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HYBRIDIZATION OF CLASSICAL INEQUALITIES WITH EQUIVALENT DYNAMIC INEQUALITIES ON TIME SCALE CALCULUS

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Abstract. The aim of this paper is to present some classical inequalities and to form their symmetric dynamic versions on time scale calculus. We also present that their symmetric dynamic versions on time scales are equivalent.

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1. Introduction

We introduce here some well known classical inequalities.

The weighted power mean inequality given in [6, Theorem 10.5], [9, pp. 12–15] and [14] is defined as follows.

If $x_1, x_2, \ldots, x_n$ are nonnegative real numbers and $p_1, p_2, \ldots, p_n$ are positive real numbers, then for $\eta_2 \geq \eta_1 > 0$, we have

$$(1.1) \qquad \left(\frac{p_1 x_1^{\eta_1} + p_2 x_2^{\eta_1} + \cdots + p_n x_n^{\eta_1}}{p_1 + p_2 + \cdots + p_n}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{p_1 x_1^{\eta_2} + p_2 x_2^{\eta_2} + \cdots + p_n x_n^{\eta_2}}{p_1 + p_2 + \cdots + p_n}\right)^{\frac{1}{\eta_2}}.$$

The inequality (1.2) is called Schlömilch's Inequality in literature as given in [9, p. 26].

If $x_1, x_2, \ldots, x_n$ are nonnegative real numbers and $0 < \eta_1 < \eta_2$, then

$$(1.2) \qquad \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\eta_1}\right)^{\frac{1}{\eta_1}} < \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\eta_2}\right)^{\frac{1}{\eta_2}},$$

unless the x_k for $k \in \mathbb{N}$ are all equal.

Rogers-Hölder's Inequality, which is commonly known as Hölder's Inequality, is an important and well known inequality. This inequality has many applications. History of Rogers-Hölder's Inequality is given in [13]. Rogers-Hölder's Inequality was first found by Rogers in 1888 and then by Hölder in 1889 as given in [11].

Jacob Bernoulli in 1689 proved the classical Bernoulli's Inequality as given in [9], [13].

Radon's Inequality is widely studied by many authors as it has many applications as given in [16]. Radon's inequality in generalized form is given in [8].

The upcoming theorem shows that Rogers-Hölder's Inequality, Bernoulli's Inequality, Radon's Inequality and generalized Radon's Inequality are equivalent as given in [8].

THEOREM 1.1 *The following inequalities are equivalent:*

(1) *Rogers-Hölder's Inequality*

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *where* x_k *and* y_k *for all* $k = 1, 2, \dots, n$ *are positive reals, then the discrete version of Rogers-Hölder's Inequality is*

$$\sum_{k=1}^{n} x_k y_k \le \left(\sum_{k=1}^{n} x_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q \right)^{\frac{1}{q}}. \tag{1.3}$$

(2) *Bernoulli's Inequality*

If $\varphi(x) \ge 0$, $x \in [a, b]$, *where* $p > 1$, *then*

$$\varphi^p(x) \ge 1 + p(\varphi(x) - 1). \tag{1.4}$$

If $\varphi(x) \ge 0$, $x \in [a, b]$, *where* $0 < \xi < 1$, *then the reversed version of Bernoulli's Inequality is*

$$\varphi^{\xi}(x) \le 1 + \xi(\varphi(x) - 1). \tag{1.5}$$

(3) *Radon's Inequality*

If $n \in \mathbb{N}$, $x_k \ge 0$ *and* $y_k > 0$, $k \in \{1, 2, \dots, n\}$ *and* $\gamma \ge 0$, *then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k \right)^{\gamma+1}}{\left(\sum\limits_{k=1}^{n} y_k \right)^{\gamma}} \le \sum_{k=1}^{n} \frac{x_k^{\gamma+1}}{y_k^{\gamma}}. \tag{1.6}$$

(4) *Generalized Radon's Inequality*

If $n \in \mathbb{N}$, $x_k \ge 0$ *and* $y_k > 0$, $k \in \{1, 2, \dots, n\}$, $\gamma \ge 0$ *and* $\zeta \ge 1$, *then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k y_k^{\zeta-1} \right)^{\gamma+\zeta}}{\left(\sum\limits_{k=1}^{n} y_k^{\zeta} \right)^{\gamma+\zeta-1}} \le \sum_{k=1}^{n} \frac{x_k^{\gamma+\zeta}}{y_k^{\gamma}}, \tag{1.7}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$.

We will prove these results on time scale calculus. The theory of time scale calculus is applied to reveal the symmetry of continuous and discrete and to combine them in one comprehensive form. In time scale calculus, results are unified and extended. This hybrid theory is also widely applied to dynamic inequalities. Research work on dynamic inequalities was done by R. Agarwal, G. Anastassiou, M. Bohner, A. Peterson, D. O'Regan, S. Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of the real interval with the given time scale.

2. Preliminaries

Time scale calculus was initiated by S. Hilger as given in [10]. It is studied as delta calculus, nabla calculus and diamond-α calculus. A *time scale* is an arbitrary nonempty closed subset of the set of real numbers.

We need here basic concepts of delta calculus. The results of delta calculus are adapted from [4, 5].

For $t \in \mathbb{T}$, *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows.

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be *delta differentiable at* t, and $f^\Delta(t)$ is called the *delta derivative* of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.1. [4, 5] A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [3, 4, 5].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$, a function f is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that for any given $\epsilon > 0$, there exists a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

Definition 2.2. [3, 4, 5] A function $G : \mathbb{T} \to \mathbb{R}$ is called the nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present a short introduction of diamond-α derivative as given in [1, 19].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, *diamond-$\alpha$ dynamic derivative* $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha)f^{\nabla}(t), \quad 0 \leq \alpha \leq 1.$$

Thus, f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.3. [19] *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then*

(*i*) *$f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(*ii*) *$fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1 - \alpha)f^{\rho}(t)g^{\nabla}(t).$$

(*iii*) *For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1 - \alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 2.4. [19] Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha) \int_a^t h(s)\nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.5. [19] *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_{\mathbb{T}}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \ \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;

(ii) $\int_a^t cf(s) \diamond_\alpha s = c \int_a^t f(s) \ \diamond_\alpha s$;

(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;

(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \ \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;

(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

To proceed further, we need the following result.

Theorem 2.6. [1] *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \diamond_\alpha s > 0$. If $F \in C((c, d), \mathbb{R})$ is convex, then the generalized Jensen's Inequality is*

$$F\left(\frac{\int_a^b |h(s)| g(s) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}\right) \le \frac{\int_a^b |h(s)| F(g(s)) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}. \tag{2.1}$$

If F is strictly convex, then the inequality $\le$ can be replaced by $<$.

Definition 2.7. [7] A function $f : \mathbb{T} \to \mathbb{R}$ is called *convex* on $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval of $\mathbb{R}$ (open or closed), if

$$f(\lambda t + (1-\lambda)s) \le \lambda f(t) + (1-\lambda) f(s), \tag{2.2}$$

for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0, 1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if the inequality (2.2) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\lambda \in (0, 1)$.

The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

3. Main results

In order to present our main results, we first present an extension of generalized Radon's inequality by applying Bernoulli's Inequality via time scales.

Theorem 3.1. *Let $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x), g(x) \neq 0$.*

(i) If $\gamma \geq 0$ *and* $\zeta \geq 1$*, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x\right)^{\gamma+\zeta}}{\left(\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x\right)^{\gamma+\zeta-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \diamond_\alpha x. \tag{3.1}$$

(ii) If $0 < \gamma + \zeta < 1$*, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x\right)^{\gamma+\zeta}}{\left(\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x\right)^{\gamma+\zeta-1}} \geq \int_a^b \frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \diamond_\alpha x. \tag{3.2}$$

Equality occurs in (3.1) and (3.2) if and only if $f(x) = cg(x)$*, where* c *is a real constant.*

Proof. We prove this result by applying Bernoulli's Inequality. The inequality (3.1) can be rearranged as

$$\int_a^b \frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \diamond_\alpha x \frac{\left(\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x\right)^{\gamma+\zeta-1}}{\left(\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x\right)^{\gamma+\zeta}} \geq 1.$$

Let us consider

$$\begin{aligned}
&\frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \frac{\left(\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x\right)^{\gamma+\zeta-1}}{\left(\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x\right)^{\gamma+\zeta}} \\
&= \frac{|w(x)||g(x)|^{\zeta}}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x} \left(\frac{|f(x)| \int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x}{|g(x)| \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}\right)^{\gamma+\zeta} \\
&= \frac{|w(x)||g(x)|^{\zeta}}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x} \times \\
&\left(1 + \frac{|f(x)| \int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x - |g(x)| \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}{|g(x)| \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}\right)^{\gamma+\zeta} \\
&\geq \frac{|w(x)||g(x)|^{\zeta}}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x} \times \\
&\left(1 + (\gamma+\zeta) \frac{|f(x)| \int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x - |g(x)| \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}{|g(x)| \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}\right) \\
&= \frac{|w(x)||g(x)|^{\zeta}}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x} \\
&\quad + (\gamma+\zeta) \left\{\frac{|w(x)||f(x)||g(x)|^{\zeta-1}}{\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x} - \frac{|w(x)||g(x)|^{\zeta}}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x}\right\}.
\end{aligned}$$

Then, integrating from a to b, we get

$$\int_a^b \frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \diamond_\alpha x \frac{\left(\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x\right)^{\gamma+\zeta-1}}{\left(\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x\right)^{\gamma+\zeta}} \geq 1 + (\gamma+\zeta)(1-1).$$

We get the required result given in (3.1). The inequality given in (3.1) is reversed if we set $0 < \gamma + \zeta < 1$.

It is clear that equality holds in (3.1) and (3.2), if and only if $f(x) = cg(x)$, where c is a real constant number. ∎

In the following corollary, we give dynamic Radon's Inequality on time scales, which is actually the reduced form of generalized Radon's Inequality.

Corollary 3.2. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x)$, $g(x) \neq 0$. If $\gamma \geq 0$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\gamma+1}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\gamma}} \leq \int_a^b \frac{|w(x)||f(x)|^{\gamma+1}}{|g(x)|^{\gamma}} \diamond_\alpha x. \tag{3.3}$$

Proof. For $\zeta = 1$ (3.1) reduces to (3.3). ∎

Next we prove that generalized Radon's Inequality given in (3.1) and Radon's Inequality given in (3.3) are equivalent.

Theorem 3.3. *The following inequalities are equivalent on dynamic time scale calculus:*

(1) *Generalized Radon's Inequality,*

(2) *Radon's Inequality.*

Proof. It is clear from Corollary 3.2 that generalized Radon's Inequality implies Radon's Inequality. To prove the converse, replace γ by $\gamma + \zeta - 1$, where $\zeta \geq 1$ and also replace $|w(x)|$ by $|w(x)||g(x)|^{\zeta-1}$ for $g \in (C[a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ in (3.3) and we get (3.1). ∎

The upcoming corollary gives us the weighted power mean inequality on time scales.

Corollary 3.4. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x) \neq 0$. If $\eta_2 \geq \eta_1 > 0$, then*

$$\left(\frac{\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\frac{1}{\eta_2}}. \tag{3.4}$$

Proof. Set $\gamma \geq 0$, $\zeta = 1$, $1+\gamma = \frac{\eta_2}{\eta_1} \geq 1$ and $g(x) = 1$. Then (3.1) becomes

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} \leq \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x. \tag{3.5}$$

Dividing (3.5) by $\int_a^b |w(x)| \diamond_\alpha x$ and replacing $|f(x)|$ by $|f(x)|^{\eta_1}$, then taking power $\frac{1}{\eta_2} > 0$, we get our claim. ■

The following theorem shows that the generalized Radon's Inequality given in (3.1) and the weighted power mean inequality given in (3.4) are equivalent.

Theorem 3.5. *The following inequalities are equivalent on dynamic time scale calculus:*

(1) *Generalized Radon's Inequality,*

(2) *The weighted power mean inequality.*

Proof. It is clear from Corollary 3.4 that generalized Radon's Inequality implies the weighted power mean inequality. In order to prove the converse, replace $|f(x)|$ by $|f(x)|^{\frac{1}{\eta_1}}$, and take power $\eta_2 > 0$ and $\frac{\eta_2}{\eta_1} = \gamma + \zeta \geq 1$, where $\gamma \geq 0$ and $\zeta \geq 1$. Then (3.4) takes the form

$$\left(\frac{\int_a^b |w(x)||f(x)| \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\gamma+\zeta} \leq \frac{\int_a^b |w(x)||f(x)|^{\gamma+\zeta} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}. \tag{3.6}$$

Now we replace $|w(x)|$ by $|w(x)||g(x)|^{\zeta}$ and also replace $|f(x)|$ by $\frac{|f(x)|}{|g(x)|}$ for $g \in (C[a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ in (3.6). Then we get

$$\left(\frac{\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x}\right)^{\gamma+\zeta} \leq \frac{\int_a^b \frac{|w(x)||f(x)|^{\gamma+\zeta}}{|g(x)|^{\gamma}} \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x}. \tag{3.7}$$

Multiplying (3.7) by $\int_a^b |w(x)||g(x)|^{\zeta} \diamond_\alpha x$, we get the required claim. ■

Remark 3.6. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, and $|w(x)|$ is replaced by p_k for $k \in \{1, 2, \ldots, n\}$, where $\{p_k\}$ is a set of positive real numbers and $|f(x)|$ is replaced by x_k for $k \in \{1, 2, \ldots, n\}$, where $\{x_k\}$ is a set of nonnegative real numbers, then (3.4) reduces to (1.1).

Further, if we set $w(x) = 1$, then (3.4) reduces to (1.2).

Now we give Schlömilch's Inequality on time scales.

Corollary 3.7. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x) \neq 0$. If $\eta_2 \geq \eta_1 > 0$, then*

$$\left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \leq \left(\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}}. \tag{3.8}$$

Proof. If we set $\int_a^b |w(x)| \diamond_\alpha x = 1$ in (3.4), then we get (3.8). ■

Next we show that weighted power mean inequality given in (3.4) is equivalent to Schlömilch's Inequality given in (3.8).

Theorem 3.8. *The following inequalities are equivalent on dynamic time scale calculus:*

(1) *The weighted power mean inequality,*

(2) *Schlömilch's Inequality.*

Proof. It is clear from Corollary 3.7 that the weighted power mean inequality implies Schlömilch's Inequality. Replacing $|w(x)|$ by $\frac{|w(x)|}{\int_a^b |w(x)| \diamond_\alpha x}$, we obtain (3.4) from (3.8). ■

Remark 3.9. Let $w, f \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$. Then for $\alpha = 1$, the inequality given in (3.8) takes the form

$$\left(\int_a^b w(x) f^{\eta_1}(x) \Delta x\right)^{\frac{1}{\eta_1}} \leq \left(\int_a^b w(x) f^{\eta_2}(x) \Delta x\right)^{\frac{1}{\eta_2}}, \tag{3.9}$$

as given in [12, Lemma A].

Let $w, f \in C([a,b]_{\mathbb{T}}, [0,\infty))$. Then (3.8) takes the form

$$\left(\int_a^b w(x) f^{\eta_1}(x) \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \leq \left(\int_a^b w(x) f^{\eta_2}(x) \diamond_\alpha x\right)^{\frac{1}{\eta_2}}, \tag{3.10}$$

as given in [20, Lemma 3.4].

Now we give another form of Schlömilch's Inequality on time scales.

Corollary 3.10. *Let $f \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be a $\diamond_\alpha$-integrable function. If $\eta_2 \geq \eta_1 > 0$, then*

$$\left(\frac{\int_a^b |f(x)|^{\eta_1} \diamond_\alpha x}{b-a}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{\int_a^b |f(x)|^{\eta_2} \diamond_\alpha x}{b-a}\right)^{\frac{1}{\eta_2}}. \tag{3.11}$$

Proof. Set $w(x) = 1$ in (3.4), and we get (3.11). ■

Example 3.11. Set $\eta_1 = 1$. Then (3.11) takes the form

$$\left(\frac{1}{b-a}\right)^{\eta_2 - 1} \left(\int_a^b |f(x)| \diamond_\alpha x\right)^{\eta_2} \leq \int_a^b |f(x)|^{\eta_2} \diamond_\alpha x. \tag{3.12}$$

In the upcoming result, using generalized Jensen's Inequality, we find Rogers-Hölder's Inequality on time scales. Rogers-Hölder's Inequality is also given in [1, 15].

COROLLARY 3.12. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then*

$$\begin{aligned}(3.13)\quad &\int_a^b |w(x)|\,|f(x)g(x)| \diamond_\alpha x \\ &\le \left(\int_a^b |w(x)|\,|f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)|\,|g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The inequality given in (3.1) can be rearranged as

$$\begin{aligned}(3.14)\quad &\left(\int_a^b \frac{|w(x)||g(x)|^\zeta}{\int_a^b |w(x)||g(x)|^\zeta \diamond_\alpha x} \left|\frac{f(x)}{g(x)}\right| \diamond_\alpha x\right)^{\gamma+\zeta} \\ &\le \frac{\int_a^b |w(x)||g(x)|^\zeta \left|\frac{f(x)}{g(x)}\right|^{\gamma+\zeta} \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^\zeta \diamond_\alpha x}.\end{aligned}$$

Let $\gamma > 0$, $\zeta = 1$ and set $\gamma + 1 = p > 1$. The function $F : [0, \infty) \to \mathbb{R}$ defined by $F(x) = x^{\gamma+1}$ is convex for $x \in [0, \infty)$, and (3.14) is similar to generalized Jensen's Inequality given in (2.1). Then (3.14) takes the form

$$\begin{aligned}(3.15)\quad &\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^p \\ &\le \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{p-1} \left(\int_a^b |w(x)||f(x)|^p |g(x)|^{1-p} \diamond_\alpha x\right).\end{aligned}$$

Further, we replace $|w(x)|$ by $|w(x)||g(x)|^{\frac{q}{p}}$ and $|f(x)|$ by $|f(x)||g(x)|^{1-\frac{q}{p}}$. Taking the power $\frac{1}{p} > 0$, (3.15) reduces to (3.13). ■

Using the previously obtained results, we conclude that several dynamic inequalities are equivalent.

THEOREM 3.13. *The following inequalities are equivalent on time scale calculus:*

(1) *Generalized Radon's Inequality,*
(2) *Radon's Inequality,*
(3) *The weighted power mean inequality,*
(4) *Schlömilch's Inequality,*
(5) *Rogers-Hölder's Inequality,*
(6) *Bernoulli's Inequality.*

Proof. It follows from Theorems 3.1, 3.3, 3.5, 3.8 and Corollary 3.12 that (6) $\Longrightarrow$ (1), the inequalities (1), (2), (3) and (4) are equivalent, and that (1) $\Longrightarrow$ (5). It remains to prove that (5) $\Longrightarrow$ (6).

Without loss of generality, we may suppose that

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \neq 0.$$

Then Rogers-Hölder's Inequality given in (3.13) takes the form

$$\begin{aligned}\int_a^b |w(x)| &\frac{|f(x)|}{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}} \frac{|g(x)|}{\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}} \diamond_\alpha x \\ &\leq 1 = \frac{1}{p} + \frac{1}{q} \\ &= \frac{1}{p} \frac{\int_a^b |w(x)||f(x)|^p \diamond_\alpha x}{\int_a^b |w(x)||f(x)|^p \diamond_\alpha x} + \frac{1}{q} \frac{\int_a^b |w(x)||g(x)|^q \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^q \diamond_\alpha x}.\end{aligned}$$

So,

$$\begin{aligned}\frac{|f(x)|}{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}} &\frac{|g(x)|}{\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}} \\ &\leq \frac{1}{p} \frac{|f(x)|^p}{\int_a^b |w(x)||f(x)|^p \diamond_\alpha x} + \frac{1}{q} \frac{|g(x)|^q}{\int_a^b |w(x)||g(x)|^q \diamond_\alpha x}.\end{aligned} \tag{3.16}$$

Let $u(x) = \frac{|f(x)|}{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}}$ and $v(x) = \frac{|g(x)|}{\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}}$. Then (3.16) can be written as

$$u(x)v(x) \leq \frac{1}{p} u^p(x) + \frac{1}{q} v^q(x), \tag{3.17}$$

where $u(x)$ and $v(x)$ are nonnegative real functions.

Taking $\frac{1}{q} = 1 - \frac{1}{p}$, where $\frac{1}{p} = \xi < 1$, (3.17) can be written as

$$u^\xi(x) v^{1-\xi}(x) \leq \xi u(x) + (1-\xi) v(x). \tag{3.18}$$

Dividing (3.18) by $v(x)$ and taking the substitution $\varphi(x) = \frac{u(x)}{v(x)}$, we get

$$\varphi^\xi(x) \leq \xi \varphi(x) + (1-\xi),$$

which is the required Bernoulli's Inequality. ■

Remark 3.14. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ and $g(x) = y_k$ for $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, where x_k and y_k are positive reals, then (3.13) reduces to (1.3).

If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ and $g(x) = y_k$ for $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, where x_k and y_k are nonnegative reals and $y_k \neq 0$, then the discrete version of (3.1) reduces to (1.7) for $\gamma \geq 0$ and $\zeta \geq 1$. If $\zeta = 1$, then the discrete version of (1.7) reduces to (1.6) for $\gamma \geq 0$.

It is clear from Theorem 3.13 that Rogers-Hölder's Inequality, Bernoulli's Inequality, Generalized Radon's Inequality and Radon's Inequality are also equivalent. So Theorem 1.1 is just a part of Theorem 3.13.

In the following example, we present some dynamic inequalities such as Radon's Inequality, the weighted power mean inequality and Schlömilch's Inequality on quantum calculus.

EXAMPLE 3.15. If we set $[a,b]_{\mathbb{T}} = [q^m, q^n]_{q^{\mathbb{N}_0}}$ for $q > 1$ and $m < n$, where $m, n \in \mathbb{N}_\circ$ and $\mathbb{N}_0$ is the set of nonnegative integers, then

$$\int_{q^m}^{q^n} f(x)\Diamond_\alpha x = (q-1)\sum_{i=m}^{n-1} q^i[\alpha f(q^i) + (1-\alpha)f(q^{i+1})].$$

If $\gamma \geq 0$, then (3.3) takes the form

$$\frac{\left[\sum\limits_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)||f(q^i)| + (1-\alpha)|w(q^{i+1})||f(q^{i+1})|\right\}\right]^{\gamma+1}}{\left[\sum\limits_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)||g(q^i)| + (1-\alpha)|w(q^{i+1})||g(q^{i+1})|\right\}\right]^{\gamma}}$$
$$\leq \sum_{i=m}^{n-1} q^i\left\{\alpha\frac{|w(q^i)||f(q^i)|^{\gamma+1}}{|g(q^i)|^\gamma} + (1-\alpha)\frac{|w(q^{i+1})||f(q^{i+1})|^{\gamma+1}}{|g(q^{i+1})|^\gamma}\right\}.$$

If $\eta_2 \geq \eta_1 > 0$, then (3.4) takes the form

$$\left[\frac{\sum\limits_{i=m}^{n-1} q^i\{\alpha|w(q^i)||f(q^i)|^{\eta_1} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})|^{\eta_1}\}}{\sum\limits_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)| + (1-\alpha)|w(q^{i+1})|\right\}}\right]^{\frac{1}{\eta_1}}$$
$$\leq \left[\frac{\sum\limits_{i=m}^{n-1} q^i\{\alpha|w(q^i)||f(q^i)|^{\eta_2} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})|^{\eta_2}\}}{\sum\limits_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)| + (1-\alpha)|w(q^{i+1})|\right\}}\right]^{\frac{1}{\eta_2}}.$$

If $\eta_2 \geq \eta_1 > 0$, then (3.8) takes the form

$$\left[(q-1)\sum_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)||f(q^i)|^{\eta_1} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})|^{\eta_1}\right\}\right]^{\frac{1}{\eta_1}}$$
$$\leq \left[(q-1)\sum_{i=m}^{n-1} q^i\left\{\alpha|w(q^i)||f(q^i)|^{\eta_2} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})|^{\eta_2}\right\}\right]^{\frac{1}{\eta_2}}.$$

Finally, we present integral dynamic inequalities in two dimensions. Diamond-α integral for a function of two variables is defined in [1].

THEOREM 3.16. *The following dynamic inequalities in two dimensions are equivalent:*

(1) *Generalized Radon's Inequality*

Let $w(x_1,x_2), f(x_1,x_2), g(x_1,x_2) \in C([a_i,b_i]^2_{\mathbb{T}}, \mathbb{R})$ $(i=1,2)$ *be* $\diamond_\alpha$*-integrable functions, where* $w(x_1,x_2), g(x_1,x_2) \neq 0$. *If* $\gamma \geq 0$ *and* $\zeta \geq 1$, *then*

$$\frac{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)||g(x_1,x_2)|^{\zeta-1} \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\gamma+\zeta}}{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||g(x_1,x_2)|^{\zeta} \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\gamma+\zeta-1}} \leq \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{|w(x_1,x_2)||f(x_1,x_2)|^{\gamma+\zeta}}{|g(x_1,x_2)|^{\gamma}} \diamond_\alpha x_1 \diamond_\alpha x_2.$$

(2) *Radon's Inequality*

Let $w(x_1,x_2), f(x_1,x_2), g(x_1,x_2) \in C([a_i,b_i]^2_{\mathbb{T}}, \mathbb{R})$ $(i=1,2)$ *be* $\diamond_\alpha$*-integrable functions, where* $w(x_1,x_2), g(x_1,x_2) \neq 0$. *If* $\gamma \geq 0$, *then*

$$\frac{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)| \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\gamma+1}}{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||g(x_1,x_2)| \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\gamma}} \leq \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{|w(x_1,x_2)||f(x_1,x_2)|^{\gamma+1}}{|g(x_1,x_2)|^{\gamma}} \diamond_\alpha x_1 \diamond_\alpha x_2.$$

(3) *The weighted power mean inequality*

Let $w(x_1,x_2), f(x_1,x_2) \in C([a_i,b_i]^2_{\mathbb{T}}, \mathbb{R})$ $(i=1,2)$ *be* $\diamond_\alpha$*-integrable functions, where* $w(x_1,x_2) \neq 0$. *If* $\eta_2 \geq \eta_1 > 0$, *then*

$$\left(\frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)|^{\eta_1} \diamond_\alpha x_1 \diamond_\alpha x_2}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)| \diamond_\alpha x_1 \diamond_\alpha x_2}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)|^{\eta_2} \diamond_\alpha x_1 \diamond_\alpha x_2}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)| \diamond_\alpha x_1 \diamond_\alpha x_2}\right)^{\frac{1}{\eta_2}}.$$

(4) *Schlömilch's Inequality*

Let $w(x_1,x_2), f(x_1,x_2) \in C([a_i,b_i]^2_{\mathbb{T}}, \mathbb{R})$ $(i=1,2)$ *be* $\diamond_\alpha$*-integrable functions, where* $\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)| \diamond_\alpha x_1 \diamond_\alpha x_2 = 1$ *and* $w(x_1,x_2) \neq 0$. *If* $\eta_2 \geq \eta_1 > 0$. *Then*

$$\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)|^{\eta_1} \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\frac{1}{\eta_1}} \leq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |w(x_1,x_2)||f(x_1,x_2)|^{\eta_2} \diamond_\alpha x_1 \diamond_\alpha x_2\right)^{\frac{1}{\eta_2}}.$$

(5) *Rogers-Hölder's Inequality*

Let $w(x_1,x_2), f(x_1,x_2), g(x_1,x_2) \in C([a_i,b_i]^2_{\mathbb{T}},\mathbb{R})$ $(i=1,2)$ *be* $\diamond_\alpha$*-integrable functions and* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$*. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|w(x_1,x_2)||f(x_1,x_2)g(x_1,x_2)|\diamond_\alpha x_1\diamond_\alpha x_2$$
$$\leq\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}|w(x_1,x_2)||f(x_1,x_2)|^p\diamond_\alpha x_1\diamond_\alpha x_2\right)^{\frac{1}{p}}$$
$$\times\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}|w(x_1,x_2)||g(x_1,x_2)|^q\ \diamond_\alpha x_1\diamond_\alpha x_2\right)^{\frac{1}{q}}.$$

(6) *Bernoulli's Inequality*

If $\varphi(x_1,x_2)\in C([a_i,b_i]^2_{\mathbb{T}},[0,\infty))$ $(i=1,2)$*, where* $p>1$*, then*

$$\varphi^p(x_1,x_2)\geq 1+p(\varphi(x_1,x_2)-1).$$

If $\varphi(x_1,x_2)\in C([a_i,b_i]^2_{\mathbb{T}},[0,\infty))$ $(i=1,2)$*, where* $0<\xi<1$*, then the reversed version of Bernoulli's Inequality is*

$$\varphi^\xi(x_1,x_2)\leq 1+\xi(\varphi(x_1,x_2)-1).$$

Proof. Similar to the proof of Theorem 3.13. ■

4. Conclusion and future work

In this research article, we have generalized and extended some classical inequalities. Our work shows that many classical inequalities such as generalized Radon's Inequality, Radon's Inequality, the weighted power mean inequality, Schlömilch's Inequality, Rogers-Hölder's Inequality and Bernoulli's Inequality are equivalent on diamond-α calculus. If we set $\alpha=1$, then we get delta versions of dynamic inequalities and if we set $\alpha=0$, then we get nabla versions of dynamic inequalities. Also we get discrete versions of dynamic inequalities, if we put $\mathbb{T}=\mathbb{Z}$ and we get continuous versions of dynamic inequalities, if we put $\mathbb{T}=\mathbb{R}$.

In future, we can find more equivalent dynamic inequalities on diamond-α calculus. We can generalize dynamic inequalities using functional generalization such as Schlömilch's and Rogers-Hölder's Inequalities are given in [20]. Dynamic inequalities can also be generalized in a similar fashion as fractional dynamic inequalities are generalized using convex functions given in [2, 17], Riemann-Liouville fractional integral given in [2, 18] and fractional derivatives given in [2]. It will be interesting to present dynamic inequalities in three or more dimensions in more generalized form. To present these dynamic inequalities on quantum calculus will also be an interesting work.

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