. If $\Gamma\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is the gamma function, then $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ defined as $f(x) = \Gamma(x+1)$ is a B-function.

Indeed,

$$\lim_{x\to+\infty} \frac{f(x+1)}{xf(x)} = \lim_{x\to+\infty} \frac{\Gamma(x+2)}{x\,\Gamma(x+1)} = \lim_{x\to+\infty} \frac{(x+1)\Gamma(x+1)}{x\,\Gamma(x+1)} = \lim_{x\to+\infty} \frac{x+1}{x} = 1,$$

so $r = 0$ and $a = 1$, and

$$\lim_{x\to+\infty} \frac{\big(f(x)\big)^{1/x}}{x} = \lim_{x\to+\infty} \frac{\big(\Gamma(x+1)\big)^{1/x}}{x} \lim_{\mathbb{N}\ni n\to\infty} \frac{\big(\Gamma(n+1)\big)^{1/n}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \Big(\frac{n}{n+1}\Big)^n = \frac{1}{e}.$$

Example 2. The function $h : \mathbb{R}_+^* \to \mathbb{R}_+^*$, $h(x) = x^{x+1}$ is a B-function.

Indeed,

$$\lim_{x\to+\infty} \frac{h(x+1)}{xh(x)} = \lim_{x\to+\infty} \frac{(x+1)^{x+2}}{x\cdot x^{x+1}} = \lim_{x\to+\infty} \Big(\frac{x+1}{x}\Big)^{x+2} = e, \text{ so } r = 0,\ a = e,$$

$$\lim_{x\to+\infty} \frac{\big(h(x)\big)^{1/x}}{x} = \lim_{x\to+\infty} \frac{x^{\frac{x+1}{x}}}{x} = \lim_{x\to+\infty} x^{1/x} = 1.$$

Example 3. The sequence $(a_n)_{n\geqslant 1}$, $a_n = (2n-1)!!$ is a B-sequence.

Indeed,

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = \lim_{n\to\infty} \frac{(2n+1)!!}{n\cdot(2n-1)!!} = \lim_{n\to\infty} \frac{2n+1}{n} = 2, \text{ so } r = 0,\ a = 2,$$

$$\lim_{n\to\infty} \frac{(a_n)^{1/n}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n}$$

$$= \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \lim_{n\to\infty} \frac{2n+1}{n+1}\Big(\frac{n}{n+1}\Big)^n = \frac{2}{e}.$$

(Here, the equality of Cauchy and D'Alembert limits was used.)

Example 4. The function $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $g(x) = x^2$ is a G-function.

Indeed,

$$\lim_{x\to+\infty} \frac{g(x+1)-g(x)}{x} = \lim_{x\to+\infty} \frac{(x+1)^2 - x^2}{x} = \lim_{x\to+\infty} \frac{2x+1}{x} = 2,$$

so $s = 1$ and $b = 2$,

$$\lim_{x\to+\infty} \frac{g(x)}{x^{s+1}} = \lim_{x\to+\infty} \frac{x^2}{x^2} = 1.$$

Example 5. The sequence $(b_n)_{n\geqslant 1}$, $b_n = n^{s+1}$, $s \in \mathbb{R}_+^*$ is a G-sequence.

Indeed,

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{n^s}=\lim_{n\to\infty}\frac{(n+1)^{s+1}-n^{s+1}}{n^s}=s+1,$$

$$\begin{aligned}\lim_{n\to\infty}\frac{b_n}{n^{s+1}}&=\lim_{n\to\infty}\frac{b_{n+1}-b_n}{(n+1)^{s+1}-n^{s+1}}\\&=\lim_{n\to\infty}\frac{b_{n+1}-b_n}{n^s}\cdot\lim_{n\to\infty}\frac{n^s}{(n+1)^{s+1}-n^{s+1}}=(s+1)\cdot\frac{1}{s+1}=1.\end{aligned}$$

(Here, Stolz Theorem was used.)

In what follows, we present some properties of the classes of functions and sequences defined above.

1. If $f\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a B-function, and r and a are as in Definition 1, then

$$\lim_{x\to+\infty}\frac{\big(f(x)\big)^{1/x}}{x^{r+1}}=\frac{a}{e^{r+1}}.$$

Indeed,

$$\begin{aligned}\lim_{x\to+\infty}\frac{\big(f(x)\big)^{1/x}}{x^{r+1}}&=\lim_{\mathbb{N}\ni n\to\infty}\frac{\big(f(n)\big)^{1/n}}{n^{r+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{f(n)}{n^{n(r+1)}}}\\&=\lim_{n\to\infty}\Big(\frac{f(n+1)}{(n+1)^{(n+1)(r+1)}}\cdot\frac{n^{n(r+1)}}{f(n)}\Big)\\&=\lim_{n\to\infty}\frac{f(n+1)}{n^{r+1}f(n)}\cdot\Big(\frac{n}{n+1}\Big)^{(n+1)(r+1)}=\frac{a}{e^{r+1}}.\end{aligned}$$

2. If $f\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a B-function, and r and a are as in Definition 1, then

$$\lim_{x\to+\infty}\frac{\big(f(x+1)\big)^{1/(x+1)}}{\big(f(x)\big)^{1/x}}=1\quad\text{and}\quad\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{1/(x+1)}}{\big(f(x)\big)^{1/x}}\right)^x=e^{r+1}.$$

Indeed,

$$\begin{aligned}\lim_{x\to+\infty}\frac{\big(f(x+1)\big)^{1/(x+1)}}{\big(f(x)\big)^{1/x}}&=\lim_{x\to+\infty}\frac{\big(f(x+1)\big)^{1/(x+1)}}{(x+1)^{r+1}}\cdot\frac{x^{r+1}}{\big(f(x)\big)^{1/x}}\cdot\Big(\frac{x+1}{x}\Big)^{r+1}\\&=\frac{a}{e^{r+1}}\cdot\frac{e^{r+1}}{a}\cdot1=1,\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{1/(x+1)}}{\big(f(x)\big)^{1/x}}\right)^x&=\lim_{x\to+\infty}\frac{f(x+1)}{f(x)}\cdot\frac{1}{\big(f(x+1)\big)^{1/(x+1)}}\\&=\lim_{x\to+\infty}\left(\frac{f(x+1)}{x^{r+1}f(x)}\cdot\frac{(x+1)^{r+1}}{\big(f(x+1)\big)^{1/(x+1)}}\cdot\Big(\frac{x}{x+1}\Big)^{r+1}\right)\\&=a\cdot\frac{e^{r+1}}{a}\cdot1=e^{r+1}.\end{aligned}$$

3. If $(a_n)_{n\geqslant 1}$ is a B-sequence and r and a are as in Definition 2, then

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{r+1}}=\frac{a}{e^{r+1}}\quad\text{and}\quad\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n=e^{r+1}.$$

Indeed,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{r+1}}&=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{n(r+1)}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)(r+1)}}\cdot\frac{n^{n(r+1)}}{a_n}\\&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{r+1}}\left(\frac{n}{n+1}\right)^{(n+1)(r+1)}=\frac{a}{e^{r+1}},\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\\&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{r+1}}\cdot\frac{(n+1)^{r+1}}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^{r+1}\\&=a\cdot\frac{e^{r+1}}{a}\cdot 1=e^{r+1}.\end{aligned}$$

4. If $g\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a G-function, and s and b are as in Definition 3, then

$$\lim_{x\to+\infty}\frac{g(x)}{x^{s+1}}=\frac{b}{s+1},\quad\lim_{x\to+\infty}\frac{g(x+1)}{g(x)}=1\quad\text{and}\quad\lim_{x\to+\infty}\left(\frac{g(x+1)}{g(x)}\right)^x=e^{s+1}.$$

Indeed,

$$\begin{aligned}\lim_{x\to+\infty}\frac{g(x)}{x^{s+1}}&=\lim_{\mathbb{N}\ni n\to\infty}\frac{g(n)}{n^{s+1}}=\lim_{n\to\infty}\frac{g(n+1)-g(n)}{(n+1)^{s+1}-n^{s+1}}\\&=\lim_{n\to\infty}\frac{g(n+1)-g(n)}{n^s}\cdot\frac{n^s}{(n+1)^{s+1}-n^{s+1}}=\frac{b}{s+1},\end{aligned}$$

$$\lim_{x\to+\infty}\frac{g(x+1)}{g(x)}=\lim_{x\to+\infty}\frac{g(x+1)}{(x+1)^{s+1}}\cdot\frac{x^{s+1}}{g(x)}\cdot\left(\frac{x+1}{x}\right)^{s+1}=\frac{b}{s+1}\cdot\frac{s+1}{b}\cdot 1=1,$$

and then

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{g(x+1)}{g(x)}\right)^x&=\lim_{x\to+\infty}\left(\left(1+\frac{g(x+1)-g(x)}{g(x)}\right)^{\frac{g(x)}{g(x+1)-g(x)}}\right)^{\frac{g(x+1)-g(x)}{x^s}\cdot\frac{x^{s+1}}{g(x)}}\\&=e^{b\cdot\frac{s+1}{b}}=e^{s+1}.\end{aligned}$$

5. If $(b_n)_{n\geqslant 1}$ is a G-sequence and s and b are as in Definition 4, then

$$\lim_{n\to\infty}\frac{b_n}{n^{s+1}}=\frac{b}{s+1},\quad\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=1\quad\text{and}\quad\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^n=e^{s+1}.$$

Indeed,

$$\lim_{n\to\infty}\frac{b_n}{n^{s+1}}=\lim_{n\to\infty}\frac{b_{n+1}-b_n}{(n+1)^{s+1}-n^s}=\lim_{n\to\infty}\frac{b_{n+1}-b_n}{n^s}\cdot\frac{n^s}{(n+1)^{s+1}-n^{s+1}}=\frac{b}{s+1},$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{s+1}}\cdot\frac{n^{s+1}}{b_n}\cdot\left(\frac{n+1}{n}\right)^{s+1}=\frac{b}{s+1}\cdot\frac{s+1}{b}\cdot 1=1,$$

and then

$$\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^n=\lim_{n\to\infty}\left(\left(1+\frac{b_{n+1}-b_n}{b_n}\right)^{\frac{b_n}{b_{n+1}-b_n}}\right)^{\frac{b_{n+1}-b_n}{n^s}\cdot\frac{n^{s+1}}{b_n}}=e^{b\cdot\frac{s+1}{b}}=e^{s+1}.$$

Theorem 1. *Let $m,p\in\mathbb{R}_+$ and $f,g\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$ be a B-function and a G-function, respectively, where a,r and b,s are as in Definitions 1 and 3, respectively. Then*

$$\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^p}{(x+1)^{mr+ps+m+p-1}}-\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}{x^{mr+ps+m+p-1}}\right)=\frac{a^mb^p}{(s+1)^pe^{m(r+1)}}.$$

Proof. For $x\in\mathbb{R}_+^*$, we denote

(1)
$$\begin{aligned}
B(x)&=\frac{\big(f(x+1)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^p}{(x+1)^{mr+ps+m+p-1}}-\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}{x^{mr+ps+m+p-1}}\\
&=\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}{x^{mr+ps+m+p-1}}(u(x)-1)=\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}{x^{mr+ps+m+p-1}}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\\
&=\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}{x^{mr+ps+m+p}}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln\big(u(x)\big)^x\\
&=\left(\frac{\big(f(x)\big)^{1/x}}{x^{r+1}}\right)^m\cdot\left(\frac{g(x)}{x^{s+1}}\right)^p\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln\big(u(x)\big)^x,
\end{aligned}$$

where $u(x)=\left(\dfrac{\big(f(x+1)\big)^{\frac{1}{x+1}}}{\big(f(x)\big)^{\frac{1}{x}}}\right)^m\left(\dfrac{g(x+1)}{g(x)}\right)^p\left(\dfrac{x}{x+1}\right)^{mr+ps+m+p-1}$. Then

$\lim\limits_{x\to+\infty}u(x)=1\cdot1\cdot1=1$ implies that $\lim\limits_{x\to+\infty}\dfrac{u(x)-1}{\ln u(x)}=1$ and

$$\begin{aligned}
&\lim_{x\to+\infty}\big(u(x)\big)^x\\
&=\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{\frac{1}{x+1}}}{\big(f(x)\big)^{\frac{1}{x}}}\right)^{mx}\cdot\lim_{x\to+\infty}\left(\frac{g(x+1)}{g(x)}\right)^{px}\lim_{x\to+\infty}\left(\frac{x}{x+1}\right)^{(mr+ps+m+p-1)x}\\
&=e^{m(r+1)}\cdot e^{p(s+1)}\cdot e^{-(mr+ps+m+p-1)}=e.
\end{aligned}$$

Hence, we obtain from (1) that

$$\lim_{x\to+\infty} B(x) = \frac{a^m b^p}{e^{m(r+1)}\cdot(s+1)^p}\cdot 1\cdot \ln e = \frac{a^m b^p}{e^{m(r+1)}\cdot(s+1)^p}. \qquad \blacksquare$$

Theorem 2. *Let $m, p \in \mathbb{R}_+$ and $f, g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a B-function and a G-function, respectively, where a, r and b, s are as in Definitions 1 and 3, respectively. Then*

$$\lim_{x\to+\infty}\left(\frac{(x+1)^{mr+ps+m+p+1}}{\big(f(x+1)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^p} - \frac{x^{mr+ps+m+p+1}}{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\right) = \frac{(s+1)^p e^{m(r+1)}}{a^m b^p}.$$

Proof. For $x \in \mathbb{R}_+^*$, we denote

(2)

$$\begin{aligned}
G(x) &= \frac{(x+1)^{mr+ps+m+p+1}}{\big(f(x+1)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^p} - \frac{x^{mr+ps+m+p+1}}{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\\
&= \frac{x^{mr+ps+m+p+1}}{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\big(v(x)-1\big) = \frac{x^{mr+ps+m+p+1}}{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\,\frac{v(x)-1}{\ln v(x)}\cdot \ln v(x)\\
&= \frac{x^{mr+ps+m+p}}{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\,\frac{v(x)-1}{\ln v(x)}\cdot \ln\big(v(x)\big)^x\\
&= \left(\frac{x^{r+1}}{\big(f(x)\big)^{1/x}}\right)^m\left(\frac{x^{s+1}}{g(x)}\right)^p\cdot\frac{v(x)-1}{\ln v(x)}\ \ln\big(v(x)\big)^x,
\end{aligned}$$

where $v(x) = \left(\frac{x+1}{x}\right)^{mr+ps+m+p+1}\left(\frac{\big(f(x)\big)^{\frac{1}{x}}}{\big(f(x+1)\big)^{\frac{1}{x+1}}}\right)^m\left(\frac{g(x)}{g(x+1)}\right)^p$. Then

$\lim_{x\to+\infty} v(x) = 1\cdot 1\cdot 1 = 1$ implies that $\lim_{x\to+\infty}\frac{v(x)-1}{\ln v(x)} = 1$ and hence

$$\begin{aligned}
&\lim_{x\to+\infty}\big(v(x)\big)^x\\
&= \lim_{x\to+\infty}\left(\frac{x+1}{x}\right)^{(mr+ps+m+p+1)x}\left(\frac{f(x)}{f(x+1)}\right)^m\big(f(x+1)\big)^{\frac{m}{x+1}}\left(\left(\frac{g(x)}{g(x+1)}\right)^x\right)^p\\
&= e^{mr+ps+m+p+1}\times\\
&\quad \lim_{x\to+\infty}\left(\frac{f(x)\cdot x^{r+1}}{f(x+1)}\right)^m\left(\frac{\big(f(x+1)\big)^{\frac{1}{x+1}}}{(x+1)^{r+1}}\right)^m\left(\frac{x+1}{x}\right)^{m(r+1)}\left(\left(\frac{g(x)}{g(x+1)}\right)^x\right)^p\\
&= e^{mr+ps+m+p+1}\cdot\frac{1}{a^m}\cdot\frac{a^m}{e^{m(r+1)}}\cdot 1\cdot\frac{1}{e^{p(s+1)}} = e.
\end{aligned}$$

Then, from (2), we deduce that

$$\lim_{x\to+\infty} G(x) = \frac{e^{m(r+1)}}{a^m}\cdot\frac{(s+1)^p}{b^p} = \frac{(s+1)^p e^{m(r+1)}}{a^m b^p}. \qquad \blacksquare$$

In a similar way, the following two results can be proved.

THEOREM 3. *Let $m, p \in \mathbb{R}_+$ and $f, g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a B-function and a G-function, respectively, where a, r and b, s are as in Definitions 1 and 3, respectively. Then*

$$\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{\frac{m}{x+1}}}{\big(g(x+1)\big)^{p}}(x+1)^{ps+p-mr-m+1}-\frac{\big(f(x)\big)^{\frac{m}{x}}}{\big(g(x)\big)^{p}}\cdot x^{ps+p-mr-m+1}\right) = \frac{a^m(s+1)^p}{b^p\, e^{m(r+1)}}.$$

THEOREM 4. *Let $m, p \in \mathbb{R}_+$ and $f, g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a B-function and a G-function, respectively, where a, r and b, s are as in Definitions 1 and 3, respectively. Then*

$$\lim_{x\to+\infty}\left(\frac{\big(g(x+1)\big)^{p}}{\big(f(x+1)\big)^{\frac{m}{x+1}}}(x+1)^{mr+m-ps-p+1}-\frac{\big(g(x)\big)^{p}}{\big(f(x)\big)^{\frac{m}{x}}}\cdot x^{mr+m-ps-s+1}\right) = \frac{b^p\, e^{m(r+1)}}{a^m(s+1)^p}.$$

2. Applications

The following applications are particular cases of limits calculated above; they can also be used to compute other limits of functions and sequences of this type.

A1. If $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = \Gamma(x+1)$, then by Example 1, we deduce that f is a B-function with $a = 1$, $r = 0$. If $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a G-function, then by Theorem 1,

(3)

$$\lim_{x\to+\infty}\left(\frac{\big(f(x+1)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^{p}}{(x+1)^{mr+ps+m+p-1}}-\frac{\big(f(x)\big)^{\frac{m}{x}}\big(g(x)\big)^{p}}{x^{mr+ps+m+p-1}}\right)$$
$$= \lim_{x\to+\infty}\left(\frac{\big(\Gamma(x+2)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^{p}}{(x+1)^{ps+m+p-1}}-\frac{\big(\Gamma(x+1)\big)^{\frac{m}{x}}\big(g(x)\big)^{p}}{x^{ps+m+p-1}}\right) = \frac{b^p}{(s+1)^p e^m}.$$

If we take $p = 0$, then by (3) we deduce

$$\lim_{x\to+\infty}\left(\frac{\big(\Gamma(x+2)\big)^{\frac{m}{x+1}}}{(x+1)^{m-1}}-\frac{\big(\Gamma(x+1)\big)^{\frac{m}{x}}}{x^{m-1}}\right) = \frac{1}{e^m}. \tag{4}$$

If we take $m = 1$, then by (4) we obtain

$$\lim_{x\to+\infty}\left(\big(\Gamma(x+2)\big)^{\frac{1}{x+1}}-\big(\Gamma(x+1)\big)^{\frac{1}{x}}\right) = \frac{1}{e}.$$

If we take $x = n \in \mathbb{N} \setminus \{1\}$, then $\Gamma(x+1) = \Gamma(n+1) = n!$, hence

$$(5) \qquad \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}\cdot\big(g(x+1)\big)^p}{(n+1)^{ps+m+p-1}} - \frac{\sqrt[n]{n!}\cdot\big(g(n)\big)^p}{n^{ps+m+p-1}}\right) = \frac{b^p}{(s+1)^p e^m}.$$

If we take $p = 0$ and $m = 1$, then (5) yields that

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e},$$

i.e., we obtain the limit of T. Lalescu's sequence (Problem 579 from Romanian Mathematical Gazette, Vol. VI (1900–1901)).

A2. $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = x^{x+1}$, then by Example 2, we deduce that f is a B-function with $a = e$, $r = 0$. If $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a G-function, then by Theorem 1, (6)

$$\lim_{x\to+\infty}\left(\frac{(x+1)^{\frac{m(x+2)}{x+1}}\big(g(x+1)\big)^p}{(x+1)^{ps+m+p-1}} - \frac{x^{\frac{m(x+1)}{x}}\big(g(x)\big)^p}{x^{ps+m+p-1}}\right) = \frac{e^m b^p}{(s+1)^p e^m} = \frac{b^p}{(s+1)^p}.$$

If we take $p = 0$, then by (6) we deduce

$$(7) \qquad \lim_{x\to+\infty}\left(\frac{(x+1)^{\frac{m(x+2)}{x+1}}}{(x+1)^{m-1}} - \frac{x^{\frac{m(x+1)}{x}}}{x^{m-1}}\right) = \lim_{x\to+\infty}\left((x+1)^{1+\frac{m}{x+1}} - x^{1+\frac{m}{x}}\right) = 1.$$

If we take $m = 1$, then by (7) we infer that

$$(8) \qquad \lim_{x\to+\infty}\left((x+1)^{1+\frac{1}{x+1}} - x^{1+\frac{1}{x}}\right) = \lim_{x\to+\infty}\left((x+1)(x+1)^{\frac{1}{x+1}} - x\cdot x^{\frac{1}{x}}\right) = 1.$$

If we take $x = n \in \mathbb{N} \setminus \{1\}$, then by (8) we obtain

$$\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\right) = 1,$$

i.e., the limit of R.T. Ianculescu's sequence (Problem 2042 from Romanian Mathematical Gazette, Vol. XIX (1913–1914)).

A3. $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = \Gamma(x+1)$, then by Example 1, we deduce that f is a B-function with $a = 1$, $r = 0$. If $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a G-function, then by Theorem 2,

$$(9) \qquad \lim_{x\to+\infty}\left(\frac{(x+1)^{ps+m+p+1}}{\big(\Gamma(x+2)\big)^{\frac{m}{x+1}}\big(g(x+1)\big)^p} - \frac{x^{ps+m+p+1}}{\big(\Gamma(x+1)\big)^{\frac{m}{x}}\big(g(x)\big)^p}\right) = \frac{(s+1)^p e^m}{b^p}.$$

If we take $p = 0$, then by (9) we deduce that

$$(10) \qquad \lim_{x\to+\infty}\left(\frac{(x+1)^{m+1}}{\big(\Gamma(x+2)\big)^{\frac{m}{x+1}}} - \frac{x^{m+1}}{\big(\Gamma(x+1)\big)^{\frac{m}{x}}}\right) = e^m.$$

If we take $m = 1$ in (10), we get that

$$\lim_{x\to+\infty}\left(\frac{(x+1)^2}{\big(\Gamma(x+2)\big)^{\frac{1}{x+1}}} - \frac{x^2}{\big(\Gamma(x+1)\big)^{\frac{1}{x}}}\right) = e. \tag{11}$$

If we take $x = n \in \mathbb{N} \setminus \{1\}$, then $\Gamma(x+1) = \Gamma(n+1) = n!$, hence, by (11) we obtain

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = e,$$

i.e., the limit of Bătinetu-Giurgiu's sequence (Problem C:890 from Romanian Mathematical Gazette, Vol. XCIV (1989)).

Remark. For certain values of the numbers $m, p, r, s \in \mathbb{R}_+$ and for particular functions f, g, using Theorems 1–4, the reader can get the limits from many problems published in various mathematical journals – see for example the list of problems from reference [1].

REFERENCES

[1] D.M. Bătineţu-Giurgiu, N. Stanciu, A. Kotronis, *Calculating the limits of some real sequences*, Math. problems, Math. Journal **4**, 1 (2014), 252–257.

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