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SOME REMARKS ON PTOLEMY'S THEOREM AND ITS APPLICATIONS

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Abstract. The subject matter of this article is a beautiful theorem developed by Ptolemy (about 100–178) in Chapters 10 and 11 of the first book of Almagest, the Great Collection of Astronomy in 13 books. In order to solve astronomical problems, Ptolemy used mathematical tools to calculate the length of the chord in the circle of radius 60 as a function of the central angle. The application of Ptolemy's theorem is presented by giving some interesting examples. The paper analyzes the results in solving problems from the competitions in which Ptolemy's theorem is applied.

MathEduc Subject Classification: G44

MSC Subject Classification: 97G40

Key words and phrases: Ptolemy's theorem; cyclic quadrilateral.

1. Introduction

Claudius Ptolemaeus or Ptolemy (about 100–178 AD) was a Greek mathematician, astronomer, geographer and astrologer who lived in the city of Alexandria, in the Roman province of Egypt, under the rule of the Roman Empire. Somewhere between 146 and 170 AD, he created the Great Mathematical Collection of Astronomy in 13 books or the Great Syntax, which was later named Almagest by connecting the Arab "al" with the Greek "megiste" [8, p. 310].

When the Arabs occupied Alexandria, they became heirs to much of its intellectual heritage. Many of the finest works of the Greek mathematicians were translated into Arabic (in many cases that was the only form in which they could survive). The Ptolemy's work was translated as *Al magest* (the greatest) and this term was latter adopted by the Romans under the Latin name *Almagestum*. In English, we refer to it still as Ptolemy's Almagest [6].

In Almagest, he used, not only astronomical models, but also mathematical tools of the elementary geometry, among them trigonometry, that were needed by astronomy.

The theory that provides numerical solutions to geometrical problems, involving angles, is called trigonometry. This literally means measurement of triangles. Ptolemy developed this subject in Chapters 10 and 11 of the first book of the Almagest [1, p. 103].

The geometry of the circle was of vital importance for the study of astronomy and geography (the earth is round, the observer's position in relation to the sphere).

The Greeks did not use modern trigonometric functions, but they used chords. For the purpose of astronomical calculations, Ptolemy needed to compare the distance between the points on the circle, or to find the length of the chord over a given central angle. Ptolemy made a table of chords, that was divided into three columns. The first column showed the angles of $(1/2)^\circ$ to 180° at intervals of $(1/2)^\circ$. The second column showed the values of the chord and the third column contained the sixtieths, giving the $1/30$ increments from one line to the next one.

Considering the mathematical steps followed by Ptolemy in order to develop the chord table, the impression is that they were not new to him, but are based on previous known theories (Euclid, Archimedes, Aristarchus, Hipparchus) [5, p. 29]. But his contribution in this direction was unquestionable, which cannot be missed by any cautious reader of Almagest. The truth can be reached by different ways, or methods.

His table gives the length of the chord in the circle, as a function of the central angle (Fig. 1). It is clear that this, in its essence, is the sine function. Ptolemy used a circle of radius 60, most likely because it is the basis of the Vavilonian number system, which was used for calculating. Contemporary authors denote the length of the chord by crd (from the English word chord).

2. The general form and a proof of Ptolemy's theorem

THEOREM 1. *In each oriented and convex quadrilateral $ABCD$ the following inequality holds*

$$AB \cdot CD + DA \cdot BC \geq AC \cdot BD,$$

where the equality is valid if and only if the quadrilateral $ABCD$ is cyclic.

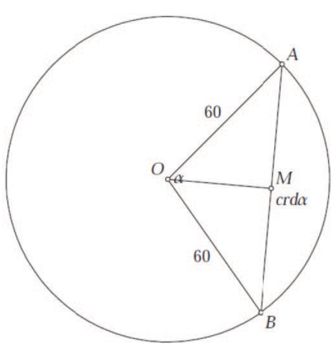

Fig. 1

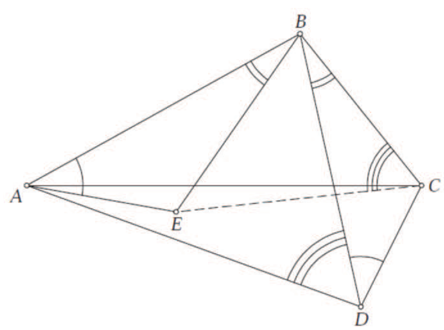

Fig. 2

$$\sin\frac{\alpha}{2} = \frac{AM}{60} = \frac{2AM}{120} = \frac{\operatorname{crd}\alpha}{120}$$
$$\operatorname{crd}\alpha = 120\sin\frac{\alpha}{2}.$$

Proof. Let E be a point in the plane of the given quadrilateral, such that the triangles AEB and DCB are similar, i.e., such that $\measuredangle EBA = \measuredangle CBD$ and $\measuredangle BAE = \measuredangle BDC$, and have the same orientation (the point E is on the same side of the line AB as the points C and D, Fig. 2). Obviously, the point E is different from the points A, B and C, since $\measuredangle CBA > \measuredangle CBD$ (for the first two points it is obvious).

Thus, since $\triangle AEB \sim \triangle DBC$, we obtain $\dfrac{AE}{CD} = \dfrac{EB}{BC} = \dfrac{AB}{BD}$, i.e.

$$BD = \frac{AB \cdot CD}{AE}, \tag{1}$$

and, since $\measuredangle DBA = \measuredangle CBE$ and $\dfrac{EB}{BC} = \dfrac{AB}{BD}$, we get $\triangle ADB \sim \triangle ECB$. Therefore, $\dfrac{DA}{EC} = \dfrac{BD}{BC}$, i.e.

$$BD = \frac{DA \cdot BC}{EC}. \tag{2}$$

Finally, by applying the triangle inequality to the triple of points A, E and C, we obtain $AE + EC \geqslant AC$, which is equivalent to

$$AE \cdot BD + EC \cdot BD \geqslant AC \cdot BD,$$

and then, by using (1) and (2), we get the desired inequality.

Observe that the equality holds if and only if the points A, E and C are collinear, which is equivalent to $\measuredangle BAC = \measuredangle BAE = \measuredangle BDC$, i.e., if and only if the quadrilateral $ABCD$ is cyclic. ∎

Mathematician Carl Anton Bretschneider (1808–1878), from Gotha (Germany), carried out the following generalization of Ptolemy's theorem, which refers to the product of diagonals in a convex quadrilateral.

Theorem 2. *Assume that $ABCD$ is a convex quadrilateral. Then*

$$AC^2 \cdot BD^2 = AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2 \cdot AB \cdot BC \cdot CD \cdot DA \cdot \cos(\measuredangle B + \measuredangle D),$$

is valid, where $\measuredangle B$ and $\measuredangle D$ are the inner angles at the vertices of the given quadrilateral.

Proof. Denote the lengths of the sides AB, BC, CD, DA, AC, BD of the given quadrilateral by a, b, c, d, e and f, respectively. Consider a point E, outside of the quadrilateral $ABCD$, such that $\measuredangle CDE = \measuredangle CBA$ and $\measuredangle DCE = \measuredangle BCA$. Then, $\triangle CDE \sim \triangle CBA$ (Fig. 3).

Thus, $\dfrac{CD}{CB} = \dfrac{DE}{BA} = \dfrac{EC}{AC}$, i.e., $\dfrac{c}{b} = \dfrac{DE}{a} = \dfrac{EC}{e}$, which implies $DE = \dfrac{ac}{b}$ and $EC = \dfrac{ce}{b}$. From the equality of the angles $\measuredangle BCD = \measuredangle ACE$ and side proportion $\dfrac{BC}{AC} = \dfrac{CD}{CE}$, it follows $\triangle BCD \sim \triangle ACE$. Hence, $\dfrac{BC}{AC} = \dfrac{CD}{CE} = \dfrac{BD}{AE}$, i.e., $\dfrac{b}{e} = \dfrac{c}{CE} = \dfrac{f}{AE}$. So, $CE = \dfrac{ce}{b}$ and $AE = \dfrac{ef}{b}$.

By using the law of cosines for the triangle ADE, we get

$$AE^2 = AD^2 + DE^2 - 2 \cdot AD \cdot DE \cdot \cos(\measuredangle ADE),$$

that is

$$\frac{e^2 f^2}{b^2} = d^2 + \frac{a^2 c^2}{b^2} - 2 \cdot d \cdot \frac{ac}{b} \cdot \cos(\measuredangle B + \measuredangle D),$$

hence

$$e^2 f^2 = b^2 d^2 + a^2 c^2 - 2abcd \cos(\measuredangle B + \measuredangle D). \quad \blacksquare$$

Note that if the quadrilateral $ABCD$ is cyclic, its opposite angles are supplementary, so

$$e^2 f^2 = b^2 d^2 + a^2 c^2 - 2abcd \cos 180^\circ = (ac + bd)^2$$

holds, i.e. $ef = ac + bd$. In general case, since $\cos(\measuredangle B + \measuredangle D) \geqslant -1$, we get $ef \geqslant ac + bd$, that also proves Ptolemy's theorem.

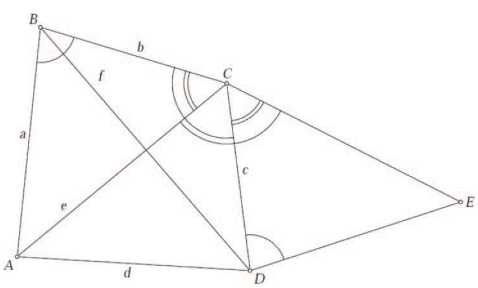

Fig. 3

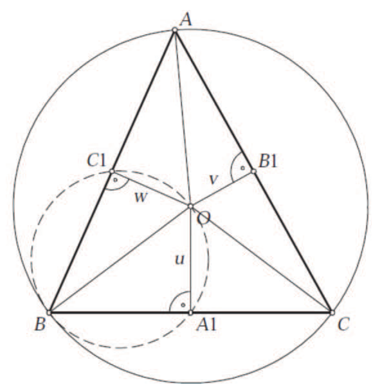

Fig. 4

3. Applications of Ptolemy's theorem

Example 1. Let R and r be the radii of the circumscribed circle and the inscribed circle, respectively, of an arbitrary acute-angled triangle ABC. If the distance from the circumcenter O to the related sides of that triangle is denoted by u, v and w, prove that

$$u + v + w = R + r.$$

Proof. Let a, b and c be the lengths of the sides of a given triangle and let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB, respectively. These points are also the feet of the perpendiculars from the center O to the sides (Fig. 4) of $\triangle ABC$.

The segments A_1B_1, B_1C_1 i C_1A_1 are the middle lines of the triangle ABC, so we have the following lengths equalities: $A_1B_1 = \dfrac{c}{2}$, $B_1C_1 = \dfrac{a}{2}$ and $C_1A_1 = \dfrac{b}{2}$. Also, it is obvious that $OB = OC = OA = R$.

By applying Ptolemy's theorem to the cyclic quadrilaterals BA_1OC_1, CB_1OA_1 and AC_1OB_1 (sums of the pairs of opposite angles are equal to 180°), we get

$$\frac{a}{2}\cdot w+\frac{c}{2}\cdot u=\frac{b}{2}\cdot R,$$
$$\frac{b}{2}\cdot u+\frac{a}{2}\cdot v=\frac{c}{2}\cdot R,$$
$$\frac{c}{2}\cdot v+\frac{b}{2}\cdot w=\frac{a}{2}\cdot R.$$

By adding these equalities, we get

$$\frac{a}{2}\cdot w+\frac{c}{2}\cdot u+\frac{a}{2}\cdot v+\frac{c}{2}\cdot v+\frac{b}{2}w=\frac{1}{2}(a+b+c)\cdot R. \tag{3}$$

Since O is the inner point of the triangle ABC, its area is equal to the sum of the areas of $\triangle BCO$, $\triangle CAO$ and $\triangle ABO$, i.e.,

$$P(\triangle ABC)=P(\triangle BCO)+P(\triangle CAO)+P(\triangle ABO),$$

and we obtain

$$\frac{a\cdot u}{2}+\frac{b\cdot v}{2}+\frac{c\cdot w}{2}=\frac{1}{2}\cdot(a+b+c)\cdot r. \tag{4}$$

Thus, from (3) and (4), we get

$$\frac{1}{2}\cdot(a+b+c)\cdot(u+v+w)=\frac{1}{2}(a+b+c)\cdot(R+r),$$

i.e., $u+v+w=R+r$. △

Remarks. 1° If $\triangle ABC$ is obtuse-angled with the obtuse angle at the vertex A (Fig. 5), then

$$\frac{b}{2}R+\frac{c}{2}u=\frac{a}{2}w,\quad \frac{c}{2}R+\frac{b}{2}u=\frac{a}{2}v,\quad \frac{b}{2}w+\frac{c}{2}v=\frac{a}{2}R.$$

Thus,

$$\frac{b}{2}R+\frac{c}{2}u+\frac{c}{2}R+\frac{b}{2}u+\frac{a}{2}R=\frac{a}{2}w+\frac{a}{2}v+\frac{b}{2}w+\frac{c}{2}v,$$

i.e.,

$$\frac{1}{2}(a+b+c)R=w\Big(\frac{a}{2}+\frac{b}{2}\Big)+v\Big(\frac{a}{2}+\frac{c}{2}\Big)-u\Big(\frac{b}{2}+\frac{c}{2}\Big). \tag{5}$$

Since O lies in the exterior of triangle ABC, the area of triangle ABC is equal to the sum of the areas of triangles BAO and OCA, reduced by the area of triangle BOC, so

$$\frac{cw}{2}+\frac{bv}{2}-\frac{au}{2}=\frac{1}{2}(a+b+c)r. \tag{6}$$

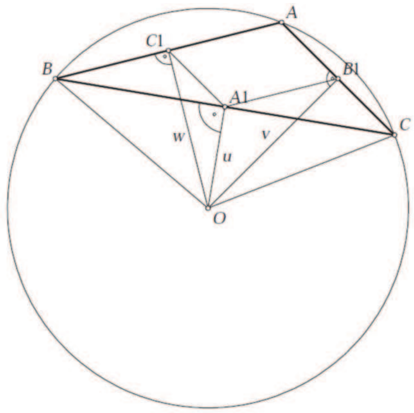

Fig. 5

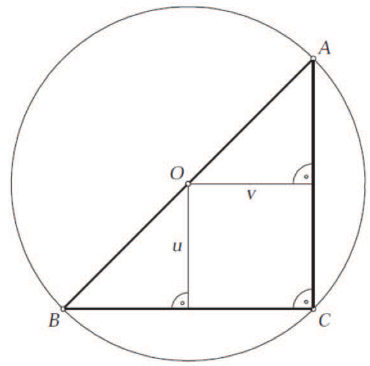

Fig. 6

By adding equalities (5) and (6), we obtain

$$\frac{1}{2}(a+b+c)\cdot(-u+v+w)=\frac{1}{2}(a+b+c)\cdot(R+r),$$

i.e., $-u+v+w=R+r$. Observe that, in the case that the triangle ABC is obtuse-angled, with the obtuse angle at the vertices B or C, respectively, then $u-v+w=R+r$ or $u+v-w=R+r$.

2° If the triangle ABC is right-angled, with the right angle at the vertex C (Fig. 6), then, by using well known relations $c=2R$ and $a+b=2(R+r)$, which are valid in right-angled triangles, we obtain ($\omega=0$, since the point O belongs to the hypotenuse of $\triangle ABC$)

$$u+v=\frac{b}{2}+\frac{a}{2}=R+r.$$

Example 2. Let $ABCD$ be an arbitrary cyclic and convex quadrilateral. If r_1, r_2, r_3 and r_4 are the lengths of the radii of inscribed circles in triangles ABC, BCD, CDA and DAB, respectively, then

$$r_1+r_3=r_2+r_4.$$

Proof. Let R be the length of radius of the circumscribed circle, with the center at the point O, of the quadrilateral $ABCD$ and let the distances from the point O to the sides AB, BC, CD, and DA, as well as to the diagonals AC and BD, be denoted by a, b, c, d, e and f, respectively (Fig. 7).

Without loss of the generality, let us assume that in the given quadrilateral there does not exist a pair of opposite right angles and that the angles $\measuredangle ABC$ and $\measuredangle BCD$ are obtuse, for example. Then, the angles $\measuredangle CDA=180^\circ-\measuredangle ABC$ and $\measuredangle DAB=180^\circ-\measuredangle BCD$ are acute. Thus, by applying the results obtained in Example 1 (and in the corresponding remarks), to the triangles ABC, BCD, CDA

and DAB, respectively, we get,

$$a+b-e=R+r_1, \qquad b+c-f=R+r_2,$$
$$c+d+e=R+r_3, \qquad a+d+f=R+r_4.$$

By adding up the first and the third relations, and than, the second and the fourth, we obtain the claim, i.e.

$$r_1+r_3=r_2+r_4. \quad \triangle$$

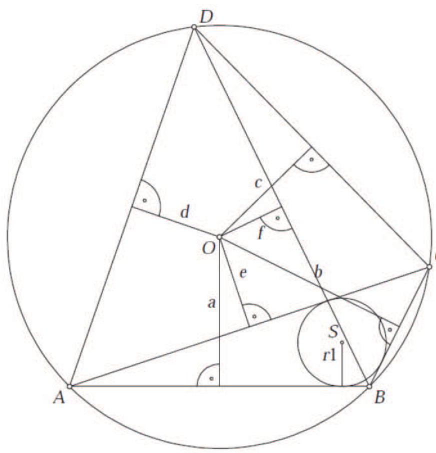

Fig. 7

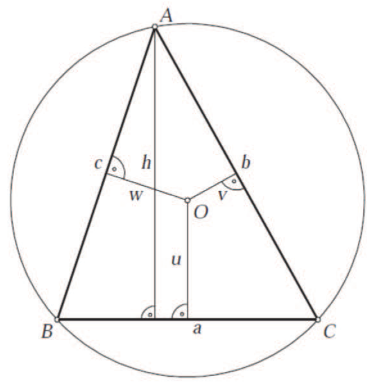

Fig. 8

Example 3. [Yugoslavia – State competition, PG 1992] If h is the length of the longest height of an acute-angled triangle ABC, then

$$R+r \leqslant h, \tag{7}$$

where R and r are radii of circumscribed and inscribed circles of the triangle, respectively.

Proof. Let us denote by a the length of the side of triangle ABC, corresponding to the longest height h (Fig. 8).

Since the shortest side of the triangle corresponds to the longest height, a is the shortest side, i.e., $a \leqslant b$ and $a \leqslant c$, where b and c are the lengths of other two sides of this triangle. Also, as $\triangle ABC$ is acute-angled, so the center of its circumscribed circle, i.e., the point O, lies in the interior of this triangle. Then, the area of triangle ABC is equal to the sum of areas of triangles BCO, CAO and ABO, i.e.,

$$P(\triangle ABC) = P(\triangle BCO) + P(\triangle CAO) + P(\triangle ABO).$$

Hence, $\dfrac{ah}{2} = \dfrac{au}{2} + \dfrac{bv}{2} + \dfrac{cw}{2}$, i.e.,

$$a(R+r) = a(u+v+w) \leqslant au+bv+cw = ah,$$

therefore (7) holds. $\triangle$

Remarks. 1° Note that the equality holds in (7) iff the triangle is equilateral.

2° This problem was assigned for the 12th grade, at the 1992 State high school student competition. Since it was not obvious that the task was a trivial consequence of the relations obtained by Ptolemy's theorem, only one of the students completely solved the problem.

4. Some problems from competitions with solutions and students' results

1. (AMC[1] 2016, 10A, Problem 24) [2] The quadrilateral is inscribed into a circle of radius $200\sqrt{2}$. The three sides of this triangle have length 200. What is the length of the fourth side?

(A) 200 (B) $200\sqrt{2}$ (C) $200\sqrt{3}$ (D) $300\sqrt{2}$ (E) 500

Solution. Let O be the center of circle and, for example, let $AB = BC = CD = 200$. Thus, $OA = OB = OC = OD = 200\sqrt{2}$ (Fig. 9).

Let us denote by X the intersection point of diagonals AC and OB of the quadrilateral $OABC$. This quadrilateral is a deltoid, so $CA = 2 \cdot CX$. Let us express the length h_a of the height, that corresponds to the side BC of the triangle OBC, by using Pythagorean theorem. Thus, from the isosceles triangle OBC, one can get $h_a = \sqrt{(200\sqrt{2})^2 - 100^2} = 100\sqrt{7}$. Further, we get $\dfrac{CX \cdot OB}{2} = \dfrac{CB \cdot h_a}{2}$, i.e.,

$$\frac{CX \cdot 200\sqrt{2}}{2} = \frac{200 \cdot 100\sqrt{7}}{2},$$

and, also, $CX = 50\sqrt{14}$ and $AC = 100\sqrt{14}$. By applying Ptolemy's theorem to the cyclic quadrilateral $ABCD$ we obtain

$$CA^2 = AD \cdot BC + AB \cdot CD.$$

That is $(100\sqrt{14})^2 = AD \cdot 200 + 200 \cdot 200$. So, $AD = 500$. △

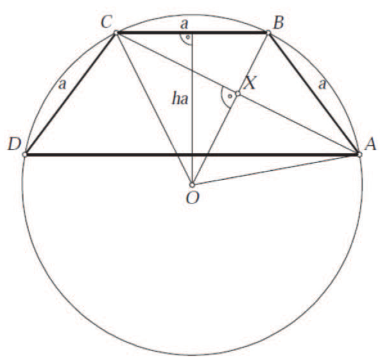

Fig. 9

2. (AMC 2017, 12A, Problem 24) [3] The quadrilateral $ABCD$ is inscribed in a circle with the center O and has the side lengths $AB = 3$, BC=2, $CD = 6$ and $DA = 8$. Let X and Y be points of the segment BD such that $\dfrac{DX}{BD} = \dfrac{1}{4}$ and $\dfrac{BY}{BD} = \dfrac{11}{36}$. Let the point E be the intersection of the line AX and the line through Y parallel to AD. Let F be the point of intersection of the line CX and line through E parallel to AC. Let G be a point on the circle, different from C, that belongs to the line CX. What is the value of $XF \cdot XG$?

[1] American Mathematics Competitions

(A) 17 (B) $\dfrac{59-5\sqrt{2}}{3}$ (C) $\dfrac{91-12\sqrt{3}}{4}$ (D) $\dfrac{67-16\sqrt{2}}{3}$ (E) 18

Solution. Let Z be the intersection of segments AC and BD. Then, $\triangle BCZ \sim \triangle ADZ$, so $\dfrac{BC}{AD} = \dfrac{BZ}{AZ}$, i.e.,

$$\frac{2}{8} = \frac{BZ}{AZ}. \tag{8}$$

Also, by using similarity $\triangle BZA \sim \triangle CZD$, we obtain

$$\frac{CD}{BA} = \frac{CZ}{BZ} \quad \text{and} \quad \frac{CD}{BA} = \frac{DZ}{AZ},$$

i.e.,

$$\frac{6}{3} = \frac{CZ}{BZ} \quad \text{and} \quad \frac{6}{3} = \frac{DZ}{AZ}. \tag{9}$$

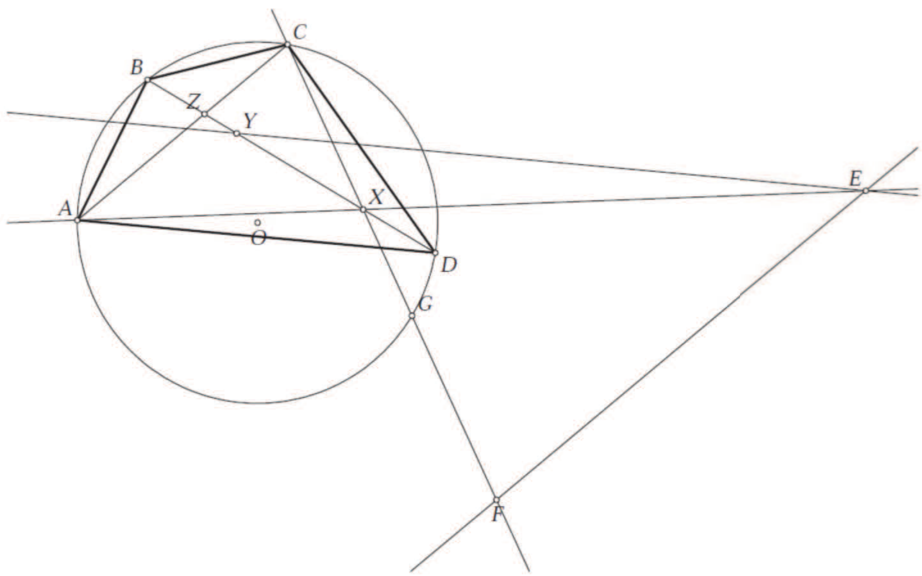

Fig. 10

From equalities (8) and (9), we find that $AZ = 4BZ$, $CZ = 2BZ$ and $DZ = 2AZ = 8BZ$. Finally, by applying Ptolemy's theorem to the cyclic quadrilateral $ABCD$, it follows that

$$AB \cdot CD + AD \cdot BC = AC \cdot BD = (AZ + ZC) \cdot (BZ + ZD).$$

That is,

$$3 \cdot 6 + 8 \cdot 2 = (4BZ + 2BZ) \cdot (BZ + 8BZ) = 54\, BZ^2.$$

Thus, $BZ^2 = \dfrac{34}{54}$.

Using the power of point X, with respect to the given circle, we get

$$GX \cdot XC = BX \cdot XD = \frac{3}{4}BD \cdot \frac{1}{4}BD = \frac{3}{16}BD^2 = \frac{3}{16}(BZ + ZD)^2$$
$$= \frac{3}{16}(BZ + 8BZ)^2 = \frac{3}{16} \cdot 81BZ^2 = \frac{9 \cdot 17}{16},$$

i.e., $GX = \dfrac{9 \cdot 17}{16 \cdot XC}$.

On the other hand,

$$XY = BD - BY - XD = BD - \frac{11}{36}BD - \frac{1}{4}BD = \frac{16}{36}BD = \frac{4}{9}BD,$$

so, by using the similarity of triangles AXD and EXY, we get $\dfrac{AD}{EY} = \dfrac{XD}{XY}$, i.e.

$$\frac{8}{EY} = \frac{\frac{1}{4}BD}{\frac{4}{9}BD} = \frac{9}{16}.$$

Hence, $EY = \dfrac{128}{9}$. Further we have $\dfrac{AD}{EY} = \dfrac{AX}{EX}$, i.e., $\dfrac{AX}{EX} = \dfrac{9}{16}$. Also, from the similarity of triangles ACX and EFX it holds $\dfrac{XF}{XC} = \dfrac{EF}{CA} = \dfrac{EX}{AX} = \dfrac{16}{9}$. Thus, $XF = \dfrac{16}{9}XC$ and, finally, $XF \cdot XG = \dfrac{16}{9}XC \cdot \dfrac{9 \cdot 17}{16 \cdot XC} = 17$. △

3. (AMC 2018, 12A, Problem 20) [4] The triangle ABC is isosceles and right-angled, with side lengths $AB = AC = 3$. Let M be the midpoint of hypotenuse BC. Points I and E lie on the sides AC and AB, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. The triangle EMI has area 2 and the length of the CI can be written as $\dfrac{a - \sqrt{b}}{c}$, where a, b and c are positive integers and b is not divisible by the square of any prime number. What is the value of $a + b + c$?

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

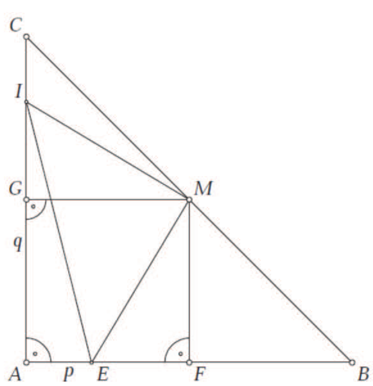

Fig. 11

Solution. Denote by F and G the points on the sides AB and AC, respectively, such that $MF \perp AB$ and $MG \perp AC$ (Fig. 11).

The point M is on the bisector $\measuredangle BAC$, so $MF = MG$. Since $AIME$ is a cyclic quadrilateral, it follows that $\measuredangle EMI = 90°$. Next, $\measuredangle EMF = 90° - \measuredangle GME = \measuredangle IMG$. From the congruent triangles EMF and IMG, we get $ME = MI$. Therefore, the triangle EMI is isosceles and right-angled. From the area of triangle EMI, we find that $MI = ME = 2$ and $EI = 2\sqrt{2}$ and, since M is the midpoint of hypotenuse BC, $AM = MB = \dfrac{BC}{2} = \dfrac{3\sqrt{2}}{2}$.

Let us denote by p and q the lengths of segments AE and AI, respectively. By applying the Ptolemy's theorem to the cyclic quadrilateral $AIME$, we obtain $MI \cdot AE + ME \cdot AI = EI \cdot AM$, i.e., $2 \cdot p + 2 \cdot q = 6$, i.e., $p + q = 3$. Further, by applying Pythagorean theorem to the triangle AEI, we get $p^2 + q^2 = 8$. Thus, we need to solve the system of equations $p + q = 3$ and $p^2 + q^2 = 8$, that leads us to $q = \dfrac{3+\sqrt{7}}{2}$ and $p = \dfrac{3-\sqrt{7}}{2}$. Hence, the required length of CI (by using the assumption) is $CI = 3 - q = \dfrac{3-\sqrt{7}}{2}$, so $a + b + c = 12$. △

In 2019, after the approval of a competent higher institution in the Republic of Serbia, we conducted a testing (which lasted 30 minutes) of all students of the 10th grade of one selected secondary school in Belgrade, bearing in mind that during the previous education they were introduced to the formulation, proof and applications of Ptolemy's theorem. We performed three types of tests, by dividing students into three (equal in number of students) groups in a random manner. As expected, each group contained almost equal number of students with similar achievements. The first group of students was solving Problem 1, without any pre-given information. Another group of students was solving Problem 2, and none of the students in that group was told that they should apply Ptolemys theorem to solve that problem. Finally, a third group of students was solving Problem 3, knowing in advance that each of them should apply Ptolemy's theorem when solving that task.

The first table below gives results on their achievements during this testing. From the first column we can conclude that even in the case of a simple request, unless the students are told in advance which theorem (property, relation, principle, method) they should use, they cannot achieve the expected results (which are in accordance with their grades). A similar situation, although Problem 2 is significantly more difficult, also occurred in the case of the second group of students. The third group of students, considering that they were aware in advance which concept of the acquired knowledge in elementary geometry should be applied, certainly achieved the best result.

Table 1 shows the percentage representation of students' responses when solving Problems 1, 2 and 3.

Answer	A	B	C	D	E	No answer	Unreadable
Problem 1	21.27	6.83	7.05	6.67	4.14	54.01	0.02
Problem 2	1.87	3.94	4.17	3.68	1.71	84.62	0.02
Problem 3	2.72	3.20	4.16	34.54	2.10	53.26	0.02

Table 1. Students' responses expressed in percentages

4. (IMO[2] 1995, Problem 5) [7] Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\measuredangle BCD = \measuredangle EFA = 60°$. Points G and H, inside the hexagon, are such that the angles AGB and DHE are equal to 120°.

[2] International Mathematical Olympiad

Prove that

$$AG + GB + GH + DH + HE \geqslant CE.$$

Solution. Triangles BCD and AFE are equilateral, because they are both isosceles and have one angle of 60°. The line BE is the axis of symmetry of quadrilateral $ABDE$ ($BA = BD$ and $EA = ED$). By symmetry, one can map the triangles BCD and AEF, with respect to the line BE, to the triangles $BC'A$ and DEF' (Fig. 12).

Since $\measuredangle AGB + \measuredangle BC'A = 180^\circ$, then $AC'BG$ is a cyclic quadrilateral and by applying the Ptolemy's theorem we obtain $AC' \cdot BG + BC' \cdot AG = AB \cdot C'G$, i.e., $BG + AG = C'G$. Similarly, $EF' \cdot HD + F'D \cdot EH = ED \cdot HF'$, i.e., $HD + EH = HF'$. It follows that

$$AG + GB + GH + DH + HE = C'G + GH + HF' \geqslant C'F' = CF.$$

Equality holds if and only if the points G and H belong to the line $C'F'$. △

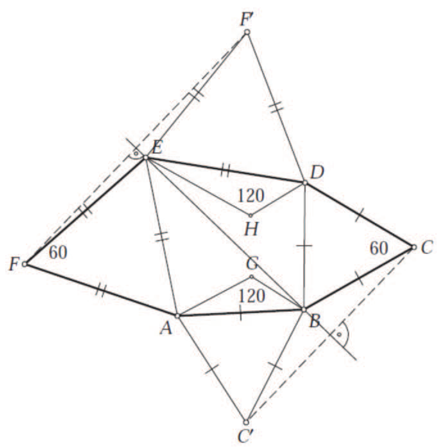

Fig. 12

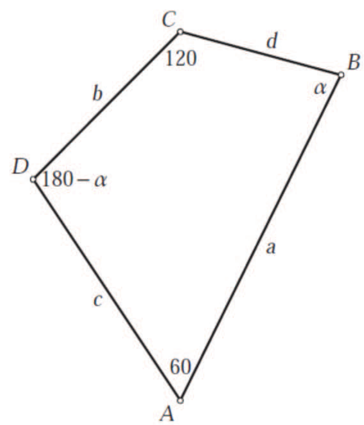

Fig. 13

5. (IMO 2001, Problem 6) [7] Let a, b, c and d be natural numbers such that $a > b > c > d$ and let

$$ac + bd = (b + d + a - c) \cdot (b + d - a + c).$$

Prove that $ab + cd$ is not a prime number.

Solution. We transform the given equality into

$$ac + bd = b^2 + 2bd + d^2 - a^2 + 2ac - c^2.$$

Thus, the equality $ac + bd = (b + d + a - c) \cdot (b + d - a + c)$ is equivalent to

$$a^2 - ac + c^2 = b^2 + bd + d^2. \tag{10}$$

Let $ABCD$ be the quadrilateral with the sides $AB = a$, $BC = d$, $AD = c$ and let $\measuredangle BAD = 60^\circ$ and $\measuredangle BCD = 120^\circ$ (Fig. 13). On the basis of (10) and the law of cosines, such a quadrilateral exists.

Denote $\measuredangle ABC = \alpha$. Then $\measuredangle CDA = 180^\circ - \alpha$. By applying the law of cosines to the triangle ABC, we obtain

$$AC^2 = a^2 + d^2 - 2ad\cos\alpha. \tag{11}$$

Similarly, from the triangle ACD, we find

$$AC^2 = b^2 + c^2 - 2bc\cos(180^\circ - \alpha) = b^2 + c^2 + 2bc\cos\alpha. \tag{12}$$

Then, by using the equalities (11) and (12), we get

$$b^2 + c^2 + 2bc\cos\alpha = a^2 + d^2 - 2ad\cos\alpha,$$

i.e., $2\cos\alpha \cdot (bc + ad) = a^2 + d^2 - b^2 - c^2$. Thus, $2\cos\alpha = \dfrac{a^2 + d^2 - b^2 - c^2}{bc + ad}$. Hence,

$$\begin{aligned} AC^2 &= a^2 + d^2 - ad\frac{a^2 + d^2 - b^2 - c^2}{bc + ad} \\ &= \frac{ab(ac + bd) + cd(bd + ac)}{bc + ad} = \frac{(ac + bd)(ab + cd)}{bc + ad}. \end{aligned}$$

Since the quadrilateral $ABCD$ is cyclic ($\measuredangle BAD + \measuredangle BCD = 180^\circ$), then, by using Ptolemy's theorem, we obtain $AC^2 \cdot BD^2 = (ab + cd)^2$, i.e.,

$$\frac{(ac + bd)(ab + cd)}{bc + ad} \cdot (a^2 + c^2 - 2ac\cos 60^\circ) = (ab + cd)^2$$

hence

$$(ac + bd) \cdot (a^2 + c^2 - ac) = (bc + ad) \cdot (ab + cd). \tag{13}$$

Since $a > b > c > d$, then $(a - d) \cdot (b - c) > 0$, i.e., $ab + cd > ac + bd$. Also, $(a - b) \cdot (c - d) > 0$, i.e., $ac + bd > ad + bc$. Thus,

$$ab + cd > ac + bd > ad + bc.$$

Finally, let $ab + cd$ be a prime number. Then, from (14), we get that $ab + cd$ and $ac + bd$ are mutually prime. So, from (13), it must be that $ac + bd$ divides $bc + ad$, but this cannot be true by (14). $\triangle$

Table 2 shows the achievement of IMO students, in percentages, in solving Problems 4 and 5. The maximum number of points, that can be won on each task, is 7. Since usually brilliant students, with extraordinary capabilities, compete on IMO, we could assume, in the case that they were told in advance that (in Problems 4 and 5) they should apply Ptolemy's Theorem, that they would all solve this type of problems.

Number of points	0	1	2	3	4	5	6	7
Problem 4	34.22	11.65	7.28	1.94	0.24	1.70	1.21	41.75
Problem 5	80.33	4.23	3.38	1.69	2.11	1.90	0.63	5.71

Table 2. Students' success rate expressed in percentages

5. Conclusion

Ptolemy's theorem is one of the most advanced theorems in the stream of elementary geometry, over the centuries. Ptolemy used the principles of similar triangles to prove the first version of the theorem. The consequences are significant and are seen through the linking of areas in the secondary school curriculums. Ptolemy's theorem and its proof introduces many geometric facts into the system, such as Pythagorean theorem, some of trigonometric identities, etc. This theorem makes it possible to connect algebraic relations with elementary geometry, where the notion of cyclic quadrilateral surely occupies one of the crucial places. Its greater application in secondary school can be considered useful in linking, developing and deepening the students' knowledge.

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