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COMBINING MATH IDEAS VIA PROBABILITY PROBLEMS

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Abstract. The paper presents a collection of problems from the Chernorizec Hrabar math tournament that takes place in Bulgaria during the last 28 years. The formulation of a question as a probability problem is used for different math ideas to be linked in a complex test unit. The complexity of the problem requires searching for the correct answer by applying not only advanced math knowledge and skills, but also a synthetic-type thinking, including some sense of mathematics. The multiple-choice format of the tournament problems allows students to skip a lot of routine work and to focus on the purified mathematical ideas. On one hand, the expected reasoning for the answer choice appears as an extract of the comprehensive problem solution. On the other hand, the reasoning forms the carcass of a sophisticated math task that challenges students synthetic competence.

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MSC Subject Classification: 97U40, 97K50

Key words and phrases: Probability test items; advanced students in mathematics; sense of mathematics; synthetic competence.

1. The ideology of the ChH math tournament

The Chernorizec Hrabar Tournament (ChH) is a competition for advanced students in mathematics [4]. The ChH competition papers often include multiple-choice test items that combine different math ideas. When designing a complex test item, we are searching for a proper shape that allows presenting an idea in such a form, which facilitates the links with the other ideas included in that test item. In other words, we are searching for a common denominator for different math ideas.

The ChH rules[1] also permit including math ideas beyond the curriculum. However, the argumentation of choosing the answer should stay in the scholar syllabus framework. The formulation of problems in a probability mode provides an opportunity to meet the listed above challenges. In the following, we will illustrate how it works in ChH.

[1]The competition papers for grades 9–12 of ChH are composed of 30 test items and the time allowed is 90 minutes. Any test item has one correct answer and 4 distractors. Scoring: the correct answer is worth 5, 7 or 9 points (depending on difficulty), leaving blank is worth 3 points, and there are no points given for a wrong answer.

2. Examples of combining school curricula topics

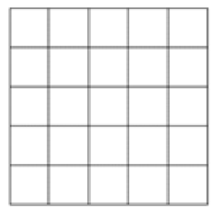
In this section, we are going to present a couple of examples, in which at least two different ideas are combined into one test-item. The coding of problems (YYYY-A,B) stands for (year-and-grades-from-A-to-B).[2]

2.1 Combinatorics to a probability

The classical probability is usually defined in combinatorics terms. This is why the transfer of combinatorial ideas into probability problem is common and easy. In the next example, combining two types of counting provide a quick way to the answer.

(2018-9,10) Rectangle composed of unit squares is randomly and equiprobably chosen in the grid on the right. What is the probability the area of the rectangle to occur not greater than 8?

(A) 8/25 (B) 16/225 (C) 32/625 (D) 64/625
(E) none of these



Reasoning.[3] The rectangle with area 8 must be 4×2 or 2×4 in size in order to be accommodated in the grid. The left-up unit square of the rectangle uniquely determines its location. There are 8 variants for choosing the left-up square of a 4×2 rectangle (this square is chosen among the unit squares, which form the 2×4 left-up rectangle in the grid). The case with choosing a 2×4 rectangle is similar. Therefore, the favorable outcomes are 16 in total.

We count the total number of outcomes as follows. Any rectangle in the figure could be determined by its diagonal. Choosing a vertex X, there are 25 options for the other endpoint Y of the diagonal (it could be any knot that does not lie on the lines of the grid passing through X). One can choose X among 36 knots, so we count 25×36 diagonals. In this number, any rectangle is counted 4 times (X could be any of the four vertices). Thus the total number of outcomes equals $\frac{25 \cdot 36}{4} = 225$.

The desired probability is $\frac{16}{225}$.

Answer (B).

Comment. Any type of the counting applied here has advantages in its particular case and is not very much appropriate for the other case. The reviewer of the article proposed another two ideas for finding the total number of outcomes n:

[2]The quoted problems from the ChH competition papers are published by the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences in a series of annual booklets, edited by the author. We skip the reference of them because of: a) saving space, and b) the booklets are in Bulgarian.

[3]As far as the ChH rules require just pointing the answer, we give reasoning instead of a complete rigorous solution. Sometimes even rejecting the distractors is a winning strategy [3].

1) $n = 25 + \frac{25\cdot 8}{2} + \frac{25\cdot 16}{4} = 225$, where: 25 is the number of unit squares; $\frac{25\cdot 8}{2}$ is the number of rectangles of the form $1 \times k$ or $k \times 1$, where $k > 1$; $\frac{25\cdot 16}{4}$ is the number of rectangles of the form $k \times l$, where $k > 1$, $l > 1$.

2) $n = 15^2$, where 15 is the number of possible orthogonal projections of the chosen rectangles on any side of the given square.

(2013-11,12)[4] Ten couples of graduating students went to the school ball. After the ball, any boy accompanied a girl to her home. What is the probability exactly 5 boys to go back with the same girl with whom they came to the ball?

(A) more than 0.5

(B) between 0.1 and 0.5

(C) between 0.01 and 0.05

(D) between 0.001 and 0.005

(E) below 0.001

Reasoning. The number of the possible couples on the way home is 10!, which is the total number of outcomes. One can choose 5 couples among 10 in $C_{10}^5 = \frac{10!}{(5\cdot 4\cdot 3\cdot 2\cdot 1)^2}$ ways. Suppose these are the loyal ones. Then the other 5 couples may be deranged in $!\,5 = 44$ ways. So, the number of the favorable outcomes is $C_{10}^5 \cdot !\,5 = \frac{10!}{(5\cdot 4\cdot 3\cdot 2\cdot 1)^2} \cdot 44$ and the desired probability equals

$$\frac{\dfrac{10!}{(5\cdot 4\cdot 3\cdot 2\cdot 1)^2}\cdot 44}{10!} = \frac{44}{120^2} = \frac{11}{120\cdot 30} = \frac{11}{3600}.$$

To choose the answer, one could roughly evaluate this fraction as $\frac{11}{3600} \approx \frac{1}{360}$, which is between $\frac{1}{1000}$ and $\frac{1}{200}$.

Answer (D).

Comment. The number of derangements is familiar for the advanced secondary school students. Usually it is calculated recurrently: the first two subfactorials are obviously $!\,1 = 0$, $!\,2 = 1$; than $\forall n \geqslant 3$

$$!\,n = (n-1)(!\,(n-1) + !\,(n-2)).$$

Complexity of the problem is in combining the advanced school combinatorics with performing some clever calculations and the ability to estimate the exact value that refers to the student's *sense-of-mathematics* [2].

2.2. Arithmetic to a probability

The divisibility ideas are quite suitable to be presented in a probability mode. In this case, the sample space usually consists of the remainders modulo a number given in advance.

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[4]This problem was submitted by Ivaylo Kortezov.

(2006-9,12) What is the probability the sum of 7 randomly chosen numbers among $\{1,2,3,4,5,6,7,8,9\}$ to occur a multiple of 3? (The choices are done without replacement.)

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{1}{5}$ (E) $\dfrac{1}{6}$

Reasoning. The sum $1+2+\cdots+9=45$ is divisible by 3. Hence, the sum of 7 among these numbers is a multiple of 3 iff the sum of the remaining 2 numbers is divisible by 3. The sum of 2 integers is divisible by 3 in two cases.

1) Both are multiples of 3, which happens when they are chosen among $\{3,6,9\}$ – 3 favorable outcomes.

2) One of the numbers gives remainder 1 modulo 3, the other one gives remainder 2 modulo 3. This happens iff the first number is chosen among $\{1,4,7\}$ and the second one among $\{2,5,8\}$ – 9 more favorable outcomes.

The total number of outcomes equals $C_9^2=36$ and the desired probability is $\frac{12}{36}=\frac{1}{3}$.

Answer (A).

Comment. The test-item examines the student's synthetic ability to combine basic arithmetic with some skills in manipulating sets: to replace a set of 7 elements (difficult to be manipulated) by its complement, which consists of 2 elements. After this, the student is expected to apply his skills in combinatorics.

(2003-11,12) Find $\lim\limits_{n\to\infty} p_n$ where p_n is the probability N^2-1 to be a multiple of 10, when N is randomly chosen among the first n naturals.

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$ (E) none of these

Reasoning. Let $n=10k+l$, $l\in\{0,1,2,3,\dots,9\}$.

1) If $l=0$ there are $2k$ numbers N, for which $N^2\equiv 1 \pmod{10}$.

2) If $l=9$ there are $2k+2$ numbers N, for which $N^2\equiv 1 \pmod{10}$.

3) If $1\leqslant l\leqslant 8$ there are $2k+1$ numbers N, for which $N^2\equiv 1 \pmod{10}$.

Hence, $p_n=\frac{2k+m}{10k+l}$, $m\in\{0,1,2\}$. Thus

$$\lim_{n\to\infty} p_n=\lim_{k\to\infty}\frac{2k+m}{10k+l}=\frac{2}{10}.$$

Answer (D).

Comment. This test-item evolves the ideas from the previous one. The limit allows the 10 cases to be unified into one asymptotic value. Ideologically, the result could be stated as: the probability for $N^2\equiv 1 \pmod{10}$, when N is randomly chosen natural, is about $\frac{1}{5}$. By the way, such formulation hints answering the question just with common sense reasoning: in 2 of 10 cases N^2 ends in 1 (!).

2.3. Basic algebra to probability

(2013-11,12) The numbers b and c are randomly chosen among the elements of the set $\{1, 2, 3, 4, 5\}$. What is the probability the equation $x^2 + bx + c = 0$ to have real roots?

(A) 0.24 (B) 0.32 (C) 0.36 (D) 0.48 (E) none of these

Reasoning. The set of the outcomes is described by the couples $(b; c)$, which are 25 in total. The number of the favorable outcomes could be obtained as follows: none when $b = 1$; 1 when $b = 2$; 2 when $b = 3$; 4 when $b = 4$; 5 when $b = 5$, that gives 12 in total. The desired probability equals $\frac{12}{25}$.

Answer (D).

Comment. This question requires routine knowledge and skills. However, a kind of synthetic competence is needed to resolve the situation in 3 minutes that is the average time for finding the answer at ChH.

2.4 Geometric series – a kind of classics in probability

(2016-11,12) The gamblers A and B throw a dice. A starts, then B throws, then A, again B and so on. The winner is the one who first throws a six. What is the chance A to win?

(A) $\frac{1}{6}$ (B) $\frac{7}{13}$ (C) $\frac{4}{7}$ (D) $\frac{7}{12}$ (E) none of these

Reasoning. The gambler A wins iff the six appears at a throw, which is odd in turn. The chance this to happen at the first throw is $\frac{1}{6}$. To occur a six for the first time at the third throw, there should be a different number thrown at the first two throws. The chance in this case equals $(\frac{5}{6})^2 \cdot \frac{1}{6}$. The chance the first six to appear at the 5th throw is $(\frac{5}{6})^4 \cdot \frac{1}{6}$, and so on. Thus, the chance of 'A to win' is

$$\frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^6 \cdot \frac{1}{6} + \cdots = \frac{1}{6} \cdot \frac{1}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11}.$$

Answer (E).

Comment. The test-item examines student's ability to describe mathematically a potentially infinite process. It is important to note that since the outcomes are infinitely many, they cannot be equally likely. Modelling the situation requires some computational thinking, which indicates high level of the student's synthetic competence.

3. Introducing geometrical probability

Any conceptual expansion is not welcome during a multiple-choice competition because of the very limited time during the competition, i.e., introduction of a new concept must be in the students' *zone of proximal development* and as close as possible to the *zone of actual development* [7]. This was our standpoint when we introduced some ideas related to the geometrical probability in ChH competition papers. Since the concept is not included in the curricula, we used discrete interpretation of the area of a figure to apply formally the classical probability. Thus, we challenge the student's sense-of-mathematics: students are expected to feel how the large numbers allow approximation of a sample set by the area of a corresponding figure.

(2017-11,12) A screen-target of high resolution consists of inner circle surrounded by a (concentric) ring. The radius of the inner circle is 1 cm. Laser gun shot hits equiprobably a pixel from the screen. A hit in the inner circle generates 5 points and a hit in the outer ring generates 1 point. Find the radius of the outer circle of the ring in centimeters if the average score (i.e. the mathematical expectation) of one shot is 1.25.

(A) 3 (B) 4 (C) 5 (D) 6 (E) none of these

Reasoning. In high resolution, the number of pixels in the circle is almost proportional to its area. Now we start examining the distractors (we can stop the check at the first matching value because there is exactly one correct answer).

A. Radius 3: the ratio of the areas of the inner circle and the whole target is $\frac{1}{9}$; hence, the chance to hit the inner circle is $\frac{1}{9}$ and the chance to hit the outer ring is $\frac{8}{9}$; the mathematical expectation is $\frac{1}{9} \cdot 5 + \frac{8}{9} \cdot 1 = \frac{13}{9} \neq 1.25$.

B. Radius 4: the chance to hit the inner circle is $\frac{1}{16}$ and the chance to hit the outer ring is $\frac{15}{16}$; the mathematical expectation is $\frac{1}{16} \cdot 5 + \frac{15}{16} \cdot 1 = \frac{20}{16} = 1.25$.

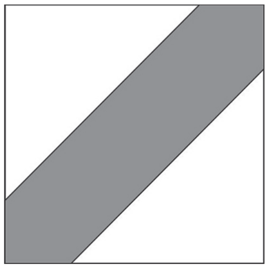
Answer (B).

Comment. The assumption that the number of pixels inside the circle can be approximated by the area of the circle refers to the Gauss's Circle problem [1, pp. 41–43]. Here the tacit ideology is to take the ratio between the number of pixels inside the circle and its area measured in square pixels.

(2003-11,12) Gamblers X and Y play the following game: any one writes a hidden natural number less than 2003; then they show the numbers to each other. If the absolute value of the difference is less than a number a, chosen in advance, then X wins; in the other case, the winner is Y. For which of the given values of a the game is most fair? We assume that any gambler chooses his number equally likely among the numbers $\{1, 2, \ldots, 2002\}$.

(A) 586 (B) 1410 (C) 501 (D) 1001 (E) 224

Reasoning. Suppose the numbers written by X and Y are x and y, respectively. Considering the knots with coordinates $(x; y)$, $0 < x, y < 2003$, we get the sample space K of the game, which is a square-grid with side of the square $b = 2002$. The successful outcomes for X correspond to the knots that satisfy the inequality $|x - y| < a$. They are packed in a hexagon F formed from K by cutting two right triangles with catheti $b - a$ (as shown in the figure on the next page).



Denote by $\{F\}$ the number of knots in F and by $[F]$ the area of F. Analogously for $\{K\}$ and $[K]$. The chance for success of X is

$$\frac{\{F\}}{\{K\}} \approx \frac{[F]}{[K]} = \frac{b^2-(b-a)^2}{b^2} = \frac{2b-a}{b^2} = \frac{2a}{b} - \left(\frac{a}{b}\right)^2.$$

The game will be fair if this ratio is as close as possible to $\frac{1}{2}$. Solving the quadratic $\frac{2a}{b} - (\frac{a}{b})^2 = \frac{1}{2}$ for $\frac{a}{b}$, we take the root $\frac{1}{2}(2-\sqrt{2}) \approx 0.3$ (the one, which is less than 1). Thus $a \approx 0.3b \approx 600$.

Answer (A).

Comment. The test item is a discrete replica of a classical model, which is modified in multiple-choice format. The approximation $\frac{\{F\}}{\{K\}} \approx \frac{[F]}{[K]}$ can be explained by the Picks formula [5]. Indeed, denoting by n and m the number of the inner and boundary knots of F, respectively, we have $[F] = n + \frac{m}{2} - 1$. Taking into account that the perimeter of F is less than $4b$, we derive that $m < 4b$. Hence, the values like $\frac{m}{[K]} < \frac{2}{b} < \frac{1}{500}$ can be neglected in $\frac{[F]}{[K]} = \frac{n+\frac{m}{2}-1}{b^2}$, i.e., $\frac{[F]}{[K]} \approx \frac{n}{b^2} = \frac{\{F\}}{\{K\}}$. The multiple-choice format of ChH allows leaving many argumentation details to the student, challenging his synthetic competence and sense-of-mathematics. In the same time, the reasoning does not leave the student's zone of proximal development. The original problem was stated without the assumption that any gambler chooses his number equally likely among the numbers $\{1, 2, \ldots, 2002\}$. As it was mentioned by the reviewer of the paper,

> 'without this assumption the gamblers could think about possible strategies that allow gambler X to obtain a small value of the difference $|x-y|$ with high probability, and allow gambler Y to obtain a large value of $|x-y|$ with high probability. For example, one of the players can consider choosing among 1 and 2002 (or among 1001 and 1002) with probability $\frac{1}{2}$.'

(2019-9,12) The function `RandomNumber` produces a random number from $(-1; 1)$. What is the output value with the accuracy 2 digits after the decimal point when running the following program:

```
p ← 0, q ← 0
Repeat 1 000 000 times
    x ← RandomNumber,   y ← RandomNumber
    If x^2 + y^2 < 1 then
        q ← q+1
        If 2√(1 − x^2 − y^2) < 1 then
            p ← p+1
Print p/q
```

(A) 0.2 (B) 0.25 (C) 0.33 (D) 0.4 (E) 0.5

Reasoning. The line $x \leftarrow$ `RandomNumber`, $y \leftarrow$ `RandomNumber` generates a randomly chosen point $(x;y)$ in the square $|x|<1$, $|y|<1$. The check $x^2+y^2<1$ screens the points that are inside the unite circle centered at $(0;0)$. These points are counted by q. Taking the point $(x;y)$ inside the unit circle one determines the unique chord, which has this point as its midpoint. The check $2\sqrt{1-x^2-y^2}<1$ screens the chord with length less than 1. These chords are counted by p.

It is easy to see that the locus of the midpoints of the chords shorter than 1 is the ring between the circles centered at $(0;0)$ and having radii $\frac{\sqrt{3}}{2}$ and 1.

Since the point $(x;y)$ is randomly (and hence uniformly) chosen among a large number of points inside the circle, then the fraction p/q equals approximately the ratio of the areas of the ring and the unit circle, i.e.,

$$\frac{p}{q} \approx \frac{[\text{ring}]}{[\text{unit circle}]} = \frac{\pi(1-\frac{3}{4})}{\pi} = \frac{1}{4}.$$

Answer (B).

Comment. This test item is of high difficulty: the knowledge and skills required are in computational thinking, geometry and approximations.[5] The pseudocode realizes a computer simulation of the probability "randomly drawn chord to be shorter than the radius". If the word randomly is not clarified, the phenomenon is not determined, which is known as the Bertrand's Paradox. The consistency of the algorithm was confirmed running a computer program written by Emil Kelevedzhiev.

4. Beyond the range of ChH

We put into practice the idea to combine two combinatorial problems in a probability-formatted mode in the 2019 issue of the Bulgarian National Math Olympiad (idea is similar to the one in the problem 2018-9,10). The following problem was included in the 7th grade final round.

(2019-MO) Consider the set of naturals that are written by no other digits but 1, 2 or 3 and have the sum of their digits equal to 12. Find the probability a randomly chosen number from this set to occur written with no digit 3.

Solution. Denote by X_n the set of the naturals written by no other digits but 1, 2 or 3 and have the sum of their digits equal to n. (Consider the sum of the digits of one-digit number equals the number itself.) Let $x_n = |X_n|$ (the number of the elements of X_n). We have

$$X_1 = \{1\} \implies x_1 = 1;$$
$$X_2 = \{11;2\} \implies x_2 = 2;$$
$$X_3 = \{111;12;21;3\} \implies x_3 = 4.$$

[5] There were 3 correct answers given in the top quartile of the variation row.

Let $n > 3$ and $\overline{Aa} = 10A + a \in X_n$. Obviously

$$a = 1 \iff A \in X_{n-1};$$
$$a = 2 \iff A \in X_{n-2};$$
$$a = 3 \iff A \in X_{n-3}.$$

Hence

$$X_n = \{\,\overline{A1} : A \in X_{n-1}\,\} \cup \{\,\overline{A2} : A \in X_{n-2}\,\} \cup \{\overline{A3} : A \in X_{n-3}\,\}$$

and the three sets in the union are disjoint. Because of

$$|\{\,\overline{A1} : A \in X_{n-1}\,\}| = x_{n-1};$$
$$|\{\,\overline{A2} : A \in X_{n-2}\,\}| = x_{n-2};$$
$$|\{\,\overline{A3} : A \in X_{n-3}\,\}| = x_{n-3},$$

we have $x_n = x_{n-1} + x_{n-2} + x_{n-3}$. Now we find consecutively

$$\begin{aligned}
x_4 &= x_3 + x_2 + x_1 = 4 + 2 + 1 = 7;\\
x_5 &= x_4 + x_3 + x_2 = 7 + 4 + 2 = 13;\\
x_6 &= x_5 + x_4 + x_3 = 13 + 7 + 4 = 24;\\
x_7 &= x_6 + x_5 + x_4 = 24 + 13 + 7 = 44;\\
x_8 &= x_7 + x_6 + x_5 = 44 + 24 + 13 = 81;\\
x_9 &= x_8 + x_7 + x_6 = 81 + 44 + 24 = 149;\\
x_{10} &= x_9 + x_8 + x_7 = 149 + 81 + 44 = 274;\\
x_{11} &= x_{10} + x_9 + x_8 = 274 + 149 + 81 = 504;\\
x_{12} &= x_{11} + x_{10} + x_9 = 504 + 274 + 149 = 927.
\end{aligned}$$

Thus, the total number of outcomes equals 927. The number of the favorable outcomes 233 could be calculated similarly with the recurrent sequence

$$y_1 = 1,\ y_2 = 2, \forall n > 2 : \quad y_n = y_{n-1} + y_{n-2}.$$

Finally, the desired probability equals 233/927.

Comment. In fact, the largest fraction of the students who solved this problem applied combinatorial approach. It works well in finding favorable outcomes: calculate the number of naturals written by 3 digits 4, then by 2 digits 4 and 4 digits 1, and so on. However, the description of the total number of outcomes becomes difficult and extremely time consuming.

5. Conclusions

Comparing ChH test items with Problem (2019-MO) one can clearly see the difference between *solving* and *reasoning*. We accept the criticism that the tournament ChH is rather a place for finding correct answers than for real problem solving.

Moreover, a deep consideration during ChH of a stated math problem following Polyas recommendations [6] could be serious obstacle for successful performance. Advocating the competition format, let us point that the scoring system does not tolerate gambling. This is why an advanced student is motivated to mobilize all his math abilities in a synthetic competence, including his sense of mathematics, for taking a complex test item. Vice versa, the complex test items provide a platform for manifestation the students synthetic abilities in finding quick response of sophisticated problems in general. Probability-mode of some test items serves this mission in ChH competition papers.

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