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A RECONSIDERATION OF: "NUMBER SYSTEMS CHARACTERIZED BY THEIR OPERATIVE PROPERTIES"

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Abstract. The starting point of our approach to the number systems is the selection of basic operative properties of the system $\mathbb{N}_0$ of natural numbers with 0. This set of properties has proved itself to be sufficient for extension of this system to the systems of integers $\mathbb{Z}$, positive rational numbers $\mathbb{Q}_+ \cup \{0\}$ and rational numbers $\mathbb{Q}$ and for the formation of basic operative properties of these extended systems.

In all these cases of number systems, the corresponding set of numbers with basic operative properties is an example of a concrete algebraic structure. These structures can be viewed abstractly as the structure $(S, +, \cdot, <)$, where S is a non-empty set, "+" and "·" are two binary operations on S and "<" is the order relation on S, which satisfy the postulated conditions that are formed according to the basic operative properties of these systems. When matched up with $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}_+ \cup \{0\}$ and $\mathbb{Q}$, the structure $(S, +, \cdot, <)$ is called ordered semifield, ordered semifield with additive inverse, ordered semifield with multiplicative inverse and ordered field, respectively. Then, these number systems are characterized as being the smallest semifield with which they fit together. Proofs of these facts require deduction of some properties of all mentioned types of this abstract structure upon which they will be clearly relied. Hence, the main aim of this paper is this deduction and some improvements of proofs contained in the paper whose reconsideration is this note.

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1. Introduction

Let us begin with the explanation of the main ideas of our approach to the number systems (see [1]). First of all, we have selected the list of basic operative properties of the system $\mathbb{N}_0$ of natural numbers with 0. These properties have proved themselves to be sufficient when the system $\mathbb{N}_0$ is extended to the systems of integers, positive rational numbers with 0 and rational numbers.

What is also specific for our approach is the deduction from the list of basic properties of $\mathbb{N}_0$ a series of its properties, which then serve as the basis for its further extension. Namely, among these properties there exist the relations which express the sum of two differences and two quotients as a difference and a quotient, respectively as well as the product of two sums and two quotients as a difference and a product, respectively. In addition, the conditions under which two differences and two quotients are related by the relation "<" are also given. As our terminology suggests it clearly all such differences and such quotients are supposed to be defined in $\mathbb{N}_0$. Thus, extending $\mathbb{N}_0$ we extend the validity of these relations and of this condition, using them for definitions of sums, products and the order relation in the extended systems. This is, of course, a significant difference between our approach and the existing approaches where such definitions are given formally.

All extensions of the system $\mathbb{N}_0$ have the form of a construction of the extended systems (see [1]). Thus, the existence of such systems is effective what also proves that their basic properties are not a contradictory set of conditions.

Taken abstractly, the systems $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}_+$ with 0 and $\mathbb{Q}$ can be viewed as the structure $(S, +, \cdot, <)$, where S is a non-empty set, "+" and "$\cdot$" are two binary operations on the set S and "<" is the order relation on S. Basic operative properties of these number systems are postulated as the properties of this abstract structure and then, their specific variables are replaced by $a, b, c, \ldots$ For example, when the basic operative properties of $\mathbb{N}_0$ are transcribed, the following list is obtained

(i) $(\forall a)(\forall b)\, a+b=b+a$

(ii) $(\forall a)(\forall b)(\forall c)\,(a+b)+c=a+(b+c)$

(iii) $(\exists 0)(\forall a)\, a+0=a$

(iv) $(\forall a)(\forall b)\, a\cdot b=b\cdot a$

(v) $(\forall a)(\forall b)(\forall c)\,(a\cdot b)\cdot c=a\cdot(b\cdot c)$

(vi) $(\exists 1)(0<1 \textit{ and } (\forall a)\, a\cdot 1=a)$

(vii) $(\forall a)(\forall b)(\forall c)\, a\cdot(b+c)=a\cdot b+a\cdot c$

(viii) $(\forall a)(\forall b)\,(a<b \iff (\exists c>0)\, a+c=b)$

(ix) $(\forall a)(\forall b)\,(a<b \textit{ or } a=b \textit{ or } b<a)$

(x) $(\forall a)(\forall b)(\forall c)$
$(a<b \iff a+c<b+c)$

(xi) $(\forall a)(\forall b)(\forall c>0)$
$(a<b \iff a\cdot c<b\cdot c)$

List 1

and the structure which satisfies conditions on this list is called the *ordered semifield*. In the same way, the structure $(S, +, \cdot, <)$ which satisfies conditions on *List 2*, Section 2 of this paper is called the *ordered semifield with additive inverse*, when this structure satisfies conditions on *List 3*, Section 3 of this paper it is called the *ordered semifield with multiplicative inverse* and finally, when this structure satisfies conditions on *List 4*, Section 4 of this paper it is called the *ordered field*. The systems $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}_+$ with 0 and $\mathbb{Q}$ are characterized as the smallest ordered semifields satisfying conditions on *List 1*, *List 2*, *List 3* and *List 4*, respectively. The clear proofs of these facts require deduction of some properties from the lists of postulated properties of the structure $(S, +, \cdot, <)$. The aim of this paper is to form a series of such properties (1_S)–(3_S), $(4_S)_a$–$(11_S)_a$, $(4_S)_m$–$(9_S)_m$ and to apply them to the proofs of the above facts. These properties are very well known and related to different algebraic structures. But it is easier to group them according to the type of a semifield and sketch their proofs than to search for references. Hence, we improve here our paper [2].

2. Characterization of the system $\mathbb{Z}$ via its operative properties

As it has already been said, the number system $\mathbb{Z}$ can be viewed as a structure $(S, +, \cdot, <)$, where S is a non-empty set, “$+$” and “$\cdot$” are two binary operations on S and “$<$” is an order relation on S. We denote identity elements in S writing 0_S and 1_S, but we leave operations and the order relation without subscript S. The list of properties of this abstract structure is given below as *List 2*, and we call this structure the *semifield with additive inverse*.

(i) $(\forall a)(\forall b)\, a + b = b + a$

(ii) $(\forall a)(\forall b)(\forall c)\, (a + b) + c = a + (b + c)$

(iii) $(\exists 0_S)(\forall a)\, a + 0_S = a$

(iv) $(\forall a)(\exists b)\, a + b = 0_S$

(v) $(\forall a)(\forall b)\, a \cdot b = b \cdot a$

(vi) $(\forall a)(\forall b)(\forall c)\, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(vii) $(\exists 1_S)(0_S < 1_S \textit{ and } (\forall a)\, a \cdot 1_S = a)$

(viii) $(\forall a)(\forall b)(\forall c)\, a \cdot (b + c) = a \cdot b + a \cdot c$

(ix) $(\forall a)(\forall a)\, (a < b \iff (\exists c > 0_S)\, a + c = b)$

(x) $(\forall a)(\forall b)\, (a < b \textit{ or } a = b \textit{ or } b < a)$

(xi) $(\forall a)(\forall b)(\forall c)$
$(a < b \iff a + c < b + c)$

(xii) $(\forall a)(\forall b)(\forall c > 0_S)$
$(a < b \iff a \cdot c < b \cdot c)$

List 2

Now we start listing the properties of the semifield with additive inverse.

In each ordered semifield the following three properties hold:

(1_S) *The identity elements* 0_S *and* 1_S *are unique.*

Suppose $0'$ is also the additive identity element. Then, $0' + 0_S$ is equal to 0_S when $0'$ is the identity element and to $0'$ when 0_S is the identity element. Hence, $0' = 0_S$. Similarly 1_S is proved to be unique.

(2_S) $(\forall a \in S)\, a \cdot 0_S = 0_S$.

From $0_S + 0_S = 0_S$, by (vii) on *List 1*, $a \cdot 0_S + a \cdot 0_S = a \cdot 0_S$. Since 0_S is unique, it follows that $a \cdot 0_S = 0_S$.

(3_S) *For each* $a, b, c \in S$, $a = b \iff a + c = b + c$.

According to (x), *List 1*, $a < b \iff a + c < b + c$ and $b < a \iff b + c < a + c$. Thus, the only possibility for $a = b$ is to be equivalent to $a + c = b + c$.

Now we use the postulated properties of the ordered semifield with additive inverse (*List 2*) to deduce some of its additional properties. The additive inverse of $a \in S$ will be denoted by $-a$.

$(4_S)_a$ $(\forall a \in S) - (-a) = a$.

Indeed, from $(-a) + a = 0_S$, it follows that $-(-a) = a$.

$(5_S)_a$ *Additive inverse is unique.*

Suppose that a' is another additive inverse of a, i.e., $a + a' = 0_S$. Then, applying (3_S), we have $(-a)+(a+a') = (-a)$, i.e., $-a = ((-a)+a)+a' = 0_S+a' = a'$.

$(6_S)_a$ $(\forall a, b \in S)\, a < b \iff -b < -a$.

According to (xi), *List 2*,

$$\begin{aligned} a < b &\iff a + (-b) < b + (-b) \iff a + (-b) < 0_S \\ &\iff (-a) + (a + (-b)) < (-a) \iff -b < -a. \end{aligned}$$

In particular, $a > 0_S \iff -a < 0_S$.

$(7_S)_a$ $(\forall a, b \in S)\, ((-a) \cdot b = a \cdot (-b) = -(a \cdot b))$ *and* $(-(a + b) = (-a) + (-b))$.

Indeed, $a \cdot b + (-a) \cdot b = (a + (-a)) \cdot b = 0_S \cdot b = 0_S$. Then, according to $(5_S)_a$, $-(a \cdot b) = (-a) \cdot b$. Similarly, $-(a \cdot b) = a \cdot (-b)$ is proved.

$(a + b) + ((-a) + (-b)) = a + b + (-a) + (-b) = 0_S$.

$(8_S)_a$ *The value of a sum of arbitrary many summands does not depend on the order of summands. The same is true for products.* (See [3]).

$(9_S)_a$ *For all* $a, b, c \in S$, *when* $c < 0_S$ *then* $a < b \iff b \cdot c < a \cdot c$.

Applying property (xii), *List 2* and then $(7_S)_a$ and $(6_S)_a$, we have

$$a < b \iff (-c) \cdot a < (-c) \cdot b \iff -(c \cdot a) < -(c \cdot b) \iff c \cdot b < c \cdot a.$$

$(10_S)_a$ *Let* $a, b \in S$. *Then,*

(i) $a > 0_S$ *and* $b > 0_S$ *implies* $a \cdot b > 0_S$.

(ii) $(a > 0_S$ *and* $b < 0_S)$ *or* $(a < 0_S$ *and* $b > 0_S)$ *implies* $a \cdot b < 0_S$.

(iii) $a < 0_S$ *and* $b < 0_S$ *implies* $a \cdot b > 0_S$.

Simple verification of these implications relies upon (xii), *List 2* and the property $(8_S)_a$.

From $(10_S)_a$ it immediately follows:

$(11_S)_a$ (i) $(\forall a, b \in S)(a \neq 0_S$ *and* $b \neq 0_S$ *implies* $a \cdot b \neq 0_S)$,

i.e.,

(ii) $(\forall a, b \in S)(a \cdot b = 0_S$ *implies* $a = 0_S$ *or* $b = 0_S)$.

When proving the following statement we use the mapping defined in Chapter 5 of [1], as well as the properties $(4_S)_a$–$(11_S)_a$. When which of these properties, the reader will find it easily.

The system $\mathbb{Z}$ of integers is the smallest ordered semifield with additive inverse.

As $\mathbb{Z}_+ \cup \{0\}$ is an isomorphic copy of $\mathbb{N}_0$, we consider the mapping $n \mapsto a_n$ to be defined on this set of non-negative integers by $a_0 = 0_S$ and $a_{n+1} = a_n + 1_S$. Let us extend it to the mapping $\varphi\colon \mathbb{Z} \to S$, taking $\varphi(n) = a_n$, $n \in \mathbb{Z}_+ \cup \{0\}$ and $\varphi(-n) = -a_n$, $-n \in \mathbb{Z}_-$.

As for each $n \in \mathbb{Z}_+ \cup \{0\}$, $a_n < a_{n+1}$, i.e., $\varphi(n) < \varphi(n+1)$, applying (6_S), it follows that $-a_{n+1} < -a_n$, i.e., $\varphi(-(n+1)) < \varphi(-n)$. Thus, it follows immediately that φ is a one to one mapping which also preserves the order relation.

Let us suppose that $m > 0$ and $n > 0$. Then, $a_{m+n} = a_m + a_n$, i.e., $\varphi(m+n) = \varphi(m) + \varphi(n)$. Being $-a_{m+n} = -(a_m + a_n) = (-a_m) + (-a_n)$, we also have $\varphi(-(m+n)) = \varphi((-m) + (-n)) = \varphi(-m) + \varphi(-n)$.

For $n > m$, $\varphi(n + (-m)) = \varphi(n - m) = a_{n-m}$ and from $a_{n-m} + a_m = a_n$, i.e., $a_{n-m} = a_n + (-a_m)$, it follows that $\varphi(n+(-m)) = \varphi(n)+\varphi(-m)$. For $n < m$,

$$\varphi(n + (-m)) = \varphi(-(m - n)) = -a_{m-n} = (-a_m) + a_n = \varphi(n) + \varphi(-m).$$

Thus, we have proved that φ preserves the operation "+".

Let m, n be positive integers. Then,

$$\varphi(m \cdot n) = a_{m\cdot n} = a_m \cdot a_n = \varphi(m) \cdot \varphi(n).$$

Furthermore,

$$\varphi((-m) \cdot n) = \varphi(-m \cdot n) = -a_{m\cdot n} = -(a_m \cdot a_n) = (-a_m) \cdot a_n = \varphi(-m) \cdot \varphi(n)$$

and

$$\varphi((-m) \cdot (-n)) = \varphi(m \cdot n) = a_{m\cdot n} = a_m \cdot a_n = (-a_m) \cdot (-a_n) = \varphi(-m) \cdot \varphi(-n).$$

Thus, we have proved that $\varphi[\mathbb{Z}]$ is an isomorphic copy of the ordered semifield $\mathbb{Z}$.

3. Characterization of the system $\mathbb{Q}_+$ with 0 via its operative properties

Similarly as in the previous chapter, the number system $\mathbb{Q}_+ \cup \{0\}$ can be viewed as a structure $(S, +, \cdot, <)$, where S is a non-empty set, "+" and "·" are two binary operations on S and "<" is an order relation on S. The list of properties of this abstract structure is given below as *List 3*. We call this structure the *semifield with multiplicative inverse.*

(i) $(\forall a)(\forall b)\, a + b = b + a$

(ii) $(\forall a)(\forall b)(\forall c)\, (a + b) + c = a + (b + c)$

(iv) $(\forall a)(\forall b)\, a \cdot b = b \cdot a$

(v) $(\forall a)(\forall b)(\forall c)\, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $(\exists 0_S)(\forall a)\, a + 0_S = a$

(vi) $(\exists 1_S)(0_S < 1_S$ *and* $(\forall a)\, a \cdot 1_S = a)$

(vii) $(\forall a \neq 0_S)(\exists b)\, a \cdot b = 1_S$

(viii) $(\forall a)(\forall b)(\forall c)\, a \cdot (b + c) = a \cdot b + a \cdot c$

(ix) $(\forall a)(\forall b)\, (a < b \iff (\exists c > 0_S)\, a + c = b)$

(x) $(\forall a)(\forall b)\, (a < b$ *or* $a = b$ *or* $b < a)$

(xi) $(\forall a)(\forall b)(\forall c)$
$(a < b \iff a + c < b + c)$

(xii) $(\forall a)(\forall b)(\forall c > 0_S)$
$(a < b \iff a \cdot c < b \cdot c)$

List 3

Now we use the postulated properties of the ordered semifield with multiplicative inverse (*List 3*) to deduce some of its additional properties. The multiplicative inverse of $a \in S$, $a \neq 0_S$ will be denoted by a^{-1}.

$(4_S)_m$ $(\forall a, b, c \in S,\, c \neq 0_S)$ $a \cdot c = b \cdot c \iff a = b.$

This follows from the existence of a multiplicative inverse.

$(5_S)_m$ (i) $(\forall a \in S \setminus \{0_S\})\, a^{-1} \neq 0_S$.
(ii) $(\forall a \in S \setminus \{0_S\})\, (a^{-1})^{-1} = a$.

Assertion (i) is a consequence of (2_S). Then, from $a^{-1} \cdot a = 1_S$, it follows that $(a^{-1})^{-1} = a$.

$(6_S)_m$ *For* $a \neq 0_S$, a^{-1} *is unique multiplicative inverse of* a.

Suppose a' is another multiplicative inverse of a, i.e., $a \cdot a' = 1_S$. Then, $a' = a' \cdot 1_S = a' \cdot (a \cdot a^{-1}) = (a' \cdot a) \cdot a^{-1} = 1_S \cdot a^{-1} = a^{-1}$.

(7_S) $a > 0_S$ $(a < 0_S)$ *implies* $a^{-1} > 0_S$ $(a^{-1} < 0_S)$.

Let $a > 0_S$. By $(4_S)_m$, $a^{-1} \neq 0_S$. If $a^{-1} < 0_S$, applying (xii), *List 3* it would follow that $a \cdot a^{-1} < a \cdot 0_S$, i.e., $1_S < 0_S$ what contradicts (vi), *List 3*. Hence, $a^{-1} > 0_S$.

$a < 0_S$ and $a^{-1} > 0_S$ implies $a \cdot a^{-1} < 0_S \cdot a^{-1}$, i.e., $1_S < 0_S$, what contradicts again (vi), *List 3*. Hence, $a^{-1} < 0_S$.

$(8_S)_m$ *If* $a, b \neq 0_S$ *then* $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$.

This follows from $(a \cdot b) \cdot (a^{-1} \cdot b^{-1}) = (a \cdot a^{-1}) \cdot (b \cdot b^{-1}) = 1_S \cdot 1_S = 1_S$.

$(9_S)_m$ (i) $a \cdot b = 0_S$ *implies* $a = 0_S$ *or* $b = 0_S$,
i.e.,
(ii) $a \neq 0_S$ *and* $b \neq 0_S$ *implies* $a \cdot b \neq 0_S$.

This property was deduced as $(11_S)_a$ in the case of a semifield with additive inverse. In the case of a semifield with multiplicative inverse it can be shown as follows. When $a \cdot b = 0_S$ and $a \neq 0_S$ then $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0_S$, i.e., $b = 0_S$. Similarly, when $a \cdot b = 0_S$ and $b \neq 0_S$ it follows that $a = 0_S$. The statement under (ii) is just logical transposition of the statement under (i).

Recall that the set $\mathbb{Q}_+ \cup \{0\}$ of positive rational numbers with zero can be defined as

$$\mathbb{Q}_+ \cup \{0\} = \{\, k : l \mid k, l \in \mathbb{N}_0,\ l \neq 0\,\},$$

with $k : l = k' : l' \iff kl' = k'l$. Then, the system $(\mathbb{Q}_+ \cup \{0\}, +, \cdot, <)$ is an ordered semifield with multiplicative inverse. Now, using the properties $(4_S)_m$–$(9_S)_m$, and again the mapping $n \mapsto a_n$ given by $a_0 = 0_S$ and $a_{n+1} = a_n + 1_S$ for $n \in \mathbb{N}_0$, the following statement will be proved.

The system of positive rational numbers with 0*,* $(\mathbb{Q}_+ \cup \{0\}, +, \cdot, <)$ *is the smallest semifield with multiplicative inverse.*

Let $(S, +, \cdot, <)$ be an arbitrary ordered semifield with multiplicative inverse and define the mapping $\psi\colon \mathbb{Q}_+ \cup \{0\} \to S$ by

$$\psi(k : l) = a_k \cdot a_l^{-1}, \qquad \text{for } k, l \in \mathbb{N}_0,\ l \neq 0.$$

We show that this mapping is injective and that it preserves operations “+” and “·” and relation “<”.

Let $(k_1 : l_1), (k_2 : l_2) \in \mathbb{Q}_+ \cup \{0\}$ be such that $\psi(k_1 : l_1) = \psi(k_2 : l_2)$. Then $a_{k_1} \cdot a_{l_1}^{-1} = a_{k_2} \cdot a_{l_2}^{-1}$. It follows that $a_{k_1} \cdot a_{l_2} = a_{k_2} \cdot a_{l_1}$, hence $a_{k_1 l_2} = a_{k_2 l_1}$. This means that $k_1 l_2 = k_2 l_1$ and finally $k_1 : l_1 = k_2 : l_2$. Thus, ψ is an injective mapping.

For arbitrary elements $(k_1 : l_1)$, $(k_2 : l_2)$ of $\mathbb{Q}_+ \cup \{0\}$ the following holds:

$$\begin{aligned}\psi((k_1 : l_1) + (k_2 : l_2)) &= \psi((k_1 l_2 + k_2 l_1) : (l_1 l_2)) = a_{k_1 l_2 + k_2 l_1} \cdot a_{l_1 l_2}^{-1}\\ &= (a_{k_1} a_{l_2} + a_{k_2} a_{l_1}) \cdot (a_{l_1} a_{l_2})^{-1} = (a_{k_1} \cdot a_{l_1}^{-1}) + (a_{k_2} \cdot a_{l_2})^{-1}\\ &= \psi(k_1 : l_1) + \psi(k_2 : l_2).\end{aligned}$$

For all $(k_1 : l_1)$, $(k_2 : l_2) \in \mathbb{Q}_+ \cup \{0\}$ we have that

$$\begin{aligned}\psi((k_1 : l_1) \cdot (k_2 : l_2)) &= \psi((k_1 k_2) : (l_1 l_2)) = a_{k_1 k_2} \cdot a_{l_1 l_2}^{-1}\\ &= (a_{k_1} a_{k_2}) \cdot (a_{l_1} a_{l_2})^{-1} = (a_{k_1} \cdot a_{l_1}^{-1}) \cdot (a_{k_2} \cdot a_{l_2})^{-1}\\ &= \psi(k_1 : l_1) \cdot \psi(k_2 : l_2).\end{aligned}$$

Let $k_1 : l_1$ and $k_2 : l_2$ be elements of $\mathbb{Q}_+ \cup \{0\}$ such that $k_1 : l_1 < k_2 : l_2$. Then $k_1 l_2 < k_2 l_1$, and it follows that $a_{k_1 l_2} < a_{k_2 l_1}$. Further, $a_{k_1} a_{l_2} < a_{k_2} a_{l_1}$, and finally $a_{k_1} \cdot a_{l_1}^{-1} < a_{k_2} \cdot a_{l_2}^{-1}$, i.e., $\psi(k_1 : l_1) < \psi(k_2 : l_2)$.

Thus, we have proved that $\psi[\mathbb{Q}_+ \cup \{0\}]$ is an isomorphic copy of $\mathbb{Q}_+ \cup \{0\}$.

4. Characterization of the system $\mathbb{Q}$ via its operative properties

An ordered semifield with both, additive and multiplicative inverses is standardly called the *ordered field*. The postulated properties of the ordered field form the list that just follows.

(i) $(\forall a)(\forall b)\, a+b=b+a$

(ii) $(\forall a)(\forall b)(\forall c)\, (a+b)+c=a+(b+c)$

(iii) $(\exists 0_S)(\forall a)\, a+0_S=a$

(iv) $(\forall a)(\exists b)\, a+b=0_S$

(v) $(\forall a)(\forall b)\, ab=ba$

(vi) $(\forall a)(\forall b)(\forall c)\, (ab)c=a(bc)$

(vii) $(\exists 1_S)(0_S<1_S \textit{ and } (\forall a)\, a\cdot 1_S=a)$

(viii) $(\forall a\neq 0_S)(\exists b)\, ab=1_S$

(ix) $(\forall a)(\forall b)(\forall c)\, a(b+c)=ab+ac$

(x) $(\forall a)(\forall b)\, (a<b \iff (\exists c>0_S)\, a+c=b)$

(xi) $(\forall a)(\forall b)\, (a<b \textit{ or } a=b \textit{ or } b<a)$

(xii) $(\forall a)(\forall b)(\forall c)$
$(a<b \iff a+c<b+c)$

(xiii) $(\forall a)(\forall b)(\forall c>0_S)$
$(a<b \iff ac<bc)$

List 4

All properties of the ordered semifields (1_S)–(3_S), $(4_S)_a$–$(11_S)_a$ and $(4_S)_m$–$(9_S)_m$ are valid, of course, for the ordered field.

The main example of an ordered field is the structure $(\mathbb{Q},+,\cdot,<)$ of rational numbers, where

$$\mathbb{Q}=\{\, z:k \mid z\in\mathbb{Z},\ k\in\mathbb{N}\,\},$$

with $z:k=z':k' \iff zk'=z'k$. Characterization of the system of rational numbers via its operational properties is a very well-known fact but, for the sake of completeness we formulate it here as the following statement which is also accompanied with a proof.

The system of rational numbers $(Q,+,\cdot,<)$ is the smallest ordered field.

Let $(S,+,\cdot,<)$ be an arbitrary ordered field and let $\varphi\colon \mathbb{Z}\to S$ be the mapping used in Section 2. Let us define the mapping $\chi\colon \mathbb{Q}\to S$ by

$$\chi(z:k)=\varphi(z)\cdot\varphi(k)^{-1}, \qquad \text{for } z\in\mathbb{Z},\ k\in\mathbb{N}.$$

We will prove that χ is an injective mapping, and that it preserves operations "+" and "·" and relation "<".

Let $(z_1:k_1),(z_2:k_2)\in\mathbb{Q}$ be such that $\chi(z_1:k_1)=\chi(z_2:k_2)$. Then $\varphi(z_1)\cdot\varphi(k_1)^{-1}=\varphi(z_2)\cdot\varphi(k_2)^{-1}$. Or else $\varphi(z_1)\cdot\varphi(k_2)=\varphi(z_2)\cdot\varphi(k_1)$, what implies that $\varphi(z_1k_2)=\varphi(z_2k_1)$. This means that $z_1k_2=z_2k_1$ and hence $z_1:k_1=z_2:k_2$. This proves that χ is an injective mapping.

The following holds for arbitrary elements $(z_1:k_1)$, $(z_2:k_2)$ in $\mathbb{Q}$:

$$\begin{aligned}\chi((z_1:k_1)+(z_2:k_2)) &= \chi((z_1k_2+z_2k_1):(k_1k_2))=\varphi(z_1k_2+z_2k_1)\cdot\varphi(k_1k_2)^{-1}\\ &= (\varphi(z_1)\varphi(k_2)+\varphi(z_2)\varphi(k_1))\cdot(\varphi(k_1)\varphi(k_2))^{-1}\\ &= (\varphi(z_1)\cdot\varphi(k_1)^{-1})+(\varphi(z_2)\cdot\varphi(k_2)^{-1})\\ &= \chi(z_1:k_1)+\chi(z_2:k_2).\end{aligned}$$

Let $(z_1 : k_1), (z_2 : k_2) \in \mathbb{Q}$; then:

$$\begin{aligned}\chi((z_1 : k_1) \cdot (z_2 : k_2)) &= \chi((z_1 z_2) : (k_1 k_2)) = \varphi(z_1 z_2) \cdot \varphi(k_1 k_2)^{-1} \\ &= (\varphi(z_1)\varphi(z_2)) \cdot (\varphi(k_1)\varphi(k_2))^{-1} = (\varphi(z_1) \cdot \varphi(k_1)^{-1}) \cdot (\varphi(z_2) \cdot \varphi(k_2)^{-1}) \\ &= \chi(z_1 : k_1) \cdot \chi(z_2 : k_2).\end{aligned}$$

Let $z_1 : k_1$ and $z_2 : k_2$ be elements of $\mathbb{Q}$ satisfying $z_1 : k_1 < z_2 : k_2$. Then $z_1 k_2 < z_2 k_1$, implying that $\varphi(z_1 k_2) < \varphi(z_2 k_1)$. It follows that $\varphi(z_1)\varphi(k_2) < \varphi(z_2)\varphi(k_1)$ and finally $\varphi(z_1) \cdot \varphi(k_1)^{-1} < \varphi(z_2) \cdot \varphi(k_2)^{-1}$, i.e., $\chi(z_1 : k_1) < \chi(z_2 : k_2)$.

Thus, we have proved that $\chi[\mathbb{Q}]$ is an isomorphic copy of $\mathbb{Q}$.

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