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SOME APPROXIMATIONS OF THE EULER NUMBER

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Abstract. In this paper, we find new approximations of the Euler number e and using Matlab we compare the existing approximations and the new approximations by testing their convergence rate to the Euler number for some terms.

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1. Introduction

The Euler number e first appeared in 1618 in contributions to the logarithms of the Scottish mathematician John Napier as abasis of logarithms. The discovery is attributed to Swiss mathematician Jakob Bernoulli, who tried to find $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, and found that its value, with the first 20 decimals, is 2,71828182845904523536. The name e for this number was given by Swiss mathematician Leonhard Euler in 1727.

The Euler number can be presented in different ways, as an infinite series or infinite product, a continuous ratio or a sequence limit, which is usually taken as a definition in mathematical analysis courses, i.e., $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. But for its calculation with the highest precision, it is most appropriate to take it as a infinite series, i.e., $e = \sum\limits_{n=0}^{\infty}\frac{1}{n!}$ which converges very quickly.

The idea of finding approximations of number e is not new, but it has attracted the attention of many mathematicians, and even today efforts are made to find the fastest and most accurate approximations. The importance of finding these approximations lies in the fact that the number e is an irrational, even transcendental number, and that it appears in many important formulas, not only in mathematics but also in other disciplines. Hence, it is important to find a series which would yield more digits of this constant with fewer summation terms.

In this paper, we present some inequalities involving the Euler number. To achieve them, we use expansion in power series, in particular the Taylor expansion, expansion in Fourier series and some known inequalities, in particular Carleman's inequality and its generalization obtained by Polya. The obtained series, which approximate the Euler number, are compared with the well-known approximations obtained by the series $e = \sum\limits_{n=0}^{\infty}\frac{1}{n!}$.

2. Auxiliary facts

In order to obtain our results, we need the following well-known facts.

$$\left(1+\frac{1}{n}\right)^n < e, \quad n \in \mathbb{N}; \tag{1}$$

$$\frac{1}{(n+1)!} < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}, \quad n = 0, 1, 2, \ldots; \tag{2}$$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1, \quad n \in \mathbb{N}. \tag{3}$$

Lemma 2.1. (Carleman's inequality) $\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n$, *where* $a_n \geqslant 0$, $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} a_n < \infty$.

A generalization of Carleman's inequality was for the first time given by Polya in the following form:

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\sigma_n}} < e \sum_{n=1}^{\infty} \lambda_n a_n, \tag{4}$$

where $\lambda_n > 0$, $\sigma_n = \sum_{k=1}^{n} \lambda_k$, $a_n \geqslant 0$, $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. It was stated in [2, 4] and its proof can be found in [2].

We will also need the following well-known facts.

Lemma 2.2. $\sum_{k=1}^{\infty} \frac{1}{\alpha^2 + k^2} = \frac{e^{\alpha\pi}\alpha\pi - \operatorname{sh}\alpha\pi}{2\alpha^2 \operatorname{sh}\alpha\pi}$, $\alpha \in \mathbb{R}$.

Lemma 2.3. $\operatorname{sgn}\sin x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$, $x \in \mathbb{R}$.

Proofs of these lemmas, using Fourier series, can be found in most of the textbooks on Mathematical analysis.

3. Main results and numerical presentations

Proposition 3.1. $e = \sum_{n=1}^{\infty} \frac{2n-1}{3(n-1)!}$.

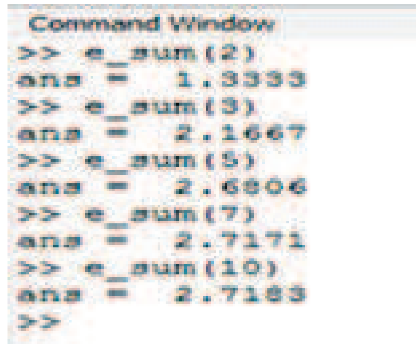
Proof. Using the Taylor expansion for e^x we have $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ and, taking derivative, $xe^{x^2} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(n-1)!}$. Taking another derivative, we obtain

$$(1+2x^2)e^{x^2} = \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-2}}{(n-1)!}, \qquad x \in \mathbb{R}.$$

For $x = 1$ we get the desired result. ■

The Matlab code to generate the partial sums of the above series and some of the results are given below.

```
function e_sum = e_sum(n)
s=0;
for i=1:n
s=s_(2*i-1)/(3*factorial(i-1));
endfor
e_sum = s;
endfunction
```



```
Command Window
>> e_sum(2)
ans =   1.3333
>> e_sum(3)
ans =   2.1667
>> e_sum(5)
ans =   2.6806
>> e_sum(7)
ans =   2.7171
>> e_sum(10)
ans =   2.7183
>>
```

Proposition 3.2.

$$\frac{(2e-1)(n+1)!-6}{(n+1)!} < \sum_{k=1}^{n} \frac{2k^2-k+3}{3k!} < \frac{(2e-1)(n+1)!-2}{(n+1)!}, \text{ for all } n \in \mathbb{N}.$$

Proof. From (2), $e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}$ and it is easily seen that also $e - \sum_{k=1}^{n} \frac{2k-1}{3(k-1)!} < \frac{3}{(n+1)!}$ for all $n \in \mathbb{N}$. Adding these inequalities, we get:

$$2e - \left(\sum_{k=0}^{n} \frac{1}{k!} + \sum_{k=1}^{n} \frac{2k-1}{3(k-1)!}\right) < \frac{6}{(n+1)!},$$

$$2e - 1 - \left(\sum_{k=1}^{n} \left(\frac{1}{k!} + \frac{2k-1}{3(k-1)!}\right)\right) < \frac{6}{(n+1)!},$$

$$2e - 1 - \frac{6}{(n+1)!} < \sum_{k=1}^{n} \frac{3+k(2k-1)}{3k!},$$

$$\frac{(2e-1)(n+1)!-6}{(n+1)!} < \sum_{k=1}^{n} \frac{2k^2-k+3}{3k!},$$

for all $n \in \mathbb{N}$. Similarly, using the inequalities

$$e - \sum_{k=0}^{n} \frac{1}{k!} > \frac{1}{(n+1)!} \quad \text{and} \quad e - \sum_{k=1}^{n} \frac{2k-1}{3(k-1)!} > \frac{1}{(n+1)!},$$

one gets

$$2e - 1 - \left(\sum_{k=1}^{n} \left(\frac{1}{k!} + \frac{2k-1}{3(k-1)!}\right)\right) > \frac{2}{(n+1)!},$$

$$2e-1-\frac{2}{(n+1)!} > \sum_{k=1}^{n} \frac{2k^2-k+3}{3k!},$$

$$\sum_{k=1}^{n} \frac{2k^2-k+3}{3k!} < \frac{(2e-1)(n+1)!-2}{(n+1)!},$$

for all $n \in \mathbb{N}$. The two obtained inequalities prove our claim. qed

Proposition 3.3. $\frac{n+1}{n} < \sqrt[n]{\frac{e}{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}}}$, *for all* $n \in \mathbb{N}$.

Proof. It follows from the relations (1) and (3) that

$$\frac{e}{\left(1+\frac{1}{n}\right)^n} > 1 > \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

and this can be easily transformed into the given inequality. ■

Proposition 3.4.

$$\sum_{n=1}^{\infty} \frac{q^n}{(2!)^{\frac{q-q^2}{1-q^n}} \cdot (3!)^{\frac{q^2-q^3}{1-q^n}} \cdot \ldots \cdot (n!)^{\frac{q^{n-1}-q^n}{1-q^n}}} < e^{1+q},$$

where $0<q<1$.

Proof. Using Polya's generalization (4) of Carleman's inequality, with $a_n = \frac{1}{n!}$, $\lambda_n = q^n$, we have that $\sigma_n = q+q^2+\cdots+q^n = q\,\frac{1-q^n}{1-q}$, and we obtain:

$$\sum_{n=1}^{\infty} q^n \left(1^q \cdot \left(\frac{1}{2!}\right)^{q^2} \cdot \left(\frac{1}{3!}\right)^{q^3} \cdot \ldots \cdot \left(\frac{1}{n!}\right)^{q^n}\right)^{\frac{1-q}{q(1-q^n)}} < e \sum_{n=1}^{\infty} \frac{q^n}{n!},$$

$$\sum_{n=1}^{\infty} q^n \left(\frac{1}{(2!)^{q^2}} \cdot \frac{1}{(3!)^{q^3}} \cdot \ldots \cdot \frac{1}{(n!)^{q^n}} \cdot\right)^{\frac{1-q}{q(1-q^n)}} < e \cdot e^q,$$

and hence the desired inequality. ■

Proposition 3.5. $e = \sqrt{\sum_{n=0}^{\infty} \frac{(2n+3)\cdot 2^{2n}}{(2n+1)!}}$.

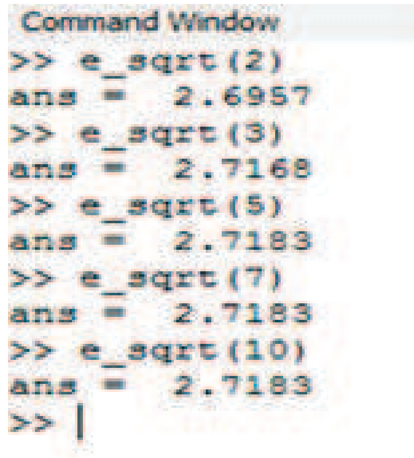
Proof. It is known that $\operatorname{ch} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ hold for every $x \in \mathbb{R}$. For $x=2$ we get $\operatorname{ch} 2 = \frac{e^2+e^{-2}}{2} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}$ and $\operatorname{sh} 2 = \frac{e^2-e^{-2}}{2} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}$. Adding these equalities, we obtain

$$e^2 = \sum_{n=0}^{\infty} \left(\frac{2^{2n}}{(2n)!} + \frac{2^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{2^{2n}(2n+1)+2\cdot 2^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+3)2^{2n}}{(2n+1)!},$$

wherefrom the given formula follows. ■

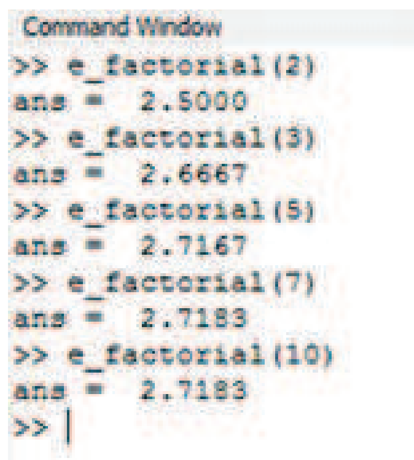
The Matlab code to generate the partial sums of the above series and some of the results are given below.

```
function e_sq = e_sqrt(n)
s=0;
for i=0:n
s=s+((2*i+3)*2^(2*i))/factorial(2*i+1);
endfor
e_sq = sqrt(s);
endfunction
```


Command Window
>> e_sqrt(2)
ans = 2.6957
>> e_sqrt(3)
ans = 2.7168
>> e_sqrt(5)
ans = 2.7183
>> e_sqrt(7)
ans = 2.7183
>> e_sqrt(10)
ans = 2.7183
>>

Now consider the well known series $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, which in Matlab has the following code, and some of the results.

```
function [e_fact] = e_factorial(n)
s=0;
for i=0:n
s=s+1/factorial(i);
endfor
e_fact = s;
endfunction
```


Command Window
>> e_factorial(2)
ans = 2.5000
>> e_factorial(3)
ans = 2.6667
>> e_factorial(5)
ans = 2.7167
>> e_factorial(7)
ans = 2.7183
>> e_factorial(10)
ans = 2.7183
>>

We can see that approximations done using partial sums of the series from Proposition 3.5 are closer to e than those of the usual sum $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. In particular, e.g., four exact decimals are obtained after 5 iterations in the case of the first sum, and after 7 iterations in the second one.

PROPOSITION 3.6. $e^{\pi} = \left(\dfrac{\sum_{k=1}^{\infty} \dfrac{k^2+1+2^{k+2}}{2^{k+2}(1+k^2)}}{\sum_{k=1}^{\infty} \left(\dfrac{k^2+1+2^{k+2}}{2^{k+2}(1+k^2)} + \dfrac{4(-1)^k}{2k-1} \right)} \right)^{\frac{1}{2}}.$

Remark. The number e^{π} is known as Gel'fond's constant.

Proof. Using Lemma 2.2 for $\alpha = 1$ we get

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} = \frac{e^{\pi}\pi}{2\operatorname{sh}\pi} - \frac{1}{2},$$

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} + \frac{1}{2} = \frac{\pi e^{\pi}}{e^{\pi} - e^{-\pi}},$$

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\pi e^{2\pi}}{e^{2\pi} - 1},$$

$$\sum_{k=1}^{\infty} \Big(\frac{1}{1+k^2} + \frac{1}{2^{k+2}} \Big) = \frac{\pi e^{2\pi}}{e^{2\pi} - 1},$$

$$1 - e^{-2\pi} = \pi \Bigg(\sum_{k=1}^{\infty} \Big(\frac{1}{1+k^2} + \frac{1}{2^{k+2}} \Big) \Bigg)^{-1},$$

$$e^{2\pi} = \frac{\sum\limits_{k=1}^{\infty} \Big(\frac{1}{1+k^2} + \frac{1}{2^{k+2}} \Big)}{\sum\limits_{k=1}^{\infty} \Big(\frac{1}{1+k^2} + \frac{1}{2^{k+2}} \Big) - \pi}.$$

Using Lemma 2.3 for $x = \frac{\pi}{2}$, i.e., $\pi = 4\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$, we get the desired result. ■

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