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A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, II

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Abstract. In this paper we start to analyze the first topics of the school arithmetic. To help the reader recognize the fundamental position and role of the system of natural numbers, a historical glance at the evolution of the number idea is given. Then, the block of numbers 1–10 is seen as a conceptual structure which dictates a series od didactical steps and procedures. As it is easy to observe, many textbooks contain rashly gathered groups of arithmetic problems without bringing before the mind of cognizing subject the effects of these steps. Since at this stage, numbers and operations as well as all their properties are perceptual entities and experiences, the role of drawings, from those showing a piece of reality to the schematic ones which condense the carrying meaning with an elegant simplicity, is particularly emphasized. A bad practice of "proving" the arithmetic rules by means of calculation of values of a few related numerical expressions is also criticized. Though, we have given the chief points only, we expect that the sketch of this block will reflect some good practice in a pragmatic way.

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7. Steps in building the system of natural numbers

Arithmetic is the most elementary branch of mathematics what also underlines its fundamental position. Understood as a school subject, the utilitarian value of arithmetic is usually reduced to the art of computing and the cultural value is more fluid and dependent on the whole educational bundle in which it is tied up. The latter also includes so called disciplinary value meaning that this subject does much towards maturing and steadying the child. Part of the cultural value of arithmetic lies in its connection with man's life, with his social environment and with the development of his occupations.

If in the by gone times arithmetic was taught to give the pupils knowledge of facts alone, in the contemporary school, that which transcends such knowledge is exceptionally important and serves to prepare them for further education. With this in mind, all innovations or better to say renovations have to be considered. And when changes are made in something what was established in the past, first we have to understand it from historical point of view.

7.1. A historical glance at arithmetic. Here we condense some of the most important facts from history of arithmetic in the way how they are widely accepted (see, for example, [10], [12], [13])[1]. So we start to retell shortly a very long tale.

Still existing grammatical forms in some contemporary languages bear the signs of a number system which consisted of one, two and many. Inflexional endings of nouns in Slavic languages, which follow numbers are changed when five is crossed what serves as evidence of the existence of a system consisting of five numbers and being developed in some preliterate cultures. Since the naming of numbers mean their abstract conception, the beginnings of arithmetic evidently belong to the very remote past. This look back at prehistory also reveals numbers as man's primary concepts.

Texts inscribed on durable clay tablets (from about 2000 B. C.) found in Mesopotamia yield much information on the Babylonian arithmetic.

Using special symbols for 1 and 10, the Babylonians combined their groups on an additive basis to denote numbers 1 to 59. Their number system had the base 60 and they used positional notation. To indicate absence of a number they often used spacing (what could cause misinterpretation) and in a latter period a separation symbol appeared (playing the role of 0, except, at the right-hand end) . To indicate addition, the Babylonians joined two numbers together and for subtraction and multiplication they had special signs denoting these operations. To divide by a number, they multiplied by its reciprocal converted to sexagesimal fractions.

The most remarkable feature of the Babylonian arithmetic is the invention of the place value system which permits an easy performance of operations on the very signs representing numbers. It is quite natural to suppose that, thanks to the caravan routes through Babylon, the Hindus had been acquainted with this system before they invented their own decimal one. Thus, the way of indication of numbers and calculating technique created in the Babylonian era remain as a permanent achievement of mankind.

From the neolithic period emerged a quite autonomous culture in Egypt, when and where, a specific corpus of mathematical knowledge was created, mostly known to us by means of two discovered papyri: the Ahmes papyrus (now in the British Museum) and the Moscow papyrus, both dating from about 1700 B. C. The Egyptian number system had the base 10 and their arithmetic was predominantly additive in character (with multiplication being reduced to repeated addition). Whereas the Babylonians used the same symbol which denoted its value by position, the Egyptians indicated each higher unit by a new symbol. Thus, there was no possibility to indicate bigger and bigger numbers (greater than 999 999 999) what is not the case with the Babylonian numeration in which, setting a symbol for 1, 2, . . . , 59 at one of the ends of a notation, another is obtained denoting a still bigger number. Besides having an enormous advantage for computation, the Babylonian numeration also suggests implicitly the potential infinity of the set of naturals. Neither in Babylon nor in Egypt, any idea of plausible arguments that might convince one of the correctness of a procedure appeared.

[1] References numbered from 1 to 9 are included in the first part of this article (this Teaching, vol. II, 1, p. 58)

The Greek way of writing numbers, known as the Alexandrian system, uses the letters of the alphabet extended by three extra signs (altogether 27 letters). In this system, nine first letters of that alphabet represent nine first natural numbers, the group of the next nine letters represents tens and the last group hundreds.

α	β	γ	δ	ϵ	ϛ	ζ	η	θ
1	2	3	4	5	6	7	8	9
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
10	20	30	40	50	60	70	80	90
ρ	σ	τ	υ	ϕ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900

Placing a stroke before a letter, thousands, tens of thousands and hundreds of thousands were indicated and, with two strokes and so on, still larger groups of numbers.

Then, each number was represented as the sum of those indicated by a single letter. Thus,

$$_{/}\lambda\eta = 30\,008, \qquad _{/}\rho\sigma\lambda\delta = 100\,234, \quad \ldots$$

In so far as the Greek arithmetic is reduced to the art of calculation, no significant improvements are seen when it is compared with the Egyptian, while the Babylonian reached a far higher level than both of them. As a matter of fact, this art was called *logistica* and the classical Greek mathematicians scorned it and they did not even considered it to be a science. *Logistica* was taught to children in schools for practical purposes, whereas the word *arithmetica* was reserved for theory of numbers (7th, 8th and 9th book of the Euclidean Elements).

In each of these three cultures, the practice of counting and measuring led to the formation of an abstract idea of number reaching the extent that we call now positive rationals. Whole numbers (and their ratios) were considered by Pythagoras to be the foundation of the universe. A great crisis was caused when the Pythagorean disciples (legend says it was Hipasus of Metapontum) discovered incommensurable lengths (a diagonal and a side of a square).

Operating with magnitude as a universal which includes lengths, areas, volumes, angles, weights and time, Eudoxus created the theory of proportions (the Elements, 5th book) overcoming so that crises. Ratios of two incommensurable magnitudes of the same kind carry a meaning which we now call irrational number. Even though ratios of whole numbers were embedded in this theory as pairs of commensurable magnitudes, lack of higher degree of abstractness kept the ideas of the discrete and the continuous separated for a long time (until 19th century A. D.). In spite of the fact that the Eudoxus theory was created exclusively on a geometric basis, he is considered to be the first founder of the real number system.

The practice of measuring led man to consider ratios of whole numbers. But only abstract conception of geometric objects, and not the practice, could lead to the discovery of incommensurable magnitudes. In that we can see one of the greatest achievements of the classical Greek thought. Thought not sole, this is also the most important way of extension of the system of natural numbers existing in the frame of mathematics.

The basic numerals in the Babylonian arithmetic are the signs denoting numbers 1, 2, … , 59. Being arrangements of only two basic signs for 1 and 10, their denotation was easy for recongnition but, say, the sign for 59 is an arrangement of 14 "wedges" what makes such a system of numeration extremely unpractical. Similarly, the Egyptian 9 is an arrangement of 9 sticklike signs, their 90 such an arrangement of 9 signs indicating 10 and so forth and so on. This unpracticalness is another unfavourable aspect of the Egyptian numeration. Letters used in the Greek numeration are purely conventional signs (though each stroke added means the multiplication by 1000). They made possible an easier manipulation by forming of written schemes and, for example, when adding the Greeks wrote numbers one below the other in order to arrange units column, tens column etc.

The early medieval period which extends from the end of 3rd century to the end of 11th century A. D. is the time of stagnation of mathematics in the European civilization. As the Church extended its influence, the antique schools disappeared and there was a complete lack of interest in the physical world as far as the ecclesiastical institutions are concerned. Disputing against the Skeptics, Saint Augustin (A. D. 354–430) in his "De civitate Dei" declares his famous principle "*si falor sum*". (If I delude myself, I exist), (see, for example, [12]). Could not we also think of this principle as expressing the essence of the medieval man, seen as a sinful being, turned to the following of the Christian values?

During approximately the same period there was a flourishing of the Hindu mathematics. Taking over and improving the Hindu positional notation of numbers, in base 10, the Arabs carried it over to Europe but more than half a millenium passed until its general acceptance. The first such endeavour was the book "Regula de abaco computi" written by French monk Gerbert (who became Pope Sylvester II in 999). In meantime the Hindu-Arabic system was used by Italian merchants in their account books, but in some cases, was also forbidden as, for example, by the university authorities in Padova ordering that the prices of books should be indicated "non per cifras sed per letteras claras" (not by ciphers but by clear letters). It was the book of Luca Paciolli "Summa de Arithmetica", published in 1494, under the influence of which this system started to be widely spread in Europe. Having only ten basic conventional signs 0, 1, … , 9 used in positional notation, the Hindu-Arabic system shows all comparable advantage over each previous one. Add also that from a latinization of the name of Arab mathematician al-Khwarizmi (c. 780–c. 850) and by means of the Latin translation of his treatise "*Algorithmi de numero Indorum*" (12th century) the word "algorithm" was derived denoting all well-known rules of calculation performed on notations which represent numbers in this system.

Whole numbers and their ratios have always been conceived as more abstract entities than the ratios of magnitudes could have been. In the latter case, the operations were not clearly determined, either and, for instance, the "product" of two lenths was area. To illustrate all awkwardness that existed, let us cite one of

Vieta's cubic equations: $3BA^2 - DA + A^3 = Z$ in his own rhetoric form

$$B3 \text{ in } A \text{ quad} - D \text{ plano in } A + A \text{ cubo equatur } Z \text{ solido},$$

from which we easily see the geometric meaning of multiplication as well as that all terms have the same dimension 3. That homogenity of measures was also a condition for feasibility of addition and subtraction. Note also that even N. Tartaglia (1500–1557) insisted on distinction between multiplying of numbers and that of magnitudes, using *multiplicare* in the former and *ducere* in the latter case in order to point out the difference.

René Descartes (1596–1650) is considered to be the first great modern philosopher. A millenium after St. Augustine, he declared his famous principle "cogito ergo sum" (I think therefore I exist). Could not we also think of this principle as expressing the essence of the modern man, seen as a being which primarily uses intellect? A relatively short appendix to his classic "Discours", "La Geometrie" is his only treatise on mathematics which influenced its development in many directions. Confining here our tale to the status of the number system, we lay a stress on the Descartes' "coordinating" of the line. (The term "coordinate" was first used by G. Leibnitz).

For Eudoxus, proportion of two magnitudes a, b (following G. Oughtred (1574–1660), denoted by $a : b1$) and that of another two c, d are in the same ratio (now we would say are equivalent by representing the same number) if for any whole numbers m, n, whenever

$$ma \gtreqless nb$$

then

$$mc \gtreqless nd.$$

To express this equivalence, the equality $a : b = c : d$ is written.We see that the same ratio (number) could be represented by an infinite class of distinct pairs of equivalent proportions. It is easy to imagine how such notations, which do not indicate numbers uniquely, had to be necessarily a clumsy tool.

Fixing a unit length and denoting it by 1, Descartes replaced each proportion $a : b$ by its unique equivalent of the form $c : 1$. Then, taking a half-line (Descartes ignored to think of negative numbers), with the end 0, he represents the lengths 1 and c by geometric segments having one of their ends at 0.

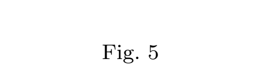

Fig. 5

Thus, an evident one-to-one correspondence between ratios and pairs of the form $c:1$ is established and, again, such a correspondence exists between all specific proportions $c:1$ and all lengths c. As a result, ratios are represented by segments of a line carrying unit length and, thereby, a geometric model is created, furnished with the meaning of, what we now call, positive real numbers.

As it also is very important to point out,this model is closed for the four elementary algebraic operations. Though it is somewhat involved in his method of solution of geometric problems by means of algebra, but Descartes apparently insists on the fact that the segments representing $a+b$, $a-b$, ab and a/b are constructible from those for a and b.

(A reader wishing to refresh his knowledge of elementary goemetry will use the proportionality of sides of similar traingles, to construct ab and a/b.)

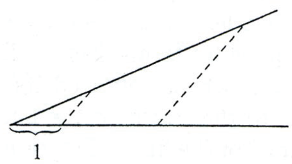

Fig. 6

(Construct x, according to $1:a=b:x$ and $b:1=a:x$.)

As a result of this improvement, the highly inconvenient condition of homogenity, still present in the writings of Vieta, was eliminated.

This geometric model influenced very much all ideas which led to the modern conception of real number.Properties of everything, amounts of which are measurable, can be transposed to a scale. (Think of thermometer, steelyard, etc.). In fact such a scale is a materialization of the coordinate line often found in real life.

Since the segments representing lengths have an end at 0 and the other determines uniquely a point on the line, then that point can also be taken as a geometric interpretation of number.Finally, when one-to-one correspondence between the set of points of a line and the set of infinite decimal notations was perceived, the Eudoxus magnitudes became real numbers as we understand them now.

At the end, let us note that tendency to have the number system rid of any specific interpretation has set up the idea of abstract set. On that idea, a logically solid basis was formed on which the modern theory of real numbers stands (R. Dedekind, second half of 19th century).

7.2. Natural numbers are more than a mere subset of reals. To indicate distinct systems of numbers, we will follow traditional notation. Use $\mathbf{N}$ when the numbers are natural, $\mathbf{Q}_+$ positive rational (plus zero), $\mathbf{R}_+$ positive real (plus zero), $\mathbf{Z}$ integral, $\mathbf{Q}$ rational and $\mathbf{R}$ when the numbers are real. This diagram

$$\mathbf{N} \to \mathbf{Q}_+ \to \mathbf{R}_+ \to \mathbf{R}$$

represents the way how the number systems were extended during a very long period in man's history.

Negative numbers, as roots of algebraic equations, appeared early in history of mathematics but they had been considered to "serve only to puzzle the whole doctrine of equations" and they were well understood only in modern times. Descartes rejected them and his "quarter" of the plane could serve only to represent parts of curves given by equations. It might be that the elimination of this deficiency encouraged mathematicians to accept these numbers. Their geometrical interpretation (together with "multiplication of signs") depends on the idea of two opposite orientations of geometrical objects and hence, it is difficult for schools.

Unrestricted feasibility of operations followed by the permanence of their properties is another plan according to which the extension of number systems goes in scientific reconstructions and didactical transformations of the historical process. One of these two ways

$$\mathbf{N} \to \mathbf{Q}_+ \to \mathbf{Q} \to \mathbf{R}$$

or

$$\mathbf{N} \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{R}$$

is usually taken, with the preference to the latter in scientific writings. In educational practice, both ways have their advocates (and me, in the role of a devil's advocate would say that a parallel following of historical development is inevitable anyway, no matter if these defenders are conscious of it or not).

We see in all these cases the initial position of **N**, what, by itself, underlines the fundamental importance of this system. Extending **N** to **Q** or **R**, which are ordered fields, both operations subtraction and division are also feasible without restriction with all operational properties preserved, but now we will be looking for specific meanings of natural numbers not shared by all reals.

If a number of dots can be arranged to form a triangle, then the first Pythagoreans called such a number triangular. According to the shape of these arrangements

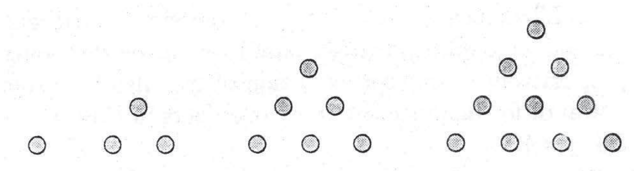

Fig. 7

triangular numbers are 1, 3, 6, 10, … . Similarly the numbers 1, 4, 9, 16, … were called square numbers.

With 9 small square jettons, a bigger square is easily composed and the same can be done with 16 jettons. Taking these jettons together and rearranging, a square composed of 25 jettons is formed. The equality $9 + 16 = 25$ can also be written as $3^2 + 4^2 = 5^2$ and such three numbers 3, 4, 5 are called a Pythagorean triple which is a solution in natural numbers of the equation $x^2 + y^2 = z^2$. The Pythagoreans already knew of infinitely many such triples and it was Diophantus (3rd century A. D.) who determined them all

$$m^2 - n^2, \quad 2mn, \quad m^2 + n^2$$

m, n being arbitrary natural numbers ($m > n$).

Using a set of small dice, say m^3 of them, a cube can be composed and with plus n^3 of them, another cube. But these dice cannot be recomposed so to form a third bigger cube, whatever the numbers m and n be. This means that there exists no triple m, n, p of natural numbers satisfying the equation $x^3 + y^3 = z^3$ (a fact proved by L. Euler in 18th century).

Many interesting problems (of arrangement and, generally, combinatorial in character) can be exclusively formulated and solved using natural numbers. The reader surely remembers their role in school combinatorics as well as he knows Fundamental Theorem of Arithmetic: Each composite number has the unique prime factorization (a fact, which would loose its meaning in **Q** or **R**).

Summarizing, we see that the system **N** is not only the first step in building larger systems of numbers, but that it, by itself, has many other important uses in mathematics. By the way, note that there exist some also important extensions of **N** distinct from number systems **Z**, **Q** and **R**.

Addendum 4.

As it is usually known to mathematicians, the set **N** together with addition and multiplication extends to the transfinite systems of ordinal or cardinal numbers in which, the natural numbers formally attain one of these two distinct characteristics of extensions. The objects added by extension still indicate, place in the former and potencies of sets in the latter case, but the operations loose some of their properties, what is, for instance, seen from $\omega + 1 > 1 + \omega$, $\aleph_0 + \aleph_0 = \aleph_0$, etc.

As an additional example, take natural numbers as topological types of finite (Hausdorff) spaces. Then, say, the set of types of compact, metric, zero dimensional spaces, with addition and multiplication based on disjoint topological sum and direct product respectively is another extension of **N**. Also here some operational properties are lost and, for instance, there exist objects in this extension such that $X \neq Y$ and still $X^2 = Y^2$.

Insisting further on exceptional role of whole numbers, let us recall that some fundamental topological invariants as Euler characteristic, Brower's degrees, etc. are integers as well as that each topological property which homology groups with coefficients in an arbitrary Abelian group register, those with integral coefficients also do (Universal Coefficient Theorem).

Conceived either as ordinal or as cardinal, natural numbers always represent experiences of discrete realities. On the contrary, everything considered to be a magnitude is continuous in nature and, when intelligibly conceived, becomes reduced to its gestalt—a line segment, as each procedure of measuring proves it so evidently. But when two segments a, b are compared, the number n_1 is found so that $n_1 b \leqslant a$ and $(n_1 + 1)b > a$. If $r_1 = a - n_1 b > 0$, then n_2 is found so that $n_2 r_1 \leqslant b$ and $(n_2+1)r_1 > b$. If $r_2 = b - n_2 r_1 > 0$, then n_3 is found so that $n_3 r_2 \leqslant r_1$ and $(n_3 + 1)r_2 > r_1$ and so on, this process is continued (possibly infinitely many times if a and b are incommensurable). Thus, a fully developed concept of the ratio of magnitudes inevitably leads to a (possibly infinite) sequence of natural numbers (and in its most elementary case is reduced to counting of copies of unit measure).

There exists a textbook in arithmetic from a country known by high educational standards and for which I do not know if, and how widely, is accepted in the current school practice. My interest provokes the fact that the authors of the book evidently take the naive idea of magnitude as a starting point in developing the number ideas. Not being ready to analyze it here in detail, I can't help thinking of it as a complete failure.

Following the meanings suggested by pictures, say, of bottles with some amount of milk in each, articles of clothing of different size, etc., letters of an alphabet are used and involved in relations as

$$\text{Л} = \text{К}, \quad \text{К} = \text{Б} + \text{Д} + \text{Ж}, \quad \text{Р} - \text{Б} = \text{Н} - \text{Б}, \quad \text{Р} > \text{Г}$$

and so on. Then, natural numbers as ratios of magnitudes, appear and, for example, we see them introduced this way

$$\frac{\text{М}}{\text{К}} = 4, \qquad \frac{\text{М}}{\text{А}} = 2, \qquad \ldots$$

Using a Piaget's term loosely, we would say that this literal arithmetic stays at an infralogical level. That numerical one, contained in this textbook as well as in its continuation (the volume for the 2nd year class) is mostly left to look after itself. But just the opposite, the operational skill, formed in numerical arithmetic is of the primary developmental meaning, what we have to care about all the time.

Being basis and ground of all mathematics, natural numbers and operations with them, in one word, arithmetic remains to be a principal occupation of educators. The processes of creation which happened once and under some circumstances can be retraced anew, but their essence should not be touched by any innovation, least of all by those ignoring historical course and structural contours of mathematics proper.

7.3. Significant innovations. Education in Christian schools in the Middle Ages was oriented towards devotion to God and religion and to acquisition of Christian wisdom. Church Fathers did suggest to educators that their work with children should be as lively as possible and to educate them in secular matters as well. But ecclesiastical arithmetic was reduced to the computation of dates of holidays called *computus* and some geometry found its place in Church architecture. Mercantile economies of Italian cities (12th and 13th century) instigated the foundation of public schools with programmes subordinate to the interest of the leading class, while the exclusive aim of feudal lords was military and physical education. Dogmatism and formalism were characteristic for medieval teaching methods and overwhelming belief in inborn ideas was a widely spread doctrine.

With the Renaissance and the Reformation, new roads in education were opened and the real subjects (arithmetic, geometry, astronomy, natural history) gained in importance. In place of scholastic verbalism, pleadings for visual method spread. It was John Locke (1632–1704) who said that the mind of a child in its earliest state is *tabula rasa*, expressing so his belief that the senses are the source of knowledge. The new didactical views insisted upon visual method, the relation between things and words as well as between perception and concepts.

The best act in response to new didactical trends in 17th century was the book of Jan Amos Komensky "Orbis sensualium pictus" which stayed to be a paragon of visual method for a long time and the contemporary arithmetic books, with all colourful illustrations, are an aspect of revival of Komensky's method. Kant's philosophy is taken to be the climax of philosophy of the Englightenment, characterized by rationalism, and the principle of his theory of knowledge—from observation to conception is basic for Pestalozzi's didacties. Pestalozzi considered number, form and word to be fundamental elements of elementary education and he rejected the teaching of arithmetic which does not develop a clear idea of number.

The name of Friedrich Eberhard von Rochow (1734–1805) is associated with the important improvements in German school system. His major contribution to the teaching of arithmetic was the splitting of the natural number system into didactical blocks 1–10, 1–20, 1–100, . . . , what is an idea which has become generally accepted in modern didactics.

The fact that each natural number is an individual concept and that there exist infinitely many of them, as might be expected, the puzzling question how to develop their meaning arises.

Pestalozzi's requirement that each number has to be treated separately should not be understood litterally but, as it goes without saying, the numbers within an initial didactical block are meant. This requirement was most consistently carried out by Pestalozzi's follower A. W. Grube (1816–1884), who established so called monographic method treating number by number with all four operations involved simultaneously. As a sample, let us select a piece of his unit devoted to the number 4.

I) Measuring and comparison

a) Measuring with 1. $||||\,4$.

$$\begin{matrix} |\,1 \\ |\,1 \\ |\,1 \\ |\,1 \end{matrix} \qquad \begin{cases} 1+1+1+1=4 & (1+1=2,\ 2+1=3,\ \text{etc.}) \\ \qquad\quad 4\times 1=4 & \\ 4-1-1-1=1 & \\ \qquad\quad 4:1=4 & \end{cases}$$

b) Measuring with 2.

$$\begin{matrix} ||2 \\ ||2 \end{matrix} \quad \begin{cases} 2+2=4 \\ 2\times 2=4 \\ 4-2=2 \\ 4:2=2 \end{cases}$$

c) Measuring with 3.

$$\begin{matrix} ||| \\ | \end{matrix} \quad \begin{cases} 3+1=4,\ 1+3=4 \\ 1\times 3+1=4 \\ 4-3=1,\ 4-1=3 \\ 4:3=1\,(1),\ 3 \text{ into } 4=1 \text{ and } 1 \text{ remains} \end{cases}$$

II) Quick calcuation

$2\times 2-3+2\times 1+1-2$ doubled!

$4-1-1-1+2+1-3,$ how many times smaller than 1, etc.

Prior to Grube, the teaching matter of arithmetic was divided into species: block of numbers 1–20 with four operations going one after another: addition, subtraction, multiplication, division (1st year class), block 1–100 with the same order of operations (2nd year class), block 1–1000 (3rd year class) and so on.

Both these methods are extreme and an optimum can be expected with their reasonable combinations which are present in the contemporary treatments.

Not long ago there were illiterate villagers or those having finished only a four year elementary school in many European countries. At that time, no one could ignore crude life and needs, often of a high percent of such population either the worth of activities, such as joinery, carpentry, dressmaking and so forth. Thus, no wonder if the educational aims of such a school were utilitarianly oriented and often referring to the subject matter as an autonomous whole.

Along with the improvement of social and economic conditions, elementary education has become obligatory and has been prolonged to at least eight year period. As a result many aspects of teaching and learning have changed and new educational aims have been proclaimed but the range and skeleton of arithmetic have remained unaltered since the time of great reformers.

In teaching of arithmetic as it was, the algorithms of operating with numbers given in decimal notation were in the first plan, what often tasked a child' s brain. Modern approaches are primarily oriented to understanding. Now, within small blocks, the numerical terms (sums, products, etc.) are used to represent numbers and equalities relating them are used to express fundamental rules of arithmetic. On that basis, the algorithms (usually postponed until the 3rd year class) become intelligible and thus easier learned. Gained skill and knowledge of such arithmetic is transferred to algebra in higher classes of elementary school, what is an important objective being not present in earlier methodology of teaching.

Relics from previous generations of teachers still exist. For example, writing about the dilemma vertical or horizontal form, an author of a book on methodology of teaching arithmetic says: "People, in adding, ordinarily write the numbers in vertical form".

Deciding in favour of the vertical form, he says that it is "the one commonly used in the community". When we think of children in school of today how they write sums of fractions and proceed to operate with algebraic terms for years we easily see how much out of date are such arguments. Vertical form comes with and is bound to algorithm of adding of numbers in decimal notation when columns of units, tens, hundreds etc. are arranged.

So it is somewhat sad to see

$$\begin{array}{r} 2 \\ -\,1 \\ \hline 1 \end{array}$$

written and its blundering reproduction in drawing

$$\begin{array}{r} \bullet\ \bullet \\ -\,\bullet \\ \hline \bullet \end{array}$$

given, and all done as "a proper basis for effective learning".

8. Didactical blocks of numbers

In the majority of approaches, the study of numbers is seen to be divided into classical didactical blocks 1–10, 1–20, 1–100, etc. and when it is not so, we readily think of some zealous innovations being quite out of the way. Revealing exactly what a block contains in the way of teaching steps, grouping of facts and distribution of practice, a foundation for its existence as a set of connected didactical units is established. We leave to expose such details for each block separately. By and large, the initial blocks consist of those numbers which have to be mastered thoroughly, establishing all relations among them, expressed by means of four arithmetic operations being contained in corresponding tables.

8.1. Block 1–10. This block consists of the first ten natural numbers plus zero and with the exception of 10, each being one digit and these digits also being the whole set of basic decimal symbols. Further, two operations begin here, the addition and the subtraction, what by equating of sums and differences with their values written in decimal notation introduces the equality relation. Comparing numbers, the order relation (bigger than, smaller than) appears as well. In all, this block is a quite complex conceptual structure

$$(\mathbf{N}_{10}, +, -, =, <)$$

which is made up of eleven numbers, each being an individual concept, of two operations and two relations, what means four extra concepts each requiring a great effort to form its proper meaning. When we count, we find here fifteen signs, or half of an alphabet, what could be too much for the unsteady hand of a child. That is why we suggest here going at an easy pace and, first, acquiring of the numbers 1–5. Forbearing to speak about a still smaller block in this case is a sign of our dislike for novelties.

Quantity of elements of small sets is recognized at once or found easily by quick short counting. As being at the sensory level, sets that the child deals with are perceptual entities and, by this means, numbers and operations are as well. Discovering numbers and finding sums and differences with components and results not exceeding 5, the child does it by observing or quick counting, usually having no problem of that kind. Thus, his/her attention should be focused on meaning and its correct codification in the form of arithmetic symbolism. That is the reason we suggest mastering the numbers 1–5 as the first steps.

We take here meaning of the word "lesson" to be amount of teaching given per school hour. For each of the first five numbers 1, 2, 3, 4 and 5, five separate lessons should be organized according to the *orbis pictus* model. In that way, the teacher starts out to create them as full concepts and for that, the mere process of counting is insufficient.

Lesson "one 1" should start with pictures representing, say, one boy, one house, one bird etc. Stimulating pupils, the teacher lets them describe what they see. Then, the children use the word "one" followed by the name of objects of their observation. Below each picture a space holder stands where the children write the sign "1", without any naming.

Thus, this sign is used for denotation of an abstract concept and not for its corresponding examples. Let us use a diagram to illustrate all such examples.

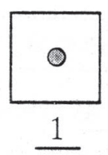

Fig. 8

In ordinary speech, the question "How many" is used when one refers to the situations where two or more things are seen and, in arithmetic, it should be intentionally used in the case of numbers 0 and 1. So this is a moment when the teacher asks questions as "How many boys are seen" and the children answer "one" ("just one") , etc.

Children also have to practice correct writting of the figure 1, moving the hand as arrows indicate it:

$$^{\nearrow}1_{\downarrow}$$

Of course, here we give only an outline of the lesson and thereby we leave out details which would model it completely.

Lesson "two 2" should begin with the pictures representing natural twos. Such are team of two horses, a pair of gloves, shoes, skis etc. Again, space holders are

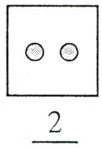

Fig. 9

attached to these pictures, where children write figure 2. Diagramically, it looks like

Of course, the practise of writing the symbol

$$\overset{\curvearrowright}{\underset{\rightarrow}{2}}$$

takes also some time.

Similarly, the lessons “three 3”, “four 4” and “five 5” are done. To preconceive addition, number pictures as

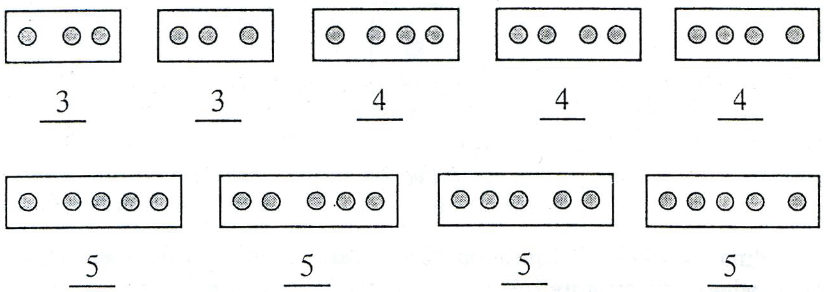

Fig. 10

should also be used in pictorial environment.

As we know it from history of mathematics, it was Robert Record (1510–1558) who, in his book “Whetstone of Witte”, introduced the equality sign “=” and where he says “noe 2 thynges can be moare equalle” (thinking of the two dashes). Thomas Harriot (1560–1621) introduced two inequality signs, usually interpreted as at one of the ends “pinched” equality sign which opens towards the bigger number.

Some children have a problem to distinguish these two signs. To help them, the drawings as in Fig. 11 could be employed, what is better than possible verbal instruction of the teacher.

$2 < 4$ $3 > 2$

Fig. 11

8.1.1. Patterns of letters designed for handwriting usally have line endings which serve to join them in groups traced by a single continuous move of the hand. On the contrary, figures are separted symbols written in one or more moves each of which traces an arc (with possibly coinciding end points). At the last end point of an arc the movement of the hand stops in order to have direction changed (corner points) or to have the tip of pencil placed at another point (points of discontinuity of writting). Thus, the figure “5” is traced in three moves

and, in this example and all others, arcs are written starting to move hand from top downwards or, in the case of horizontal arcs, from left to right.

To establish this manner of writing figures, some teachers use big copies (the size of a page) and let children move their hands in the order and direction indicated by arrows. They also claim that it helps their pupils easily overcome some initial difficulties in writing figures.

When we are ready to accept such a claim, we have in mind that more intensive motor acts are likely to be easier represented inactively and that training an activity, the responses also tend to a regular form.

8.1.2. Since the zero is a specific number, it deserves some extra attention. The symbol “0” appears in Greek papyri from the Alexandrian period to indicate missing numbers and it is generally supposed to be the initial letter of the Greek word “ouden” which means “nothing”. But the use of this symbol in decimal notation does not necessarily mean the inclusion of the zero into the system of natural numbers. In the developing of meaning of zero as a number, some difficulty consists in the right comprehension of the empty set and is also caused by the fact that the zero does not come into existence naturally, as a result of forward counting.

To form the first idea of this number, the child has to be in contact with situations where some elements have been existing and they are now absent or they still exist, but at the places on which the attention is not focused. Hence an idea of backward counting may be well exploited to instigate the activities through which zero becomes a number indicating the empty place (a materialization of the empty set).

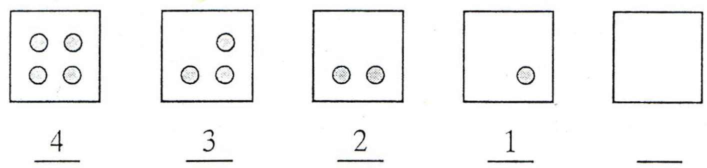

Fig. 12

As an example, we can use a picture of a series of trays, successively having four, three, two, one and none cake on them. The simplified drawings in Fig. 12 illustrate it and we suppose that the children have already been trained to fill in the blanc places writing 4, 3, 2, 1. If followed by a teacher's story, the children reactions to the case of the last tray might be "all cakes eaten", "none cake left", "nothing left", "there is nothing", "there is none", etc. Now the teacher accepts such their reactions with approval and saying that even in this case a number is written, he/she fills in the left blanc place using the symbol "0" and explaining that we read it as "zero".

Asking how many cakes here, and here, ... , the teacher moves his pointing finger from tray to tray guiding so his/her pupils count backwards: 4, 3, 2, 1, 0.

With the help of similar examples the zero starts to gather some meaning and, later when it is involved in calculation as a component of sums and differences, it slowly becomes a number as any other.

Here I expose the way how this topic is treated in the current school practice in my country and I have never met a teacher reporting any difficulties the children have with the acquiring of this concept.

Addendum 5.

In set theory, the existence of the empty set $\varnothing$ is postulated and if x is any object, then

$$x \notin \varnothing.$$

It is also postulated that there is only one empty set.

But it does not seem rational to relate the concept of empty set with the idea of an "absolute nothing". As there exists only one cardinal number "2" and many its different materializations : two cakes on a tray, two birds in a cage, etc., in the same way, there exists only one empty set and many its different materializations: a tray with no cake on it (when cakes are counted), a cage with no bird in it (when birds are counted), etc. Of course, if we count trays or cages, then such examples does not materialize the empty set any longer.

Syntactic sign of the empty set is "∅" and a pictorial one is seen as an "empty" rectangle in Fig. 12. When a closed curve that holds things together is used in pictorial representation of sets, then the empty set is represented by such a curve alone. As it goes without saying, only pictorial representation of the empty set is present at this stage.

8.1.3. In the same way as the concept of set precedes that of number, the idea of a set partitioned into two disjoint subsets goes before the addition and the subtraction. At the sensory level, this means a collection of objects in natural environment obviously partitioned by their grouping at two distinct places or by an easily observable difference which separates them into two groups. Information about cardinality of two of these sets is given (say, obtained by counting or communicateted in words) and then, the cardinality of the third set is to be found. We will call such a real world situation or its pictorial representation, or else, mental representation associated with an information given in words, *scheme (situation) at which we react adding or subtracting.* This step stresses the dependance of an operation on natural surroundings.

In the case of addition, the cardinality of two subsets is given and in the case of subtraction, the cardinality of the set and of one of its subsets is given. Thus, the first step in performing one of these operations is

(I) *comprehension of scheme.*

The step to follow is

(II) *composition of numerical expression*,

(sum in the case of addition and difference in the case of subtraction).

Finally, the third step is

(III) *calculation of numerical value of the expression*,

what means a process of equating which leads to decimal notation of that value.

Now we describe a graphical representation of this scheme which results from many real world situations or from their *orbis pictus* representations. When planed to be materialized as an item of didactical apparatus, the scheme might be a paper box or a plate, each divided by a thin wall (or line) and with the labelled tags notifying cardinality.

Perceptually, it also represents very well the idea of two distinct places which was so often verbally expressed by traditional teachers. With A and B denoting sets, the graphical scheme in Fig. 13 is associated with addition when m and n are given and subtraction when s and m or s and r are given.

Writing $m+n$ in the former and $s-m$ or $s-n$ in the latter case, we perform the second step.

If m, n and s stand for decimal notations of corresponding numbers, then, in the third step, the equalities $m+n=s$, $s-m=n$ and $s-n=m$ are written and the numbers s, n and m are then the results of calculation.

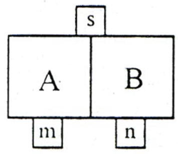

Fig. 13

This general exposition describes and emphasizes the role of a scheme (in mental imagery or as a graphical realization), but the real teaching starts, of course, with simple examples.

EXAMPLE 1. Two boys are seen on a lawn. Another two are coming.

The teacher schedules a performing following the general plan.

How many boys are there on the lawn: 2.

How many are coming: 2.

Now the teacher says that altogether we see two plus two boys and he/she writes on the blackboard

$$2 + 2$$

reading it as "two plus two". Asking: How many is it: 4, the teacher uses the equality sign for the first time and he/she writes

$$2 + 2 = 4$$

reading this relation: "two plus two equals four".

After having done in this way a couple of similar examples, the teacher lets the children do exercises of this type

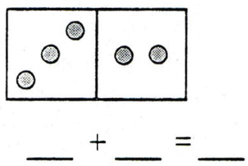

Fig. 14

varying the number of elements and their nature (going from more realistic illustrations to graphical schemes).

While the pictures reflect the structure of the scheme, the usage of space holders leads the child to compose sums and to equate them with the right results. This equating is a special manner of expressing the principle of invariance of number: the same number is obtained when a set is seen in groups, and, say, when we react writing $3 + 2$, and when we see it as a whole, and we write 5 instead. A deliberate usage of space holders is very effective at this stage of teaching and we will write about it later, in the form of a separate topic. Note also that this practice of writing sums and equalities is easily rooted and the children usually have no problem to do it, even when they are no longer helped by space holders.

Example 2. Two boys are seen on the lawn. Another two are seen to be running away. Now the teacher performs the start in subtraction.

How many boys were there on the lawn: 4.

How many are leaving: 2.

Now the teacher says that the four boys were on the lawn and that now, there are minus two.

Writing on the blackboard the expression

$$4 - 2$$

and reading it as, "four minus two", he/she asks the children how many it is: 2. Then the equality

$$4 - 2 = 2$$

is written and properly read.

In case of subtraction a graphical scheme, for example, this one

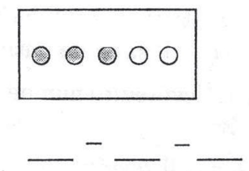

Fig. 15

could be more appropriate for the beginning (five glaces − two empty, five pears − two eaten up, etc.)

There is a large variety of different verbs suggesting addition and subtraction. Those having their meaning within the child's experience should be used in composition of simple word problems, what connects the symbolic codes of arithmetic with the surrounding reality. But when helping the child, the teacher does it best by drawing a corresponding graphical scheme not changing much its form going from a problem to another. Thus, he/she lets the underlying abstraction develop the way which is less hindered by unnecessary noise.

In this frame of the first five numbers plus zero, no efforts to stimulate practice of formal addition and subtraction should be planed and done. Nevertheless, through the lessons devoted to these two operation, all possible relations $m+n=s$ and $s-m=n$, where m, n, s belong to the set $\{0,1,2,3,4,5\}$ should be included in the form of exercises.

The Chapter 2 of Skemp's book [6] is an excellent exposition on schemata and advantage of schematic learning. Let us cite a passage from it.

> "When learning schematically—which, in the present contest, is to say intelligently—we are not only learning much more efficiently what we

are currently engaged in; we are preparing a mental tool for applying the same approach to future learning tasks in that field."

The contents of this point are evidently devoted to schematic learning of addition and subtraction with the accent on the systematic use of a graphical scheme which serves as a significant sign (Section 4 of this paper).

8.1.4. The range of number 0–5 extends on the basis of addition, using the sums whose summands stay within this range and the values exceed 5. Thus, the lesson " six 6" should start with the sum $5+1$ which has already gained a meaning through the earlier performed activities. Now, a good move from the side of teacher would be to draw the addition scheme and to ask children to fill in it with circlets to represent the sum $5+1$. Induced by previous activities and knowing to count (up to 10), the children guess easily that $5+1$ is six. Then, the teacher writes

$$5+1=6$$

using now a new symbol "6" and letting children have a drill on writing it. (Formally, a mathematician could say "6 is, by definition, $5+1$".)

In a similar way, the lessons devoted to numbers 7, 8, 9 and 10 start with $6+1$, $7+1$, $8+1$ and $9+1$, respectively.

In the course of these lessons all relations $m+n=s$ and $s-m=n$ have also to be covered in the form of exercises which should be supported by convenient iconic representations. Leaving out full details of that kind, we will cover shortly main steps, referring the reader to our paper "Schematic learning of the addition and multiplication tables—sticks as concrete manipulatives", this Teaching, vol. I, pp. 31–51.

Besides the "defining" sums $6+1, \dots, 9+1$ the following ones $5+1$, $5+2, \dots, 5+5$ are also easy to children. Those which "cross the five line" are considered to be more difficult. Therefore, they have to be accompanied with suitable illustrations. Instead of using colours, due to technical reasons, continuous and dotted line segments are drawn in Fig. 16 to form the arrangements representing such sums.

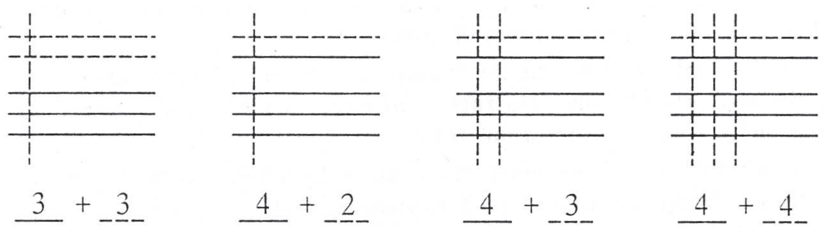

Fig. 16

To calculate, say, $4+3$ the children proceed as follows

$$4+3=4+1+2=5+2=7,$$

completing first five and then finding an easy sum $5+2$. When let to do it orally, the children, leaned upon the above pictures, do it easily.

8.1.5. In the course of this block, the meaning of some words belonging to the arithmetic vocabulary have also to be established. Using some examples, say,

$$3 + 2$$

the teacher notifies "3" as *first summand*, "2" as *second summand* and the whole expression as *sum*. By means of programmed exercises he/she also controls the assimilation of their meaning. To bring to completeness this block, two properties of addition have to be covered as well. In the way that it is easier to count up to 9, beginning at 6 than at 3, in the same way, taken as an action, it is easier for the child to find $6 + 3$ than $3 + 6$. Thus, the rule of interchange of summands (said formally, commutative law) has to be established.

A related practice found in some text-books must be criticized. Namely, some authors take a number of examples, where they calculate a sum and the interchanged one finding the same result and then, on that basis, they "derive" commutativity of addition, stating it in words or symbolically. It is easy to find fault with it. First, they use the so called principle of incomplete induction as a basis for conclusion and second, when calculating they often use the rules which they intend to derive.

A good teacher knows that fundamental rules (principles) are not to be proved but accepted and he/she leads his/her class that way.

For instance, starting with a set in picture-form consisting of blue and red flowers, the teacher directs these questions to his/her class:

How many blue: 4. How many red: 3 .

How many altogether: $4 + 3$.

Then, the teacher interchanges the questions:

How many red: 3. How many blue: 4.

How many altogether: $3 + 4$.

Saying that in both cases we have found the some number of flowers, writing once $4 + 3$ and then $3 + 4$, he/she writes the equality

$$4 + 3 = 3 + 4.$$

Let us notice that this equality is established without counting (or calculating), as well that the chosen set stands still all the time and that the procedure does not evidently depend on the number of flowers.

In this and all other similar examples, the *rule of interchange of summands* is expressed procedurally. Its rhetorical form would be: *the summands may be interchanged without altering the sum.*

At a later time, when more cases can be included, we shall write about rules of arithmetic and the ways of their expression, setting it down as a separate topic and providing much more details.

When this block is extended, the sums as, for example, "$7 + 5$" are found in this way

$$7 + 5 = 7 + 3 + 2 = 10 + 2 = 12$$

and we see that the three member sums are involved and that the rule of association of summands acts as well. Since the meaning of the sum of two numbers does not include that of longer sums, first we have to make meaningful such summation.

In the examples that follow, we use pictures of coloured jettons having different shape. We also shorten the accompanying teacher's questions.

Fig. 17

white: $2+3$, black: 4; altogether: $(2+3)+4$.
round: 2, triangular: $3+4$; altogether: $2+(3+4)$.

Saying also here that in both cases we have found the same number of jettons, the teacher writes the equality

$$(2+3)+4 = 2+(3+4).$$

Using another picture

Fig. 18

and directing the same questions, the teacher establishes the equality

$$(2+4)+3 = (2+3)+4,$$

etc. After having done a number of similar examples, the teacher expresses the *rule of association of summands* rhetorically: *the summands may be associated freely without altering the sum.*

8.1.6. *Ad notam.* In this paragraph, as well as earlier, we often use a language which does not describe ready-made products of arithmetic but the activities through which they are synthesized. Also, when we say "the teacher leads his/her class", "the teacher directs question to his/her class", etc., we assign a thing which has to be done by him/her and, of course, it does not mean we are so pleading for the frontal method. In fact, we do not treat purely pedagogical questions in any part of this paper.

As it is easily seen, longer sums and the rule of association of summands are treated here in a synthetic rather than analytic way when, in the latter case, addition is considered to be a binary operation and the sums of more than two summands are defined inductively. In the present form, the rule of association is suitable for use by children. Logically, it appears as a combination of commutative and associative laws, but at this early stage of arithmetic teaching we do not expect children to deduce formally.

The usage of brackets may also be postponed and then, the summation should go in the order the summands are written.

A number of examples used for illustration of these rules serve to practise the procedure. Mental picturing and establishing of equality present in the case of special numbers would be just the same in general, what means in the case of any others. Hence, it is in no way an instance of incomplete induction, but the formation of an intuitive basis upon which these rules are acceptable.

Activities through which the touch with addition and subtraction scheme begins should start with the lessons devoted to the counting drill and the usage of words "set", "element" (and of their natural equivalents). A usual place for symbolic coding of these two operations is after the lesson "three 3". The reason for it could be the maintained opinion that the children see the number of objects at once, when it does not go beyond three.

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