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RECURSIVE FORMULAS FOR ROOT CALCULATION INSPIRED BY GEOMETRICAL CONSTRUCTIONS

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Abstract. This article describes a method for calculating arithmetic, geometric and harmonic means of two numbers and how they can be represented geometrically. We extend these mean values to arithmetic, geometric and harmonic thirds, fourths, etc. For this we will only use the tools of the affine planar geometry. Also, we will make allusion to the more general interpretation in the projective plane.

From the relations between these means we can deduce a multitude of recursive formulas for n-th root calculation and represent them by geometric constructions. These formulas give a solution for reducing the power of the root. Surprisingly, one of these algorithms turns out to be the same as the one using Newton's tangent method for calculating zero values of functions of the form $f(x) = x^n - c$, but obtained without use of analysis. Moreover, regarding speed of convergence these algorithms are faster than Newton's tangent method.

This geometric interpretation of mean values and root calculation fits into the larger context of affine geometry, where we use multi-projections as generating transformations for building up all the affine transformations. Our focus will primarily be on mean values and roots.

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MSC Subject Classification: 97G40

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Part 1. Arithmetic, geometric and harmonic means of an interval

1. Geometric construction of the different means

In the affine plane parallel rulers (well known from nautical navigation) are a convenient tool for constructions using parallel lines only.

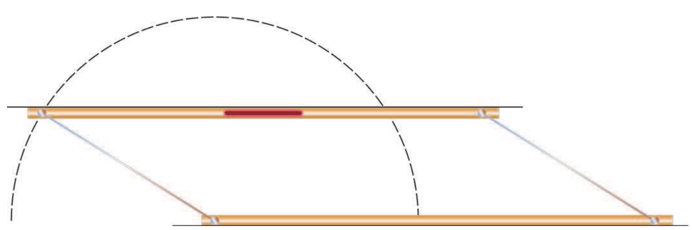


Fig. 1

For example, this instrument is a good help for constructing multiples and powers of a real number a, represented on an axis, by use of translations and homotheties.

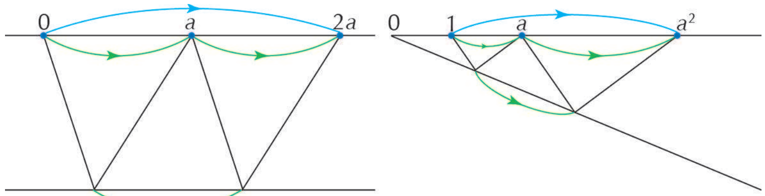


Fig. 2

When we reverse the problem and aim for a construction to divide a number in half or to extract the square root of a positive number, using parallel lines only, then the construction is possible for the first (at the left) but not for the second (at the right), as we will show soon.

Constructions with parallel rulers only allow operations defined by the group of dilatations, namely addition, subtraction, multiplication and division. Root extraction needs in addition metric transformations, for example rotations (circles). The same situation remains for the arithmetic and the geometric mean of two numbers a and b, represented on a real number axis.

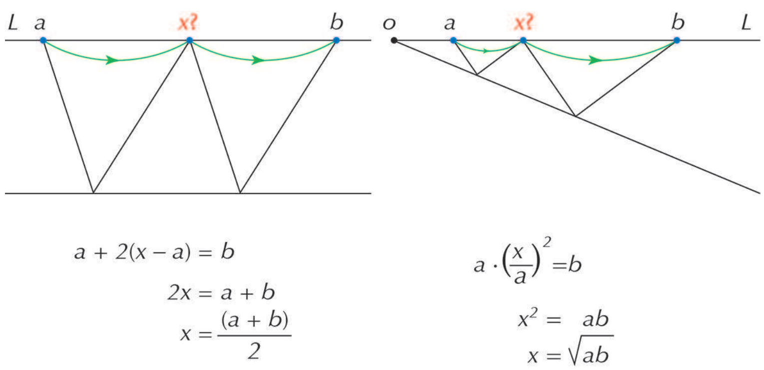


Fig. 3

We can construct the arithmetic mean as seen below on the left figure but not the geometric mean on the right figure. However, a construction with parallel lines only leads to the harmonic mean of two numbers a and b represented on a real number axis.

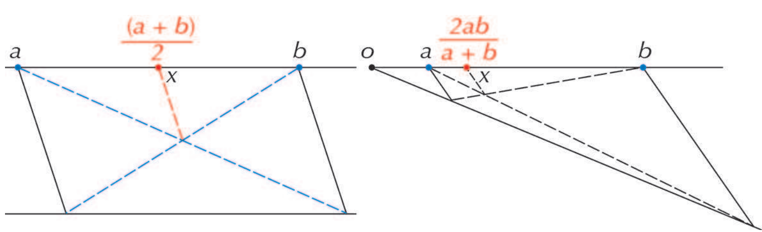


Fig. 4

Both results follow from Thales' Intercept theorem: the ratio of $x-a$ to $b-x$ is the same as that of the section parts of the diagonals:

In the parallelogram on the left: $x - a = b - x \implies x = \dfrac{a+b}{2}$;

in the trapezium on the right: the ratio of the section parts of the diagonals is the same as that of a to b, because of the similarity of triangles, so

$$\frac{x-a}{b-x} = \frac{a}{b} \implies x = \frac{2ab}{a+b}.$$

This is the harmonic mean of a and b.

There exist inequality relations between the arithmetic, the harmonic and the geometric mean of two numbers that easily can be proved algebraically but can be represented geometrically as well.

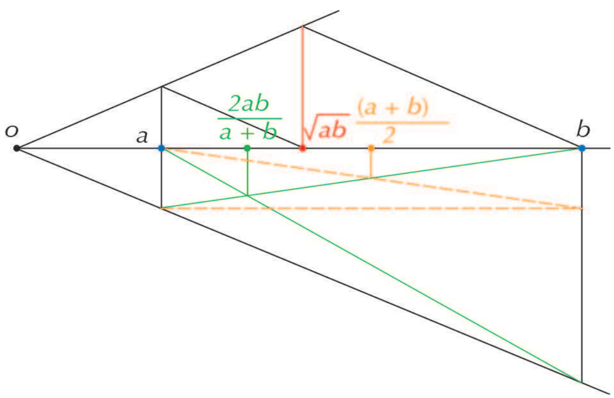


Fig. 5

We notice that for $a \neq b$,

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2}.$$

The geometric mean of a and b equals the geometric mean of the arithmetic mean and the harmonic mean of a and b:

$$\sqrt{\frac{2ab}{a+b} \cdot \frac{a+b}{2}} = \sqrt{ab}.$$

This means that we can approach the geometric mean of a and b by iterating on the arithmetic and harmonic mean of a and b, since with each step the distance between the arithmetic mean and the harmonic mean of two numbers becomes less than half of the difference between these two numbers.

$$\frac{a+b}{2} - \frac{2ab}{a+b} = \frac{(b-a)^2}{2(a+b)} < \frac{(b-a)(b+a)}{2(a+b)} = \frac{b-a}{2}.$$

So the iteration process converges to the geometric mean of a and b.

Since the square root of a strictly positive number c can be defined as the geometric mean of 1 and c, $\sqrt{1 \cdot c} = \sqrt{c}$, we can also use this iteration process for calculating the square root of c starting from the arithmetic and harmonic mean of 1 and c.

For example, the square root of 2 is the geometric mean of $a = 1$ and $b = 2$.

$\frac{a+b}{2}$	1.5	1.4166 ...	1.414215 ...	
$\frac{2ab}{a+b}$	1.33 ...	1.4111 ...	1.414211 ...	Thus, $\sqrt{2} = 1.41421\ldots$

If c is large, then the convergence starting from 1 and c will be rather slow. In that case it is better to start from c and a close estimation x of the root. We will show this later on. In order to generalise these relations, we introduce the following notations:

$HM_{1/2}$ = the harmonic mean (which we will also refer to as the harmonic half),

$GM_{1/2}$ = the geometric mean (... geometric half),

$AM_{1/2}$ = the arithmetic mean (... arithmetic half).

2. Arithmetic, geometric and harmonic thirds of an interval

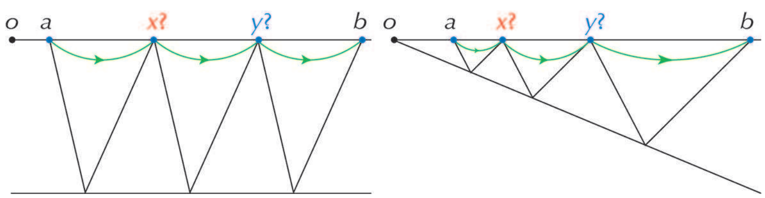


Fig. 6

From $\quad a + 3(x-a) = b \qquad a\left(\frac{x}{a}\right)^3 = b$

and $\quad a + 2(x-a) = y \qquad a\left(\frac{x}{a}\right)^2 = y$

follows

$$x = \frac{2a+b}{3} = AM_{1/3} \qquad x = \sqrt[3]{a^2b} = GM_{1/3}$$

$$y = \frac{a+2b}{3} = AM_{2/3} \qquad y = \sqrt[3]{ab^2} = GM_{2/3}.$$

We can refer to $AM_{1/3}$ as the first arithmetic third, to $AM_{2/3}$ as the second arithmetic third, $GM_{1/3}$ as the first geometric third, etc. Using the similarity of triangles and the property of the diagonals of a trapezium we can also construct the harmonic thirds of an interval (Fig. 7).

We can also construct the arithmetic thirds in an analogous way as in Fig. 7 (Fig. 8).

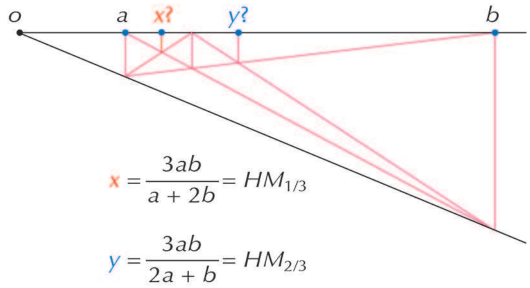
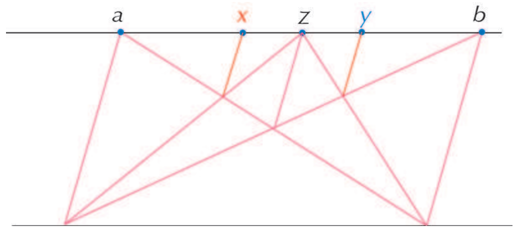


Fig. 7



$$x = AM_{1/3}, \quad z = AM_{1/2}, \quad y = AM_{2/3}$$

Also this can be proven by using the similarity of triangles and the property of the diagonals of a trapezium. For example:

$$\frac{z-x}{x-a} = \frac{1}{2} \ \text{ and } \ z = \frac{a+b}{2} \implies x = \frac{2a+b}{3} = AM_{1/3}.$$

We have the following relations:

$$AM_{1/3} \cdot H_{2/3} = ab, \quad GM_{1/3} \cdot GM_{2/3} = ab, \quad AM_{2/3} \cdot HM_{1/3} = ab.$$

3. Arithmetic, geometric and harmonic fourths of an interval

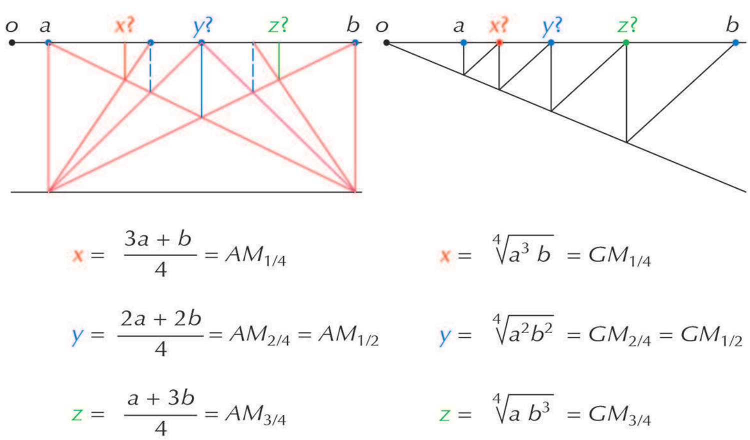


Fig. 9

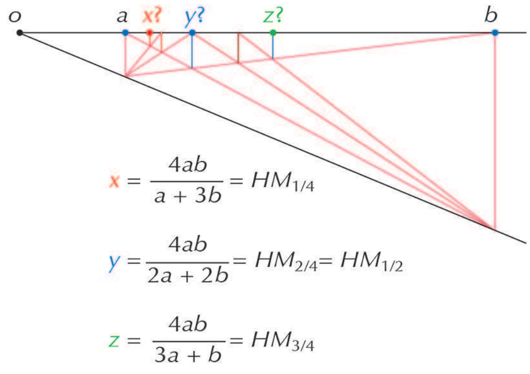


Fig. 10

We have the following relations:

$$AM_{1/4} \cdot H_{3/4} = ab, \quad AM_{2/4} \cdot HM_{2/4} = ab, \quad AM_{3/4} \cdot HM_{1/4} = ab,$$
$$GM_{1/4} \cdot GM_{3/4} = ab, \qquad GM_{2/4} \cdot GM_{2/4} = ab.$$

4. Arithmetic, geometric and harmonic n-th parts of an interval

The following formulas apply respectively to the i-th and the $(n-i)$-th arithmetic, geometric and harmonic n-th part of the interval $[a, b]$ where $0 < i < n$ and $a \neq b$:

$$AM_{i/n} = \frac{(n-i)a + ib}{n}, \quad GM_{i/n} = \sqrt[n]{a^{n-i}b^{i}}, \quad HM_{i/n} = \frac{nab}{ia + (n-i)b},$$
$$AM_{(n-i)/n} = \frac{ia + (n-i)b}{n}, \quad GM_{(n-i)/n} = \sqrt[n]{a^{i}b^{n-i}}, \quad HM_{(n-i)/n} = \frac{nab}{(n-i)a + ib}.$$

Observe that $AM_{i/n} \cdot HM_{(n-i)/n} = ab$, $AM_{(n-i)/n} \cdot HM_{i/n} = ab$, $GM_{i/n} \cdot GM_{(n-i)/n} = ab$.

5. Interpretation with double ratios

The relation between the i-th arithmetic n-th part and the $(n-i)$-th harmonic n-th part of an interval can also be expressed by means of double ratios in the projective plane. [This section is not essential for understanding the next sections, but offers an interpretation in a more general background.]

The division ratio $[a, b, x]$ of three numbers a, b, x is the quotient of $x - a$ and $x - b$. If x is infinity (∞) then we take $[a, b, \infty] = 1$.

The double ratio of four numbers $[a, b, x, y]$ is the quotient of $[a, b, x]$ and $[a, b, y]$. So we have in particular $[a, b, x, \infty] = [a, b, x] : [a, b, \infty] = [a, b, x] : 1 = [a, b, x]$.

If we divide the interval $[a,b]$ in n equal parts and the abscissa of x respective to a and b is $\frac{i}{n}$ (i and n being natural numbers), then x is the i-th arithmetic n-th of $[a,b]$: $x = AM_{i/n} = \dfrac{(n-i)a+ib}{n}$ and the double ratio $[a,b,x,\infty] = [a,b,x] = \dfrac{x-a}{x-b} = \dfrac{\frac{i}{n}-0}{\frac{i}{n}-1} = \dfrac{i}{i-n}$.

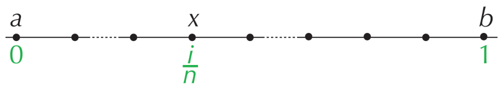


Fig. 11

The $(n-i)$-th harmonic n-th part $HM_{(n-i)/n}$ of the interval $[a,b]$ is $y = \dfrac{nab}{(n-i)a+ib}$. We calculate the double ratio

$$[a,b,y,0] = \frac{\dfrac{nab}{(n-1)a+ib}-a}{\dfrac{nab}{(n-1)a+ib}-b} : \frac{0-a}{0-b} = \frac{nab-(n-i)a^2-iab}{nab-(n-i)ab-ib^2}\cdot\frac{b}{a}$$
$$= \frac{nb-(n-i)a-ib}{na-(n-i)a-ib} = \frac{(a-b)i-(a-b)n}{i(a-b)} = \frac{i-n}{i}.$$

This precisely is the reverse of the double ratio $[a,b,x,\infty]$. Thus, $[a,b,x,\infty][a,b,y,0] = 1$, which is the projective interpretation of the relation $xy = AM_{i/n}\cdot HM_{(n-i)/n} = ab$.

Let $[a,b,x] = \dfrac{x-a}{x-b} = k$; then $x = \dfrac{a-kb}{1-k}$. From $x\cdot y = a\cdot b$ follows $y = \dfrac{(1-k)ab}{a-kb}$. We select some integer values and their reverses for parameter k and calculate the corresponding arithmetic and harmonic means $x = AM_{i/n}$ and $y = HM_{(n-i)/n}$.

k	$-\frac{1}{n}$	$\dots$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	$\dots$	$-n$
$\frac{1}{k}$	$-n$	$\dots$	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$\dots$	$-\frac{1}{n}$
x	$\frac{na+b}{n+1}$	$\dots$	$\frac{3a+b}{4}$	$\frac{2a+b}{3}$	$\frac{a+b}{2}$	$\frac{a+2b}{3}$	$\frac{a+3b}{4}$	$\dots$	$\frac{a+nb}{n+1}$
y	$\frac{(n+1)ab}{na+b}$	$\dots$	$\frac{4ab}{3a+b}$	$\frac{3ab}{2a+b}$	$\frac{2ab}{a+b}$	$\frac{3ab}{a+2b}$	$\frac{4ab}{a+3b}$	$\dots$	$\frac{(n+1)ab}{a+nb}$
xy	ab	$\dots$	ab	ab	ab	ab	ab	$\dots$	ab

We notice that in particular for $k = -1$ it shows:

$$x = \frac{a+b}{2} \text{ and thus } [a, b, x, \infty] = [a, b, \frac{a+b}{2}, \infty] = -1,$$
$$y = \frac{2ab}{a+b} \text{ and thus } [a, b, y, 0] = [a, b, \frac{2ab}{a+b}, 0] = -1.$$

We call a double ratio *harmonic* if it equals -1. The arithmetic mean of a and b is the harmonic conjugate of ∞ with respect to a and b. The harmonic mean of a and b is the harmonic conjugate of 0 with respect to a and b.

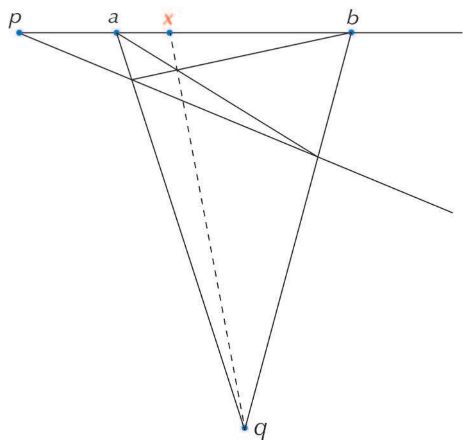


Fig. 12

In the projective plane the harmonic conjugate x of p with respect to a and b is constructed as in Fig. 12.

The constructions of the arithmetic and the harmonic mean of a and b in Section 1 are thus affine versions of this construction. In the first situation with p and q at infinity, in the second with $p = 0$ and q at infinity. Now we will use all these relations between arithmetic, geometric and harmonic means for creating quite a lot of recursive formulas for root calculation.

Part 2: Reductive formulas for root calculation

6. Square root calculation

We already learned in Section 1 that we can calculate the square root of a positive number c, different from 0, by iterating on the arithmetic and the harmonic mean of 1 and c. If c is large then the convergence starting from these might be slow. In this case it is better to start from a closer estimate x for the square root of c. Since $\sqrt{\frac{1}{c}} = \frac{1}{\sqrt{c}}$, we can restrict the problem to the case that $1 < c$.

If x is an underestimate for the square root of c, then c/x is an overestimate for this square root, because $x < \sqrt{c}$ implies $\frac{c}{x} > \frac{c}{\sqrt{x}}$, which implies $\frac{c}{x} > \sqrt{c}$. So the interval $[x, c/x]$ contains $\sqrt{c}$.

Now, $\sqrt{c}$ is the geometric mean of x and c/x, because $\sqrt{x \cdot \frac{c}{x}} = \sqrt{c}$. If we choose $a = x$ and $b = c/x$ then, with the relations from Section 1, we can calculate $\sqrt{c}$ by iterating on the arithmetic and harmonic mean of x and c/x:

$$AM_{1/2} = \frac{x + \frac{c}{x}}{2} = \frac{x^2 + c}{2x} \quad \text{and} \quad HM_{1/2} = \frac{2x \cdot \frac{c}{x}}{x + \frac{c}{x}} = \frac{2cx}{x^2 + c}.$$

We know that $HM_{1/2} = \dfrac{2cx}{x^2+c} < \sqrt{c} < \dfrac{x^2+c}{2x} = AM_{1/2}$.

The algorithm based on this two-sided approximation can be represented geometrically by the following construction that we call *nomogram.*

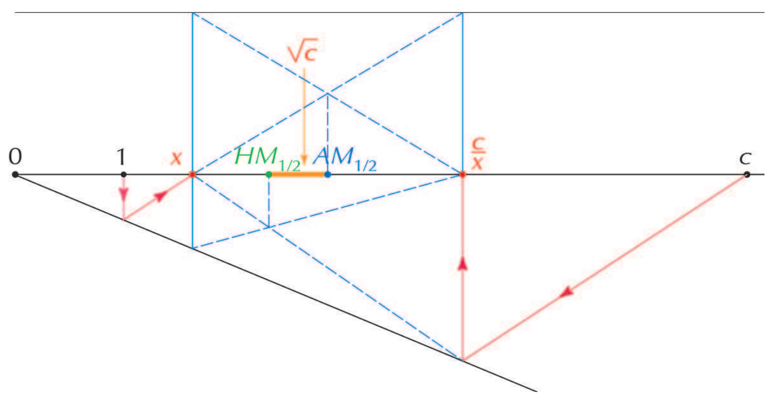


Fig. 13

For example, let $c = 4711$. We start with the estimate $x_0 = 68$ ($68^2 = 4624 < 4711$); then $\dfrac{c}{x_0} = \dfrac{4711}{68} = 69.279\ldots$,

$$HM_{1/2} = \frac{2cx_0}{x_0^2+c} = \frac{2\cdot 4711\cdot 68}{68^2+4711} = 68.633\cdots = x_1,$$

$$AM_{1/2} = \frac{x_0^2+c}{2x_0} = \frac{68^2+4711}{2\cdot 68} = 68,639\ldots$$

(Since $AM_{1/2}\cdot HM_{1/2} = c = 4711$, we can calculate $AM_{1/2}$ also by $AM_{1/2} = \dfrac{c}{HM_{1/2}} = \dfrac{4711}{68.633\ldots} = 68.639\ldots$)

After the first iteration it already shows $68.633\cdots < \sqrt{4711} < 68.639\ldots$ and thus $\sqrt{4711} = 68.63\ldots$. We continue the iteration with $x_1 = 68.633\ldots$.

$$HM_{1/2} = \frac{2cx_1}{x_1^2+c} = 68.6367238\cdots = x_2,$$

$$AM_{1/2} = \frac{x_1^2+c}{2x_1} = 68.636724\ldots.$$

After just two iterations we already know the result up to the 5th decimal after the decimal point: $\sqrt{4711} = 68.63672\ldots$. This algorithm converges very fast.

In a program running these iterative algorithms we can include a stop, based on the desired difference between $AM_{1/2}$ and $HM_{1/2}$, i.e. $AM_{1/2} - HM_{1/2} < 10^{-n}$. So we can calculate the root up to any desired decimal.

But there is something interesting with the right side approximation in the formula. The square root of c can also be interpreted as a zero value of the function

$f(x) = x^2 - c$. So we can use Newton's tangent method for approaching this zero value. Therefore we need the derivative of f, i.e. $f'(x) = 2x$. With x_0 as the first estimate and T as the tangent to the graph of f at the point $(x_0, f(x_0))$ we obtain a better approximation x_1 as the abscissa of the intersection point of T and the x-axis.

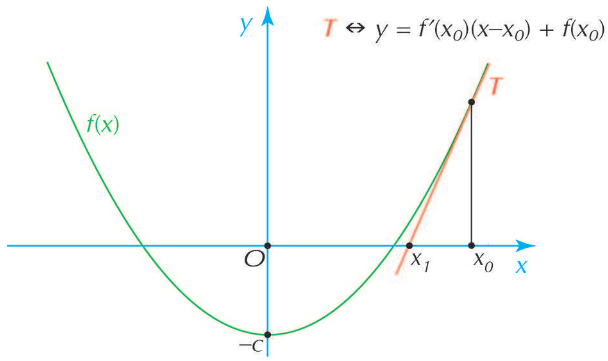


Fig. 14

The value x_1 is calculated from the equations of T and the x-axis ($y = 0$):

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - c}{2x_0} = \frac{x_0^2 + c}{2x_0}.$$

And this result is the same as $AM_{1/2}$.

But the great advantages of developing these formulas in the geometrical way are:

- we do not need analysis;
- we can use approximations not only from one side, but from both sides. So we can take control of the number of exact decimals;
- we have a geometrical interpretation in nomograms;
- but first of all, when we extract roots of higher power, we will see further on that with the geometrical procedure we get a choice of algorithms with various convergence speeds and among those, the one given by Newtons tangent method is one of the two slowest.

7. Cube root calculation

The cube root of a positive number c can be interpreted as the first geometric third $GM_{1/3}$ of 1 and c because $\sqrt[3]{1^2 \cdot c} = \sqrt[3]{c}$. Since $\sqrt[3]{\dfrac{1}{c}} = \dfrac{1}{\sqrt[3]{c}}$, we can restrict the problem to the case $1 < c$.

When x is an underestimate of the cube root of c, then $\dfrac{c}{x^2}$ is an overestimate, for:

$$x < \sqrt[3]{c} \implies x^2 < \sqrt[3]{c^2} \implies \frac{c}{x^2} > \frac{c}{\sqrt[3]{c^2}} \implies \frac{c}{x^2} > \sqrt[3]{c}.$$

So the interval $[x, c/x^2]$ contains $\sqrt[3]{c}$. The first geometric third $GM_{1/3}$ of this interval is also $\sqrt[3]{c}$ because $\sqrt[3]{x^2 \cdot \dfrac{c}{x^2}} = \sqrt[3]{c}$.

The first arithmetic third $AM_{1/3} = \dfrac{2x + \frac{c}{x^2}}{3} = \dfrac{2x^3 + c}{3x^2}$ (see Section 2) of this interval is then a new approximation of $\sqrt[3]{c}$, larger than $\sqrt[3]{c}$, i.e.,

$$\sqrt[3]{c} < AM_{1/3} = \frac{2x^3 + c}{3x^2}. \tag{1}$$

It follows from $x < \sqrt[3]{c}$ that $x^2 < \sqrt[3]{c^2}$ and $\dfrac{c}{x} > \dfrac{c}{\sqrt[3]{c}} = \sqrt[3]{c^2}$ and thus the interval $[x^2, c/x]$ contains $\sqrt[3]{c^2}$.

The second geometric third $GM_{2/3}$ of this interval is again $\sqrt[3]{c^2}$ because $\sqrt[3]{x^2 \cdot \dfrac{c^2}{x^2}} = \sqrt[3]{c^2}$ and the second harmonic third $HM_{2/3} = \dfrac{3x^2 \cdot \frac{c}{x}}{2x^2 + \frac{c}{x}} = \dfrac{3cx^2}{2x^3 + c}$ (s. Section 2) of this interval is then a new approximation of $\sqrt[3]{c^2}$, less than $\sqrt[3]{c^2}$. So, $HM_{2/3} < \sqrt[3]{c^2}$ and thus

$$\sqrt{HM_{2/3}} = \sqrt{\frac{3cx^2}{2x^3 + c}} < \sqrt[3]{c}. \tag{2}$$

From (1) and (2), we get:

$$\sqrt{HM_{2/3}} = \sqrt{\frac{3cx^2}{2x^3 + c}} < \sqrt[3]{c} < \frac{2x^3 + c}{3x^2} = AM_{1/3}. \tag{3}$$

This can be represented geometrically by the nomogram:

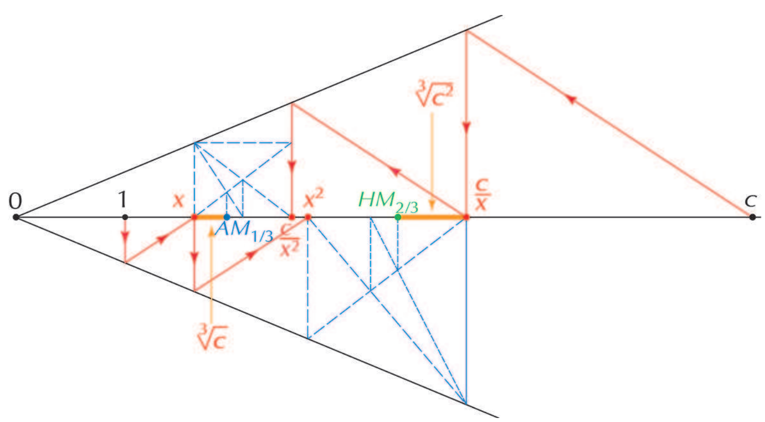


Fig. 15

The first geometric third $GM_{1/3}$ of $\sqrt{H_{2/3}}$ and $AM_{1/3}$ is again $\sqrt[3]{c}$, since

$$GM_{1/3} = \sqrt[3]{\sqrt{HM_{2/3}}^{\,2} \cdot AM_{1/3}} = \sqrt[3]{\sqrt{\frac{3cx^2}{2x^3 + c}}^{\,2} \cdot \frac{2x^3 + c}{3x^2}} = \sqrt[3]{c^2}.$$

So we can iterate on $AM_{1/3}$ as well as on $\sqrt{HM_{2/3}}$ in (3) for calculating the cube root of c. Since the length of the new interval is less than a third of the former, the iterations will converge to $\sqrt[3]{c}$. Indeed,

$$\frac{2x^3+c}{3x^2} - \sqrt{\frac{3cx^2}{2x^3+c}} < \frac{1}{3}\Big(\frac{c}{x^2} - x\Big) \iff \frac{2x^3+c}{3x^2} - \frac{1}{3}\Big(\frac{c}{x^2} - x\Big) < \sqrt{\frac{3cx^2}{2x^3+c}}$$

$$\iff x < \sqrt{\frac{3cx^2}{2x^3+c}} \iff x^2(2x^3+c) < 3cx^2$$

$$\iff 2x^3 + c < 3c \iff x^3 < c \iff x < \sqrt[3]{c},$$

which was stated from the beginning.

For example, let $c = 1789$. Take $x = 12$ as an underestimate for $\sqrt[3]{c}$ ($12^3 = 1728 < 1789$); then $\frac{c}{x^2} = \frac{1789}{12^2} = 12.4236\ldots$ is an overestimate of $\sqrt[3]{c}$. The first arithmetic third of the interval $[x, c/x^2]$ is

$$AM_{1/3} = \frac{2x^3+c}{3x^2} = \frac{2\cdot 12^3 + 1789}{3\cdot 12^2} = 12.141203\ldots$$

The square root of the second harmonic third of the interval $[x^2, c/x] = [144,\ 149.0833\ldots]$ is

$$\sqrt{HM_{2/3}} = \sqrt{\frac{3cx^2}{2x^3+c}} = \sqrt{\frac{3\cdot 1789\cdot 12^2}{2\cdot 12^3 + 1789}} = 12.1387592\ldots$$

We continue the iteration by repeating these calculations on the interval $[12.1387592\ldots,\ 12.141203\ldots]$:

$$AM_{1/3} = \frac{2\cdot 12.1387592\ldots^3 + 1789}{3\cdot 12.1387592\ldots} = 12.139574\ldots \tag{4}$$

$$\sqrt{HM_{2/3}} = \sqrt{\frac{3\cdot 1789\cdot 12.1387592\ldots^2}{2\cdot 12.1387592\ldots^3 + 1789}} = 12.139573\ldots \tag{5}$$

It follows from (4) and (5) that $\sqrt[3]{1789} = 12.13957\ldots$.

As in the case of square root calculation it is possible to obtain the formula at the right-hand side of (3) by Newton's tangent method on the function $f(x) = x^3 - c$ with $f'(x) = 3x^2$. Let x_0 be an initial estimate for the zero $\sqrt[3]{c}$ of this function. We obtain a better approximation by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - c}{3x_0^2} = \frac{2x_0^3 + c}{3x_0^2}.$$

Moreover, beside these iterations on the left-hand and on the right-hand sides of (3), we could give even more formulas for calculating $\sqrt[3]{c}$ inspired by other relations between mean values. Indeed, we have used above the relation $AM_{1/3}\cdot HM_{2/3} = c$, but we could have used the relation $AM_{2/3}\cdot HM_{1/3} = c$, as well.

When programming a one-sided approach, the geometrical way inspires us for four formulas, Newton's tangent method being one of them. On these procedures we have the same comments as for the square root in Section 4. The formula on the left-hand side of (3), based on the square root of the second harmonic third, calculates the cube root by means of the square root and therefore gives a solution for the reduction of the root exponent.

8. General recursive formulas for root calculation

If x is an underestimate for the n-th root of a positive number c, then we can prove that the interval $\left[x^i, \dfrac{c}{x^{n-i}}\right]$ contains $\sqrt[n]{c^i}$, for $i = 1, 2, \ldots, n-1$ ($n-1$ intervals).

$AM_{i/n}$ in the interval $\left[x^i, \dfrac{c}{x^{n-i}}\right]$ is an overestimate of $\sqrt[n]{c^i}$ and thus the i-th root of $AM_{i/n}$ is an overestimate of $\sqrt[n]{c}$. Complementary, $HM_{(n-i)/n}$ in the interval $\left[x^{n-i}, \dfrac{c}{x^i}\right]$ is an underestimate of $\sqrt[n]{c^{n-i}}$ and thus the $(n-i)$-th root of $HM_{(n-i)/n}$ is an underestimate of $\sqrt[n]{c}$.

We now use the relation $AM_{i/n} \cdot HM_{(n-i)/n} = c$ (see the end of Section 2) to construct the following recursive formulas for calculating $\sqrt[n]{c}$:

$$AM_{i/n} = \frac{(n-i)\cdot x^i + i \cdot \dfrac{c}{x^{n-i}}}{n} = \frac{(n-i)\cdot x^n + ic}{nx^{n-i}},$$

$$HM_{(n-i)/n} = \frac{n \cdot x^{n-i} \cdot \dfrac{c}{x^i}}{(n-i)\cdot x^{n-i} + i \cdot \dfrac{c}{x^i}} = \frac{ncx^{n-i}}{(n-i)\cdot x^n + ic}.$$

So, for $\sqrt[n]{c}$ we get

$$\sqrt[n-i]{HM_{(n-i)/n}} = \sqrt[n-i]{\frac{ncx^{n-i}}{(n-i)\cdot x^n + ic}} < \sqrt[n]{c} < \sqrt[i]{\frac{(n-i)\cdot x^n + ic}{nx^{n-i}}} = \sqrt[i]{AM_{i/n}}.$$

If we want to program an algorithm with a formula from one side for any value of $i < n$ then we have no less than $2(n-1)$ alternatives for iterative calculations for $\sqrt[n]{c}$ with a root exponent less than n. *So these iterative formulas give a solution for reduction of the root exponent.*

Nevertheless, there is a difference in convergence speed between the different algorithms. The higher the root exponent the faster the algorithm. So we obtain the fastest approximation for $\sqrt[n]{c}$ by using the formula with root exponent $n-1$. For example, in

$$\sqrt[n-1]{HM_{(n-1)/n}} = \sqrt[n-1]{\frac{ncx^{n-1}}{(n-1)\cdot x^n + c}} < \sqrt[n]{c} < \frac{(n-1)\cdot x^n + c}{nx^{n-i}} = AM_{1/n},$$

the algorithm using the formula at the left-hand side is one of the fastest while the algorithm using the formula at the right-hand side one of the slowest.

The formula on the right-hand side is essentially Newton's tangent method applied to the function $f(x) = x^n - c$, for $f'(x) = nx^{n-1}$ and

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^n - c}{nx^{n-1}} = \frac{(n-1)x^n + c}{nx^{n-1}} = AM_{1/n}.$$

All these algorithms can be represented geometrically in nomograms. A wonderful marriage between geometry and algebra, with a beautiful progeny!

We finish by presenting a test for $n = 5$ and $c = A = 32$. There are $2{\cdot}(5-1) = 8$ algorithms. In order to compare the convergence of these algorithms, we need a sufficient number of iterations; that is why we start from the roughly chosen initial approximation $x_0 = 5$. We enumerate the algorithms from X_1 to X_8:

$$\begin{array}{lll}
X_1: & (HM_{4/5})^{1/4} = ((5AX_1^4/(4X_1^5 + A))^{1/4} & \text{(root exponent 4)} \\
X_2: & AM_{1/5} = (4X_2^5 + A)/5X_2^4 & \text{(Newton, root exponent 1)} \\
X_3: & (HM_{3/5})^{1/3} = ((5AX_3^3/(4X_3^5 + 2A))^{1/3} & \text{(root exponent 3)} \\
X4: & (AM_{2/5})^{1/2} = ((3X_4^5 + 2A)/5X_4^3)^{1/2} & \text{(root exponent 2)} \\
X_5: & (HM_{2/5})^{1/2} = ((5AX_5^2/(2X_5^5 + 3A))^{1/2} & \text{(root exponent 2)} \\
X_6: & (AM_{3/5})^{1/3} = ((2X_6^5 + 3A)/5X_6^2)^{1/3} & \text{(root exponent 3)} \\
X_7: & HM_{1/5} = 5AX_7/(X_7^5 + 4A) & \text{(root exponent 1)} \\
X_8: & (AM_{4/5})^{1/4} = ((X_8^5 + 4A)/5X_8)^{1/4} & \text{(root exponent 4)}
\end{array}$$

The results obtained by a computer program are presented in the next page. We notice that

$$X_7 \leqslant X_5 \leqslant X_3 \leqslant X_1 \leqslant A^{1/5} = 2 \leqslant X_8 \leqslant X_6 \leqslant X_4 \leqslant X_2 \quad \text{(Newton)}$$

X_1, with root exponent 4, is already exact to the 8th decimal after 5 iterations, while X_2 (Newton) with root exponent 1 needs 8 iterations. The higher the root exponent, the higher the speed of convergence.

Conclusion

This material offers a favourable didactical context for application of elementary algebra and geometry, within the reach of young pupils, and with astonishingly deep mathematical results. In the Platonic world of mathematical concepts there are many pearls and diamonds waiting to be discovered, even with just simple tools.

Acknowledgement. An outline of this method was integrated in the article [1]. As an article in an article my contribution was rather brief. Since the article seems to have been read and quoted by many, the current article intends to situate it in a broader context and make the reading easier by mentioning more explicitly the several steps that led me to the results.

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A = 32 X = 5	
X1 = 1.6807182	X2 = 4.01024
X3 = 1.28440308	X4 = 2.88618065
X5 = .793925885	X6 = 3.70279798
X7 = .245926837	X8 = 3.37742733
X1 = 1.96493674	X2 = 3.23293763
X3 = 1.65722801	X4 = 3.04623254
X5 = 1.02160272	X6 = 2.78956854
X7 = .307406385	X8 = 2.40766957
X1 = 1.99968165	X2 = 2.64493568
X3 = 1.92870919	X4 = 2.45367777
X5 = 1.30385641	X6 = 2.23406723
X7 = .384249741	X8 = 2.04101579
X1 = 1.99999998	X2 = 2.24672197
X3 = 1.99733834	X4 = 2.11631719
X5 = 1.62084948	X6 = 2.02526477
X7 = .480280746	X8 = 2.00042053
X1 = 2	X2 = 2.04855618
X3 = 1.99999645	X4 = 2.00940176
X5 = 1.88440625	X6 = 2.00031647
X7 = .600231097	X8 = 2.00000002
X1 = 2	X2 = 2.0022479
X3 = 2	X4 = 2.00006588
X5 = 1.98963755	X6 = 2.00000003
X7 = .74983247	X8 = 2
X1 = 2	X2 = 2.00000504
X3 = 2	X4 = 2
X5 = 1.99991919	X6 = 2
X7 = .935558052	X8 = 2
X1 = 2	X2 = 2
X3 = 2	X4 = 2
X5 = 2	X6 = 2
X7 = 1.16293579	X8 = 2
X1 = 2	X2 = 2
X3 = 2	X4 = 2
X5 = 2	X6 = 2
X7 = 1.42990807	X8 = 2
X1 = 2	X2 = 2
X3 = 2	X4 = 2
X5 = 2	X6 = 2
X7 = 1.70763597	X8 = 2
X1 = 2	X2 = 2
X3 = 2	X4 = 2
X5 = 2	X6 = 2
X7 = 1.91707215	X8 = 2

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