THE TEACHING OF MATHEMATICS
2020, Vol. XXIII, 1, pp. 17–19

A NOTE ON BRAUER'S THEOREM

Aaron Melman

Abstract. We present an elementary proof of Brauer's theorem, which shows how knowledge of an eigenpair can be used to change a single eigenvalue of a matrix.

MathEduc Subject Classification: H65

MSC Subject Classification: 97H60, 15A18

Key words and phrases: Eigenvalue; eigenvector; rank-one modification; Brauer.

An interesting theorem by Alfred Brauer from 1952 allows one to use an eigenpair of a matrix to create a new matrix whose spectrum differs in only one eigenvalue, which can be chosen freely. It is often used in deflation techniques when computing eigenvalues [3, 4.2]. We propose a self-contained proof using only basic concepts.

Before we do, we need a little background and notation. Throughout, matrices will be square and complex, and we denote by A^* the Hermitian conjugate of the matrix A, i.e., $A^* = \bar{A}^T$. If $Av = \lambda v$, then v is an eigenvector with eigenvalue λ, also referred to as a *right eigenvector*, and if $w^*A = \mu w^*$, then w is a *left eigenvector* with eigenvalue μ. The left and right eigenvalues of a matrix are the same, but this is not necessarily true for the corresponding eigenvectors. Since $\lambda w^*v = w^*(\lambda v) = w^*Av = \mu w^*v$, $\lambda \neq \mu$ implies that $w^*v = 0$, i.e., left and right eigenvectors belonging to different eigenvalues are orthogonal. This property is called biorthogonality [2, 7.9].

Informally, Brauer's theorem states that, when (λ, v) is an eigenpair of $A \in \mathbb{C}^{n\times n}$ and $u \in \mathbb{C}^n$ is arbitrary, then the matrix $A - vu^*$ has the same eigenvalues as A, except for λ, which is replaced by $\lambda - u^*v$. When λ is a simple eigenvalue, the proof is short and straightforward. A more subtle approach is required when it is not.

The original proof in [1] relies, as does our proof, on the biorthogonality property, but proves a slightly weaker result. This is clarified in a remark after our proof below. The proof in [2, p. 51] explicitly uses the characteristic polynomial of $A - vu^*$, and shows that it can be factored, which involves the adjoint of the matrix and properties of determinants, while the proof in [2, p. 122] uses Schur triangularization in which a unitary matrix is used to triangularize $A - vu^*$, allowing the eigenvalue $\lambda - u^*v$ to be split off. These are the proofs that are typically cited in the literature.

Here, we provide a different but elementary proof based on the biorthogonality property and the trace of a matrix. Our intention is not to find the shortest proof (ours is somewhat longer than the aforementioned proofs), but rather to present a nontraditional one. A view from a different angle is always a useful tool in the classroom.

Theorem [1]. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, let v_k be an eigenvector associated with λ_k, and let $u \in \mathbb{C}^n$ be arbitrary. Then the matrix $B = A - v_k u^*$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - u^* v_k, \lambda_{k+1}, \ldots, \lambda_n$.*

Proof. We start by introducing the basic spectral properties of the matrices A and B, along with some notation. For convenience, we set $\lambda = \lambda_k$, $v = v_k$, and relabel the eigenvalues of A as $\mu_1, \ldots, \mu_s, \lambda, \ldots, \lambda$, where the (algebraic) multiplicity of λ is $n - s$, $0 \le s \le n-1$, and $\mu_j \ne \lambda$ for all j. If all the eigenvalues of A are equal to λ, we assign the value $s = 0$, and, throughout, adopt the convention that a quantity with a nonpositive subscript is not present. Define $B = A - vu^*$ and let w_j be a left eigenvector of A corresponding to one of the eigenvalues μ_j. Since $\mu_j \ne \lambda$ implies $w_j^* v = 0$, we obtain $w_j^* B = w_j^* A - w_j^* v u^* = w_j^* A = \mu_j w_j^*$, which means that μ_j is also an eigenvalue of B. Moreover, $Bv = Av - vu^*v = (\lambda - u^*v)\, v$, i.e., $\lambda - u^*v$ is an eigenvalue of B.

In view of the above, we can label the eigenvalues of B as $\nu_1, \ldots, \nu_{s'}, \lambda - u^*v, \ldots, \lambda - u^*v$, where the multiplicity of $\lambda - u^*v$ is $n - s'$, $0 \le s' \le n-1$, and $\nu_j \ne \lambda - u^*v$ for all j. Analogously, the eigenvalues ν_j of B are also eigenvalues of A.

We now define ℓ, with $0 \le \ell \le s$, as the largest integer such that the eigenvalues $\mu_1, \ldots, \mu_\ell$ of A are equal to the eigenvalues $\nu_1, \ldots, \nu_\ell$ of B, after reordering, with the remaining $\mu_{\ell+1}, \ldots, \mu_s$ necessarily all being equal to $\lambda - u^*v$, since they must be eigenvalues of B, different from any ν_j. Likewise, the eigenvalues $\nu_{\ell+1}, \ldots, \nu_{s'}$ are necessarily all equal to λ since they must be eigenvalues of A, different from any μ_j. By our convention, the value $\ell = 0$ is assigned when none of them are equal. These eigenvalues $\mu_j = \nu_j \ne \lambda$, $\lambda - u^*v$ have the same multiplicities since if, for some index i, μ_i had a multiplicity larger than that of ν_i, then an infinitesimal perturbation of the "extra" eigenvalues μ_i would send them to $\lambda - u^*v$, which is impossible since they would also be infinitesimally close to $\mu_i = \nu_i \ne \lambda - u^*v$. A similar argument holds if the multiplicity were less with the roles of μ_i and ν_i reversed. A similar argument holds if the multiplicity were less with the roles of μ_k and ν_k reversed. This means that

$$\text{spectrum of } A = \{\mu_1, \ldots, \mu_\ell, \underbrace{\lambda - u^*v, \ldots, \lambda - u^*v}_{s-\ell}, \underbrace{\lambda, \ldots, \lambda}_{n-s}, \} \,,$$

$$\text{spectrum of } B = \{\mu_1, \ldots, \mu_\ell, \underbrace{\lambda, \ldots, \lambda}_{s'-\ell}, \underbrace{\lambda - u^*v, \ldots, \lambda - u^*v}_{n-s'}\} \,.$$

Until here, the proof is similar to Brauer's proof. Now, since trace$(B) =$

trace$(A) - u^*v$ and the trace of a matrix is the sum of its eigenvalues, we have that

$$\sum_{j=1}^{\ell} \mu_j + (s' - \ell)\lambda + (n - s')(\lambda - u^*v) = \sum_{j=1}^{\ell} \mu_j + (s - \ell)(\lambda - u^*v) + (n - s)\lambda - u^*v \ ,$$

and therefore $\big((n - s') - (s - \ell)\big)u^*v = u^*v$. If $u^*v = 0$, then the eigenvalues of A and B coincide and the proof follows. If $u^*v \neq 0$, then $n - s' = s - \ell + 1$, so that $s' - \ell = n - s - 1$, implying that

$$\text{spectrum of } B = \big\{\mu_1, \ldots, \mu_\ell, \underbrace{\lambda, \ldots, \lambda}_{n-s-1}, \underbrace{\lambda - u^*v, \ldots, \lambda - u^*v}_{s-\ell+1}\big\} \ ,$$

which are the eigenvalues of A with one of its eigenvalues λ replaced by $\lambda - u^*v$. ∎

We note that the theorem we have just proved is somewhat stronger than the theorem in [1], as the latter only states that, when $\lambda_k \neq \lambda_k - u^*v_k$, its multiplicity for the matrix $A - v_k u^*$ is less than its multiplicity for the matrix A. Our theorem here, which is the same as in [2] and elsewhere in the literature, states that its multiplicity is decreased by exactly one.

REFERENCES

[1] A. Brauer, *Limits for the characteristic roots of a matrix. IV: Applications to stochastic matrices*, Duke Math. J., **19** (1952), 75–91.

[2] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Second ed., Cambridge University Press, Cambridge, 2013.

[3] Y. Saad, *Numerical methods for large eigenvalue problems*, Classics in Applied Mathematics, 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

Department of Applied Mathematics, School of Engineering, Santa Clara University, Santa Clara, CA 95053, USA

E-mail: amelman@scu.edu