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PAYING ATTENTION TO STUDENTS' IDEAS IN THE DIGITAL ERA

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Abstract. This paper demonstrates how recognition of a hidden potential of rather involved mathematical explorations in a student's unintentionally far-reaching response to an open-ended question about constructing a visual pattern allows for the development of the so-called TITE problem-solving activities that require concurrent use of computing technology and mathematical reasoning. The paper begins with the presentation of such a response by an elementary teacher candidate and it continues towards revealing the potential of the response as a springboard into the development of various TITE generalization activities with ever increasing conceptual and symbolic complexity. It is argued that whereas one of the goals of moving from particular to general is to assist in understanding special cases, the construction of workable computational algorithms for spreadsheet-supported problem solving and posing is not possible without experience in generalization. The mathematical content of the paper deals with polygonal numbers and their partial sums. Computer programs used are *Wolfram Alpha* (free interface) and Microsoft Excel spreadsheet.

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MSC Subject Classification: 97D40, 97I30

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1. Introduction

The motivation to write this paper was due to the author's interest in the idea of developing the so-called TITE mathematics curriculum [1, 2], the problems of which, while requiring technology-enabled (TE) solution strategies, are still technology-immune (TI) in the sense that they do require using mathematical reasoning not replaceable by the modern-day technology despite its various symbolic computation capabilities. The more technology grows in the scope and sophistication of symbolic computations, the more challenging for mathematics educators is to preserve TI components of traditional problem-solving activities. TITE problems cannot be automatically solved by software; yet, the role of technology in dealing with those problems is critical. Within a TITE problem-solving context, argument and computation go hand by hand, leading the way to the appropriate use of technology. In particular, by learning to solve a TITE problem, one develops the appreciation of the notion of "instrumental genesis" [16] through distinguishing between positive and negative affordances of high-level digital tools and enabling their conceptual applications. Thus, the goal of this paper is to contribute to TITE mathematics education research efforts by finding the right balance between TI and TE parts of problem-solving activities.

The content of this paper, aiming to provide an example of a proper TITE setting, was motivated by the author's experience working with teacher candidates within an elementary mathematics content and methods course. Such work can hardly be described in TITE terms as technology at the primary level is mostly limited to concrete materials (commonly known as manipulatives) and mathematical reasoning skills are almost non-existent at the beginning of the course. Therefore, the author's motivation has been of an intrinsic nature which, as noted by Biggs [4], is associated with "the intellectual pleasure of problem solving" (p. 62), something that is important to be imparted to future teachers of mathematics.

Consistent with the above description of the course and background of its participants, one of the first course's homework included the creation of different visual patterns to be described symbolically. Visual patterns, frequently supported by either physical or virtual manipulatives, are commonly introduced at the kindergarten level and are traditionally referred to as AB patterns. For example, both dog cat/dog cat/dog cat/ ... and dog-dog cat-cat/ dog-dog cat-cat / dog-dog cat-cat / ... are AB patterns. Teaching young children to appreciate the sameness in miscellany may be considered as teaching early algebra to the children. During the discussion of the homework, one of the teacher candidates presented a pattern shown in Figure 1; yet she could not describe it using the AB-language. Indeed, the repetition of colors, red [R] and blue [B], in the pattern varied by increasing monotonically at each step. Instead of saying to the teacher candidate that this is not a pattern sought by the homework and, thus, it has to be corrected to match what was sought, the class was advised to recognize the pattern as a creative one for it appeared to allow for multiple modifications and extensions, something that is the major condition for encouraging creativity in the classroom. This advice was in the spirit of Montessori [13] who emphasized the importance for education "the recognition of new phenomena, their reproduction and utilization" (p. 73). The rest of the paper is aimed at illustrating this vision of the pattern by the author and using it as a springboard into TITE activities appropriate for different levels of pre-college mathematics education.

R	B	R	R	B	B	R	R	R	B	B	B

Figure 1. A pattern offered by a teacher candidate.

2. Creativity as a foundation of success

Teacher candidates have to be prepared to recognize student creativity early. Educators see creativity as "one of the essential 21st century skills ... vital to individual and organizational success" [3, p. 1]. Teachers' ability to appreciate mathematical creativity of their students that may be hidden behind one's immature classroom performance, is critical for successful teaching and productive learning of mathematics. Such ability develops through teachers' own preparation to teach the subject matter. If students' hidden creativity is not acknowledged and supported by a teacher, it would most likely remain latent and eventually could die out. Likewise, if teacher candidates' creative potential is overlooked, they will impart such attitude to their own students. After all, the acknowledgement of creativity, whatever the context, does uplift one's excitement about the context. Needless to say, such acknowledgement requires intellectual courage of going into an uncharted territory and pedagogical skills of balancing on the border between known and unknown.

3. A question seeking information

What can be said about the pattern of Figure 1 from the perspective of creativity? How can such pattern be extended to allow for posing challenging questions, something that young children, when given a freedom of asking questions, are not hesitant to do? Learning of mathematics is more productive when questions are asked by students rather than by the teacher. Research conducted by psychologists in public schools of various parts of the Brooklyn borough of New York City in the mid 20th century provided evidence [11] that it is a forward-looking pedagogy, with its emphasis on critical thinking and experiential learning rather than on the "rule method" [12], and not a student population that results in mathematical problem solving manifesting creativity and insight. For instance, one such forward-looking school "was located in a deteriorated slum which was populated by recent immigrants who were minority group members and who were of the lower class" [11, pp. 62-63]. In particular, progressive problem-solving pedagogy, born out of mathematically-oriented ideas of Gestalt psychology [21] with genesis in the papyrus roll writings of ancient Egyptians [5], challenges belief that teachers are the only ones in charge of questions in the classroom. Perhaps influenced by the knowledge of Common Core State Standards [6] and recommendations for mathematics teacher preparation by the Conference Board of the Mathematical Sciences [7], one of the author's students, a teacher candidate, put it as follows: "*If a student asks why and a teacher cannot explain how something has come to be, the student loses interest in the subject and respect for the teacher*". This statement beautifully resonates with one of Pólya's classic writings for teachers: "No amount of courses in teaching methods will enable you to explain understandably a point that you do not understand yourself. Hence the second commandment for teachers: *Know your subject*" [15, p. 102, italics in the original].

A question about the pattern, that a teacher might be asked by a student in the era of Common Core State Standards [6] rooted in the ideas of the progressive pedagogy of the mid-20th century [11], could be as follows: *If the pattern shown in Figure 1 continues as long as one wishes, which color, red or blue, will be at the 100th place?* This question, that Isaacs [9] would have referred to as the one seeking information (being, in general, of a less sophisticated type than the one requesting explanation), immediately introduces learners of mathematics to a numeric environment and, skipping the basics of early algebra, leads to a true school algebra with its emphasis on the use of the language of functions as means of describing patterns. That is, the first step toward finding an answer to the above question is to move from the letters R and B to numbers which can serve as labels of the positions of colors in an evolving pattern.

There are several ways to associate the letters representing colors with numeric labels. For example, one can describe numerically the last red cell in each combination of equal quantities of R's and B's. Such approach brings one very close to deciding the color of the 100th place: those red cells have the labels 1, 4, 9, 16, 25, and so on. What is special about these numbers? In the author's experience, elementary teacher candidates know that these are square numbers. In particular, the numbers are squares of the ranks of "RB"-combinations in the evolving pattern. So, the number 100, a square number, will be on the list. The equality $10^2 = 100$ implies that the label 100 belongs to the "RB"-combination of rank 10. It is the use of insightful strategy that provided a quick answer to apparently a not very simple question. Note that if instead of focusing on the label for the last red cells, one focused on the labels for the first red cells, the sequence 1, 3, 7, 13, 21, and so on, would have resulted. This sequence does not include the number 100, so the focus on the label for the first color does not deliver an answer right away.

One might ask, requesting explanation: *How does one know which places, the first or the last ones, have to be given labels in order to be a successful problem solver?* Often, request for explanation stems from epistemic curiosity [19] about a method that provided information. There is no easy way to answer this question (in particular, demonstrating that the question requesting explanation is indeed more intelligent than the one seeking information). It could be that the switch from focusing on the first color to focusing on the last color occurs unconsciously once a sequence of labels 1, 3, 7, 13, 21 with no frame of reference familiar to unsophisticated learners of mathematics comes to light. Such a switch is due to insight or productive thinking [21], important intellectual tools of mathematical problem solving. Furthermore, in the spirit of transition from arithmetic to algebra, numerical evidence brings about the function $f(n) = n^2$ which maps the rank n of an "RB"-combination to the position of the last R within it. Because one has to make $n - 1$ steps back to move from the last R to the first R within the "RB"-combination of rank n, the function $g(n) = n^2 - n + 1$ may also be considered mapping the rank n of an "RB"-combination to the position of the first R within it. One can check to see that $g(1) = 1$, $g(2) = 3$, $g(3) = 7$, $g(4) = 13$, $g(5) = 21$ – familiar labels originally having no frame of reference. Now a quadratic function is their frame of reference which was developed through generalization. As an aside, note that this development is an example when solving a more general problem is easier that solving its special case [14]. Thereby, this section may be concluded with a remark that one of the goals of generalization in mathematics, in general, and in the context of this paper, in particular, is to assist one in understanding and resolving special cases.

4. Alternative responses to the question seeking information

The content of pre-college mathematics and its methods of teaching are not in the relation of dichotomy when methods do not depend on content and content has little (if any) agency for the methods. It is the diversity of methods of teaching mathematics used by a teacher stems from their knowledge of content. Moreover, the appreciation of this diversity opens a window to teaching and learning a new content. This vision of the relationship between content and methods implies that multiple ways of solving a problem lead to new concepts and ideas about their use. With this in mind, another way to explore the teacher candidate's pattern (Figure 1) is to note that the pattern can be presented as a continuous string of digits 11223344 .... The digits in the string do not represent positions of colors in the pattern; rather, the digits, going in pairs, represent the ranks of "RB"-combinations where the repetitions occur. Nonetheless, this transition is designed to demonstrate the efficiency of numbers in describing the behavior of various non-numeric patterns and it enables the study of visual patterns by the tools and methods of mathematics. Towards this end, in the attempt to answer the original question about the color of the 100th place, one can simply add the digits in the string until the sum reaches the largest number smaller than 100 (alternatively, the smallest number larger than 100). Noting that each digit enters the string twice, one can come across twice the sum of the first nine natural numbers $2 \cdot (1+2+\cdots+9) = 90$ in which the number 9 (in the left-hand side of the last equality) points at the number of R's and B's in the 9th two-color combination and the 90th position has blue color. Because the 10th "RB"-combination has ten R's, once again, one can conclude that the 100th place in the pattern of Figure 1 is colored red.

One can also see that by adding the odd number of digits beginning from the first digit, one can get the positions of the last R's in the pattern. In doing so, one adds each time an odd number to the previous sum: 1, $1 + (1 + 2) = 4$, $4 + (2 + 3) = 9$, $9 + (3 + 4) = 16$, $16 + (4 + 5) = 25$, $\ldots$, $81 + (9 + 10) = 100$, ... In this representation, one can recognize a well-known fact that the sum of consecutive odd numbers starting from one is always a square number. That is, once again, the 100th place can be recognized to be filled with the last R in the "RB" combination of rank 10.

The last two solutions were based on very important concepts of number theory: triangular numbers $t_n = n(n + 1)/2$ as the partial sums of consecutive natural numbers, and square numbers $s_n = n^2$ as the partial sums of consecutive odd numbers. Both triangular and square numbers are special cases of the so-called polygonal (or figurate) numbers. In order to understand how polygonal numbers develop, note that the sequences of consecutive natural numbers and consecutive odd numbers are arithmetic sequences with the differences one and two, respectively; both starting from the number 1. This interesting connection of arithmetic sequences to polygonal numbers known from the 3rd century mathematical work by Diophantus [8] prompts one to use several other arithmetic sequences with differences three, four, and so on in order to consider other polygonal numbers including pentagonal numbers $p_n = (3n - 1)n/2$ as the partial sums of the sequence

1, 4, 7, 10, 13, ... , $3n-2$, ... (an arithmetic sequence with difference three) and hexagonal numbers $h_n = (2n-1)n$ as the partial sums of the sequence 1, 5, 9, 13, 17, ... , $4n-3$, ... (an arithmetic sequence with difference four). Indeed, $p_n = \frac{1+(3n-2)}{2}n = \frac{(3n-1)n}{2}$ and $h_n = \frac{1+(4n-3)}{2}n = (2n-1)n$. One can see that the difference d of an arithmetic sequence is connected to its geometric characteristic m (the number of sides in a polygon that the numbers represent) through the formula $d = m-2$.

Keeping in mind that "Generalities without interesting particular cases are of little value" [15, p. 103], one can proceed to consider such a generality in the form of the arithmetic sequence 1, $1+d$, $1+2d$, $1+3d$, ... , $1+(n-1)d$, ... , with the first term one and difference d. The sum of its first n terms is equal to $\frac{1}{2}(1+[1+d(n-1)])$ and can be referred to as the polygonal number of rank n and side $m = d+2$ (or the m-gonal number of rank n). Using the notation $P(n,m)$ for such a number yields the formula

$$P(n,m) = \frac{[(m-2)(n-1)+2]n}{2}. \tag{1}$$

This connection of the pattern of Figure 1 to polygonal numbers, made possible through alternative responses to the basic question, motivates the pattern's extension and generalization to include these numbers as the pattern guides. For instance, the sequence of natural numbers served as a guide for the pattern offered by the teacher candidate (Figure 1). Another direction towards generalization is to consider the case of more than two colors, although unlike the case of the polygons the number of sides of which is not limited, the number of colors, although large, is limited. Thereby, this section may be concluded with a remark that one of the goals of generalization in mathematics is the development of new concepts and tools of mathematical exploration.

5. Generalizing towards p-color patterns guided by m-gonal numbers

To begin, consider the case of p colors, $p \geqslant 2$, and the pattern

$$\underbrace{C_1C_2\ldots C_p}_{p}\,\underbrace{C_1C_1C_1\ldots C_pC_pC_p}_{3p}\,\underbrace{\underbrace{C_1\ldots C_1}_{6}\ldots\underbrace{C_p\ldots C_p}_{6}}_{6p}\ldots\underbrace{\underbrace{C_1\ldots C_1}_{\frac{n(n+1)}{2}}\ldots\underbrace{C_p\ldots C_p}_{\frac{n(n+1)}{2}}}_{\frac{n(n+1)}{2}p}\ldots \tag{2}$$

in which the sequence of triangular numbers 1, 3, 6, ... , $\frac{1}{2}n(n+1)$ serves as the pattern guide; that is, the number of C_k's (i.e., colors) in this p-color combination varies according to this sequence. As the numbers 3 and 6 are triangular numbers of ranks two and three, respectively, we see three and six C_1's in the second and the third p-color combinations of pattern (2). This time (i.e., in the context of generalization), let us focus on the position of the first color C_1 (just because the first comes first). The first positions of color C_1 within pattern (2) guided by the triangular numbers follow the sequence 1, $p+1$, $4p+1$, $10p+1$, $20p+1$, $\ldots$, where the coefficients in p are the partial sums of consecutive triangular numbers; namely, one (1) in $p+1$, four $(1+3)$ in $4p+1$, ten $(1+3+6)$ in $10p+1$, and so on. This observation can be proved easily by mathematical induction. Indeed, assuming that the n-th coefficient in p in the sequence $p+1$, $4p+1$, $10p+1$, $20p+1$, $\ldots$, is the n-th partial sum of consecutive triangular numbers, $\sum_{k=1}^{n} \frac{1}{2}k(k+1)$, one has to prove that adding the triangular number $\frac{1}{2}(n+1)(n+2)$ in the transition from the combination of rank n to the combination of rank $n+1$ yields

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} + \frac{(n+1)(n+2)}{2} = \sum_{k=1}^{n+1} \frac{k(k+1)}{2}.$$

The last relation is obvious. Note that the n-th partial sum of triangular numbers is represented by the third-degree polynomial in n as the n-th partial sum of any quadratic sequence ak^2+bk+c, $k=1,2,3,\ldots$, is the cubic polynomial in n. Indeed, using the formulas $1+2+\cdots+n = \frac{1}{2}n(n+1)$ and $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$, yields

$$\begin{aligned}\sum_{k=1}^{n}(ak^2+bk+c) &= a\sum_{k=1}^{n} k^2 + b\sum_{k=1}^{n} k + c\sum_{k=1}^{n} 1\\ &= a\,\frac{n(n+1)(2n+1)}{6} + b\,\frac{n(n+1)}{2} + cn = \frac{a}{3}\,n^3 + \cdots .\end{aligned}$$

Note that, so far, all activities were free from the use of technology and they may be considered as TI activities. The next step requires rather complicated symbolic computations including solving systems of four linear equations and finding partial sums of polygonal numbers of arbitrary rank and side. While in the pre-digital era such problems were solvable, though not by many students of mathematics, nowadays, in the true spirit of TITE problem solving, these computations can be supported by *Wolfram Alpha*.

6. A TE part of problem-solving activities begins

Let $f_p(n,3)$ describe the first position of color C_1 in the n-th combination of the p-color pattern guided by the triangular numbers. Then $f_p(n,3) = an^3+bn^2+cn+d$, where the coefficients a,b,c, and d depend on p. In order to find these (four) coefficients, one has to plug the first four values of n into $f_p(n,3)$ and solve the following system of four linear equations

$$\begin{aligned} a+b+c+d &= 1, & 27a+9b+3c+d &= 1+4p,\\ 8a+4b+2c+d &= 1+p, & 64a+16b+4c+d &= 1+10p. \end{aligned}$$

Using *Wolfram Alpha* in solving the system of equations in a,b,c, and d, yields $a=\frac{p}{6}$, $b=0$, $c=-\frac{p}{6}$, $d=1$. That is, $f_p(n,3) = \frac{p}{6}\,n^2 - \frac{p}{6}\,n+1$. As a TI alternative to the use of *Wolfram Alpha*, one can note that $f_p(n,3) = \big(\sum_{k=1}^{n} \frac{1}{2}k(k-1)\big)p+1$

and, therefore, using formulas mentioned at the end of the last section,

$$f_p(n,3) = \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right)p + 1 = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]p + 1$$
$$= \frac{p}{6}n^3 - \frac{p}{6}n + 1.$$

One can check to see that generalization to $f_p(n,3)$ is consistent with the known special cases. Indeed, $f_p(1,3) = 1$, $f_p(2,3) = p+1$, $f_p(3,3) = 4p+1$, $f_p(4,3) = 10p+1$.

The next step is to use formula (1) and *Wolfram Alpha* in finding the partial sums of polygonal numbers as shown in Figure 2. This yields the formula

(3) $$\sum_{k=1}^{n} P(k,m) = \frac{1}{6}n(n+1)[m(n-1) - 2n + 5].$$

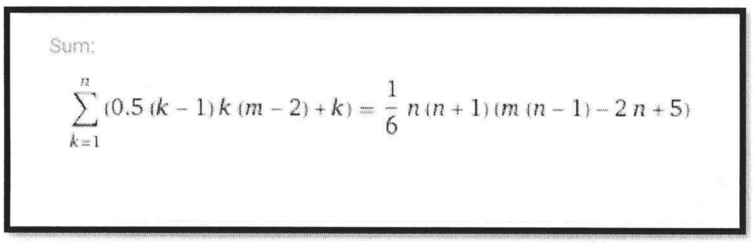

Figure 2. Finding partial sums of m-gonal numbers using *Wolfram Alpha*.

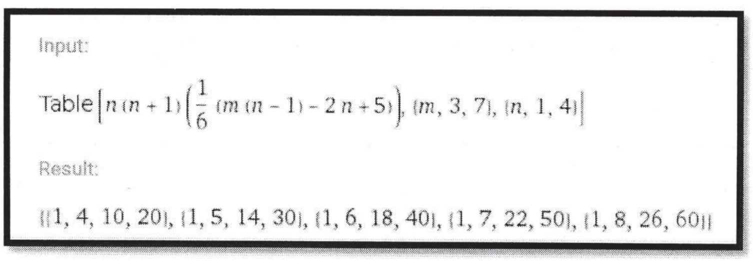

Figure 3. Finding the values of x for the equations $f_p(n,m) = 1 + xp$.

Formula (3) can be used to find a few coefficients in p similar to how they were found in a TI fashion for triangular numbers. To this end, one can enter into the input box of *Wolfram Alpha* the (three-component) expression "Table $\left[\frac{1}{6}n(n+1)\big((m(n-1) - 2n+5)\big),\ \{m,3,7\},\ \{n,1,4\}\right]$", in which, using *Wolfram* programming code, the second and the third components of the command Table are aimed at instructing the program to create a table of four n values, [1, 4], for each of the five m values, [3, 7]. As shown in Figure 3, the Result line includes five quadruples of integers each of which provides the first four coefficients in p for each of the five pattern guides – triangular, square, pentagonal, hexagonal, and heptagonal numbers. These coefficients will be used in developing special cases for the pattern guides towards the end of finding a formula for the general m-gonal pattern guide.

7. Collecting data for special cases as a TE activity

In this section it will be demonstrated how *Wolfram Alpha* can be used to develop the third degree polynomials in n that map n to the first position of the first color in the p-color combination of rank n guided by the sequences of m-gonal numbers ($m = 4, 5, 6, 7$). The special cases of these polynomials (along with the case $m = 3$) will then be generalized, considering the generalization activity as a TI part of problem solving.

To begin, let $f_p(n, 4) = an^3 + bn^2 + cn + d$ describe the first position of color C_1 in the n-th combination of the p-color pattern guided by square numbers, where, once again, the coefficients a, b, c, and d depend on p. Using the second quadruple generated by *Wolfram Alpha* in Figure 3, yields the system of equations

$$\begin{aligned} a + b + c + d &= 1, & 27a + 9b + 3c + d &= 1 + 5p, \\ 8a + 4b + 2c + d &= 1 + p, & 64a + 16b + 4c + d &= 1 + 14p. \end{aligned}$$

Once again, *Wolfram Alpha* provides the following solution: $a = \frac{p}{3}$, $b = -\frac{p}{2}$, $c = \frac{p}{6}$, $d = 1$. That is, $f_p(n, 4) = \frac{p}{3}\, n^3 - \frac{p}{2}\, n^2 + \frac{p}{6}\, n + 1$.

Let $f_p(n, 5) = an^3 + bn^2 + cn + d$ describe the first position of color C_1 in the n-th combination of the p-color pattern guided by pentagonal numbers. Using the third quadruple generated by *Wolfram Alpha* in Figure 3, yields the system of equations

$$\begin{aligned} a + b + c + d &= 1, & 27a + 9b + 3c + d &= 1 + 6p, \\ 8a + 4b + 2c + d &= 1 + p, & 64a + 16b + 4c + d &= 1 + 18p. \end{aligned}$$

Wolfram Alpha provides the following solution: $a = \frac{p}{2}$, $b = -p$, $c = \frac{p}{2}$, $d = 1$. That is, $f_p(n, 5) = \frac{p}{2}\, n^3 - pn^2 + \frac{p}{2}\, n + 1$.

Let $f_p(n, 6) = an^3 + bn^2 + cn + d$ describe the first position of color C_1 in the n-th combination of the p-color pattern guided by hexagonal numbers. Then, using the fourth quadruple generated by *Wolfram Alpha* in Figure 3, yields the system of equations

$$\begin{aligned} a + b + c + d &= 1, & 27a + 9b + 3c + d &= 1 + 7p, \\ 8a + 4b + 2c + d &= 1 + p, & 64a + 16b + 4c + d &= 1 + 22p. \end{aligned}$$

Using *Wolfram Alpha*, the following solution follows: $a = \frac{2}{3}p$, $b = -\frac{3}{2}p$, $c = \frac{5}{6}p$, $d = 1$. That is, $f_p(n, 6) = \frac{2p}{3}\, n^3 - \frac{3p}{2}\, n^2 + \frac{5p}{6}\, n + 1$.

Finally, let $f_p(n, 7) = an^3 + bn^2 + cn + d$ describe the first position of color C_1 in the n-th combination of the p-color pattern guided by heptagonal numbers.

Then, using the fifth quadruple generated by *Wolfram Alpha* in Figure 3, yields the system of equations

$$a+b+c+d=1, \qquad 27a+9b+3c+d=1+8p,$$
$$8a+4b+2c+d=1+p, \qquad 64a+16b+4c+d=1+26p.$$

Using *Wolfram Alpha*, the following solution follows: $a=\frac{5}{6}p$, $b=-2p$, $c=\frac{7}{6}p$, $d=1$. That is, $f_p(n,7)=\frac{5p}{6}n^3-2pn^2+\frac{7p}{6}n+1$. As an aside note that in order for generalization to take place, special cases have to be developed. But generalizing from special cases through empirical induction would require the use of mathematical induction.

8. TI part of the activities begins

In order to generalize to $f_p(n,m)$, all five functions found in the previous two sections, namely,

$$f_p(n,3)=\frac{p}{6}n^3-\frac{p}{6}n+1, \quad f_p(n,4)=\frac{p}{3}n^3-\frac{p}{2}n^2+\frac{p}{6}n+1,$$
$$f_p(n,5)=\frac{p}{2}n^3-pn^2+\frac{p}{2}n+1, \quad f_p(n,6)=\frac{2p}{3}n^3-\frac{3p}{2}n^2+\frac{5p}{6}n+1,$$
$$f_p(n,7)=\frac{5p}{6}n^3-2pn^2+\frac{7p}{6}n+1$$

have to be jointly compared. The goal of such comparison is to find coefficients of the cubic polynomial $f_p(n,m)=an^3+bn^2+cn+d$ as the functions of p and m.

Analyzing the coefficients of the above five functions in terms of their generalization from the cases $m=3,4,5,6,7$ to the general case of an m-gonal number, the following technology-immune result can be derived

$$f_p(n,m)=\frac{1}{6}(m-2)pn^3-\frac{1}{2}(m-3)pn^2+\frac{2m-7}{6}pn+1. \tag{4}$$

Indeed, regardless of m, we have the free term $d=1$. Furthermore, when $m=3$ we have $a=\frac{p}{6}$, when $m=4$ we have $a=\frac{p}{3}$, when $m=5$ we have $a=\frac{p}{2}$, when $m=6$ we have $a=\frac{2p}{3}$, and when $m=7$ we have $a=\frac{5p}{6}$. One can see that the values of the coefficient a are the $(m-2)$ multiples of $\frac{p}{6}$.

Likewise, when $m=3$ we have $b=0$, when $m=4$ we have $b=-\frac{p}{2}$, when $m=5$ we have $b=-p$, when $m=6$ we have $b=-\frac{3p}{2}$, and when $m=7$ we have $b=-2p$. One can see that the values of the coefficient b are the $(3-m)$ multiples of $\frac{p}{2}$.

Finally, when $m=3$ we have $c=-\frac{p}{6}$, when $m=4$ we have $c=\frac{p}{6}$, when $m=5$ we have $c=\frac{p}{2}$, when $m=6$ we have $c=\frac{5p}{6}$, and when $m=7$ we have $c=\frac{7p}{6}$. One can see that the values of the coefficient c are the $(2m-7)$ multiples of $\frac{p}{6}$.

Formula (4) was developed through empirical induction which, lacking rigor, may lead to erroneous generalization. A rigor requires proof by mathematical induction by the variable m that consists in demonstrating the transition from m to $m+1$ (the base clause $m=3$ yields $f_p(n,3)=\frac{p}{6}\,n^3-\frac{p}{6}\,n+1$). To this end note that the positions of the first C_1's in the p-color combinations of rank n guided, by $(m+1)$-gonal numbers and m-gonal numbers, respectively, differ by

$$\begin{aligned} p\sum_{i=1}^{n-1}[P(i,m+1)-P(i,m)] &= \frac{p}{2}\sum_{i=1}^{n-1}[(m-1)(i-1)+2-(m-2)(i-1)-2]i \\ &= \frac{p}{2}\sum_{i=1}^{n-1}(i-1)i=\frac{p}{2}\left[\sum_{i=1}^{n-1}i^2-\sum_{i=1}^{n-1}i\right] \\ &= \frac{p}{2}\left[\frac{(n-1)n(2n-1)}{6}-\frac{(n-1)n}{2}\right]=\frac{p}{6}n(n-1)(n-2). \end{aligned}$$

That is, one has to make $pn(n-1)(n-2)/6$ steps in order to reach the first C_1 in the p-color combination of rank n guided by $(m+1)$-gonal numbers starting from the first C_1 in the p-color combination of rank n guided by m-gonal numbers. Therefore, in order to prove formula (4) one has to show that in the course of the transition from m to $m+1$ the relation $f_p(n,m+1)-f_p(n,m)=pn(n-1)(n-2)/6$ holds true. Indeed,

$$f_p(n,m+1)-f_p(n,m)=\frac{pn^3}{6}-\frac{pn^2}{2}+\frac{pn}{3}=\frac{p}{6}\,n(n^2-3n+2)=\frac{p}{6}\,n(n-1)(n-2).$$

This completes the proof of formula (4).

Let $g_p(n,m)$ represent the function that maps n to the position of the last C_1 in the p-color combination of rank n. Then, because there are $[(m-2)(n-1)+2]n/2$ of C_1's in this combination,

$$\begin{aligned} g_p(n,m) &= f_p(n,m)+\frac{(m-2)(n-1)+2}{2}\,n-1 \\ &= \frac{(m-2)pn^3}{6}-\frac{(m-3)pn^2}{2}+\frac{(2m-7)pn}{6}+\frac{m(n-1)}{2}-n+2. \end{aligned}$$

One can check to see that $g_p(3,2)=3p+1$, whence $g_2(3,2)=7$ (cf. Figure 1).

Let $h_p(n,m)$ be a function that maps n to the position of the last C_p in the n-th p-color combination. Then

$$h_p(n,m)=f_p(n+1,m)-1=\frac{1}{6}(m-2)p(n+1)^3-\frac{1}{2}(m-3)p(n+1)^2+\frac{2m-7}{6}\,p(n+1).$$

One can check to see that $h_p(3,3)=\frac{64}{6}\,p-\frac{4}{6}\,p=10p$ – the number which is smaller by one than the position of C_1 in the p-color combination of rank four mentioned above. Note that the correctness of formula (4) implies the correctness of formulas for $g_p(n,m)$ and $h_p(n,m)$.

9. Using TI results in support of TE activities

Formula (4) makes it possible to carry out the next step in developing TITE activities. As was mentioned above, one goal of generalization is to assist in understanding and resolving specials cases. This agency of generalization was noted by Pólya [14] in the pre-digital era. Yet the idea was in the possibility to parametrize a problem situation, make a solution dependent on a parameter and then set the needed value of the parameter in order to obtain the solution for a special case. Nowadays, this parameter-extended agency of generalization can be entertained by constructing an interactive computational environment in which two parameters, p and m, can be taken into consideration and specific cases can be resolved by plugging in their particular numeric values and watching for computational response of the environment.

For example, one may consider a three-color pattern guided by square numbers (the first two combinations of which look like

$$C_1C_2C_3C_1C_1C_1C_1C_2C_2C_2C_2C_3C_3C_3C_3)$$

and ask the computer to provide information about the color of a sufficiently large place. As shown in Figure 4, in such a pattern, the 1155-th position has color C_3 (the number 3 in cell K3 of the spreadsheet). In this section, TI activities of the previous section that led to generalization, will be used to support the creation of a tool to carry out a purely TE activities. At the same time, new TI activities would be needed before the TE part becomes available.

	A	B	C	D	E	F	G	H	I	J	K
1		p		position N		f(n)	f(n+1)-1	f(n+1)-1-f(n)	N-f(n)	ratio	color number
2		3									
3		m		1155		856	1155	299	299	1	3
4		4									
5											
6	1	1	4	0	0						
7	2	4	16	0	0						
8	3	16	43	0	0						
9	4	43	91	0	0						
10	5	91	166	0	0						
11	6	166	274	0	0						
12	7	274	421	0	0						
13	8	421	613	0	0						
14	9	613	856	0	0						
15	10	856	1156	0	1	1					
16	11	1156	1519	1	1						
17	12	1519	1951	1	1						
18	13	1951	2458	1	1						

Figure 4. Generalization at work.

In order to create such a computational tool, one can use the values of m and p as parameters and determine the color of the N-th position in the pattern defined by those parameters. The environment shown in the spreadsheet of Figure 4 is organized as follows. Cell B2 is given name p and entered with the number of colors; cell B4 is given name m and entered with the side of the polygonal numbers that guide the pattern; cell D3 is given name N and entered with the position number the color of which has to be determined. Next, column A beginning from cell A6 is filled with consecutive natural numbers starting from one. In columns B and C the values of $f_p(n,m)$ and $f_p(n+1,m)$ (the first positions of color C_1 in the combinations of ranks n and $n+1$, respectively) are defined. In columns D and E the values of $f_p(n,m)$ and $f_p(n+1,m)$ are compared with the value of N and if $f_p(n,m)$ is smaller than or equal to N, the number zero (otherwise, one) is displayed in column D and if $f_p(n+1,m)$ is smaller than N, the number zero (otherwise, one) is displayed in column E.

Consequently, in column F the numbers from columns D and E are added and when the sum is equal to one (this could happen one time only), the number 1 is displayed, otherwise a cell is left blank. The next step is to display in cells F3 and G3, respectively, the values of $f_p(n,m)$ and $f_p(n+1,m)-1$ (the place of the last C_p in the n-th combination) that are located in the same row as the number 1. Such display can be done through the use of the spreadsheet function LOOKUP which has three arguments: the number to be looked up (in our case, the number 1), the range where a looked-up number is located, and the range where a number to be displayed is located. Thus, in cells F3 and G3, respectively, the spreadsheet functions =LOOKUP(1, F6:F100, B6:B100) and =LOOKUP(1, F6:F100, C6:C100) $-$ 1 are defined. The length of the n-th combination within which the place N is located is displayed in cell H3 by calculating the difference $[f_p(n+1,m)-1]-f_p(n,m)$. The value $N-f_p(n,m)$ is calculated in cell I3 and the ratio

$$r=\frac{N-f_p(n,m)}{[f_p(n+1,m)-1]-f_p(n,m)}$$

is calculated in cell J3. Finally, the number $\mathrm{INT}(rp)+1$, when N is not the last position of the combination of rank n; i.e., $N\neq f_p(n+1,m)-1$, otherwise, the number $\mathrm{INT}(rp)$, is the color of the position of N in the pattern. For example, in the case $p=2$ and $m=3$, as was mentioned above, the positions of color C_1 follow the pattern 1, $p+1$, $4p+1$, $10p+1$, $20p+1$, … ; that is, in the case $p=2$ we have the sequence 1, 3, 9, 21, 41, … , so that the 23rd position has color C_1. As 23 is not the last position in a combination, we have $r=\frac{23-21}{40-21}=\frac{2}{19}<1$ and $\mathrm{INT}(rp)+1=\ \mathrm{INT}(4/19)+1=1$ is the color number.

To explain how either the value of $\mathrm{INT}(rp)+1$ or $\mathrm{INT}(rp)$ provide the value of color number, let $N\in[f_p(n,m),f_p(n+1,m)-1]=[x_0,x_p]$. Let us divide the segment $[x_0,x_p]$ into p equal parts by $x_0<x_1<\cdots<x_{i-1}<x_i<\cdots<x_p$. The inclusion $N\in[x_{i-1},x_i]$, $1\leqslant i\leqslant p$ implies that the segment $[x_{i-1},x_i]$ is filled with color C_i. The ratio $r=\dfrac{N-x_0}{x_p-x_0}$, $0\leqslant r\leqslant 1$, represents the fraction of the segment $[x_0,x_p]$ that spans from x_0 to N. Then

$$\begin{aligned}pr&=\frac{p(N-x_0)}{x_p-x_0}=\frac{p(x_{i-1}+N-x_{i-1}-x_0)}{p(x_1-x_0)}=\frac{(i-1)(x_1-x_0)+N-x_{i-1}}{x_1-x_0}\\&=i-1+\frac{N-x_{i-1}}{x_1-x_0}=i-1+r_1,\quad 0\leqslant r_1\leqslant 1.\end{aligned}$$

If $N=x_i$ then $r_1=1$ and $\mathrm{INT}(pr)=i$; that is, the N-th place has color C_i. If

$N \neq x_i$ then $0 \leqslant r_1 < 1$ and $\mathrm{INT}(pr) = i - 1$ implying that $i = \mathrm{INT}(pr) + 1$ and the inclusion $N \in [x_{i-1}, x_i]$ means that the position N has color C_i.

10. Conclusion

This paper was written as a reflection on a single episode from the author's experience working with elementary teacher candidates within a mathematics content and methods course in a master's degree program for teachers of ages 6–11 students. The main emphasis of the paper was on the notion of TITE mathematics curriculum which rises in importance with the growth in sophistication of modern technological tools. The paper's topics included three major contexts associated with TITE problem solving: pedagogy, mathematics, and technology.

In the context of pedagogy, the paper underlined the significance of paying attention to students' ideas in the classroom and it asserted that one's thinking "outside the box" has potential for an extended mathematical discussion that bears fruit. Instead of ignoring ideas that such thinking entails, its appreciation by 'a more knowledgeable other' can lead to learning at a much higher level than it was originally anticipated. This pedagogy is especially valuable for a mathematics classroom of teacher candidates. In such a classroom, prospective teachers can learn about inherent connectivity of mathematical concepts enabling a simple idea to be used as springboard into the exploration of more complicated ideas. However, using this springboard effectively requires from a teacher to possess intellectual courage of entering an uncharted territory where a border between known and unknown is quite elusive. Perhaps the most important application of this pedagogy to mathematics teacher education is to prepare teacher candidates to answer students' questions because answering and asking questions is the major vehicle of conceptual development. Once a question is answered and the answer is assimilated, a new question is likely to be asked so that the learning process continues in the solve-reflect-pose recursive mode of learning mathematics [17].

In the context of mathematics, the paper focused on the importance of generalization as a means for understanding special cases, for the development of new tools of investigation, and for creating computational environments capable of problem solving and problem posing. Learning mathematical generalization is important for it provides "ready ascent from particular facts to generalizations and ready descent from generalizations to particular facts" [15, p. 103]. At the same time, generalization requires careful reasoning and understanding the difference between empirical induction which is based on specific cases and mathematical induction which provides generalization with a rigorous proof. It was shown how each step towards generalization required a combination of TE and TI activities in the sense that a TE activity requires a TI support and a TI activity can benefit from a TE support.

In the context of technology, the paper demonstrated integrative power of two disparate digital tools. *Wolfram Alpha* was shown as a tool capable of symbolic computations that, in particular, facilitates access of *all* students to advanced mathematics, a long-term program initiated at the end of the 20th century with the advent of computers into K-16 mathematics curricula [10]. A computer spreadsheet was shown as an instrument created (from an artifact) through the activity of generalization. According to the theory of instrumental genesis [16], the TE part of this activity can be described as a process through which the spreadsheet broadens the scope of utilization, and its TI part can be described as a process through which a user of the spreadsheet develops intellectually. The construction of such instruments for solving problems that are far beyond an original task which was at the origin of this construction contributes to the pragmatic dimension of commonly available tools integrated into the practice of the modern-day classroom [18].

It is worth noting that *Wolfram Alpha*, unlike the spreadsheet, was shown as a ready to be used instrument capable of solving multi-variable systems of linear equations constructed through a TI process and the corresponding TE part required knowledge of what the instrument can do mathematically. One can say that *Wolfram Alpha* was inserted between its user and mathematics in order for the latter to work. On the contrary, mathematics was inserted between the user and the spreadsheet in order for the latter to work. From the instrumental perspective with its origin in the seminal ideas by Vygotsky [20] about mediating cognitive processes by technical devices and psychological tools, one can see an interesting relationship between mathematics and technology. When technology is just an artifact, like a spreadsheet, it is mathematics that turns it into an instrument to mediate problem solving. When technology is an instrument, like *Wolfram Alpha*, such instruments mediate the application of mathematical methods to problem solving. In that way, a TITE mathematical problem solving may include multiple digital tools, both artifacts and instruments, with different relation to mathematics. This duality in the order of positioning and using theoretical and technological knowledge in the context of TITE activities appears to be an important pedagogical aspect of the modern-day mathematical learning. Paying attention to students' ideas makes the learning student-centered and encourages creativity as the foundation of success both within and outside mathematics.

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