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ON SOME PROPERTIES OF TRIANGLE *OIG*

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Abstract. Let O be the circumcenter of a triangle ABC, I the incenter and G the centroid of ABC. In this paper, we study properties of the triangle OIG.

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1. Introduction

Let R and r be the circumradius and the inradius of a triangle ABC, respectively, and let p be the semiperimeter of ABC. Denote the circumcenter of ABC by O, its incenter by I, the centroid of ABC by G and its orthocenter by H. Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, $a = BC$, $b = CA$, $c = AB$ and let m_a be the length of median AM, where M is the midpoint of BC.

In this paper, we will prove the following results:

(1) If two of the points O, I, G coincide then the triangle ABC is regular.

(2) Let the triangle ABC be not regular. The points O, I, G are collinear iff $\triangle ABC$ is isosceles (in this case, the point G lies on the segment IO).

(3) Let the triangle ABC be not isosceles. Then the triangle OIG is obtuse-angled; in this case,

$$\cos\angle IGO = -\frac{p^2 - 10Rr - 7r^2}{2\sqrt{(p^2 + 5r^2 - 16Rr)(9R^2 + 2r^2 + 8Rr - 2p^2)}} < 0$$

and $\angle IGO > \pi/2$.

(4) Let the triangle ABC be not isosceles. Then the triangle OIG is isosceles iff $p^2 = 3R^2 + 8Rr - r^2$ and $R \geq \frac{8}{3}r$.

(5) There is a single rectangular triangle ABC (up to similarity transformation) such that the triangle IOG is isosceles; moreover, this triangle ABC is similar to the triangle with sides $3+\sqrt{2}+\sqrt{1+2\sqrt{2}}$, $3+\sqrt{2}-\sqrt{1+2\sqrt{2}}$ and $4+2\sqrt{2}$.

(6) There does not exist a triangle ABC with an angle $\pi/3$ such that the triangle IOG is isosceles.

(7) Let the triangle ABC be not isosceles. Then the area of $\triangle IOG$ is

$$S(\triangle IOG) = \frac{1}{12}\sqrt{4R(R-2r)^3 - (p^2 - (2R^2 + 10Rr - r^2))^2}.$$

In the book [3], it is proved that the triangle ABC is uniquely determined by parameters p, R, r. These numbers cannot be arbitrary; they have to satisfy the so-called fundamental inequality

$$(p^2 - 2R^2 - 10Rr + r^2)^2 \le 4R(R - 2r)^3. \tag{1}$$

Moreover, arbitrary positive real numbers p, R, r satisfying the inequality (1) are the semiperimeter, the circumradius and the inradius, respectively, of some triangle ABC (see [3]). Further, in the book [1], it is shown that $IO^2 = R^2 - 2Rr$, $OG^2 = R^2 - \dfrac{a^2+b^2+c^2}{9}$ and $IG^2 = \dfrac{9r^2 - 3p^2 + 2(a^2+b^2+c^2)}{9}$. Since $a^2+b^2+c^2 = 2(p^2 - r^2 - 4Rr)$ (see [3]), we have

$$OG^2 = \frac{9R^2 + 2r^2 + 8Rr - 2p^2}{9}, \quad IG^2 = \frac{p^2 + 5r^2 - 16Rr}{9}.$$

2. Auxiliary statements

Before proving the main result, let us consider some auxiliary statements.

Proposition 1. *If two of the points O, I, G coincide in a triangle ABC then this triangle is regular.*

Proof. Let $O = I$ in a triangle ABC. Consider the isosceles triangle AOB. We have that $\alpha/2 = \beta/2$, hence $\alpha = \beta$. In the same way, we can prove that $\alpha = \gamma$.

Now let $O = G$. Consider again the isosceles triangle AOB. Since the point of intersection of the medians divides them in ratio $2:1$, we have $\frac{2}{3}m_a = \frac{2}{3}m_b = R$. Hence, $m_a = m_b$. Since $m_a^2 = \dfrac{b^2+c^2}{2} - \dfrac{a^2}{4}$, and similarly for m_b (see [1]), it follows that $\dfrac{b^2+c^2}{2} - \dfrac{a^2}{4} = \dfrac{a^2+c^2}{2} - \dfrac{b^2}{4}$, hence $a = b$. In the same way, we can show that $a = c$.

Finally, let $I = G$. Then the bisectrix AA_1 is a median of the triangle ABC (Fig. 1) and $BA_1 = A_1C = \dfrac{a}{2}$. By [1], $\dfrac{c}{b} = \dfrac{BA_1}{A_1C} = \dfrac{a/2}{a/2} = 1$, i.e., $b = c$. Similarly, $a = c$ and the triangle ABC is regular. ∎

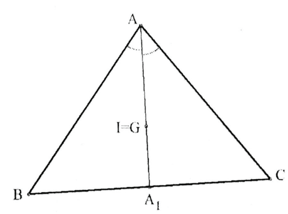

Fig. 1

Assume now that a triangle ABC is not regular. In this case, by Proposition 1, the points I, O, G are pairwise distinct. Now we study the case when these points are collinear.

Lemma 1. *For any triangle ABC the following inequalities hold:*

(1) *$R \ge 2r$; moreover, $R = 2r$ iff the triangle ABC is regular;*

(2) *$16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$, with equalities iff the triangle ABC is regular (see [3]).*

Lemma 2. *If a triangle ABC is not regular then $OI > OG$ and $OI > IG$.*

Proof. Assume that a triangle ABC is not regular. Since $R > 2r$ (Lemma 1), we have

$$\begin{aligned} OI^2 - OG^2 &= R^2 - 2Rr - \frac{9R^2 + 2r^2 + 8Rr - 2p^2}{9} = \frac{2}{9}(p^2 - 13Rr - r^2) \\ &\geq \frac{2}{9}(16Rr - 5r^2 - 13Rr - r^2) = \frac{2}{3}(R - 2r)r > 0. \end{aligned}$$

Hence, $OI > OG$. Further,

$$\begin{aligned} OI^2 - IG^2 &= R^2 - 2Rr - \frac{p^2 + 5r^2 - 16Rr}{9} = \frac{9R^2 - 2Rr - p^2 - 5r^2}{9} \\ &\geq \frac{(9R^2 - 2Rr - 5r^2) - (4R^2 + 4Rr + 3r^2)}{9} \\ &= \frac{5R^2 - 6Rr - 8r^2}{9} = \frac{(R - 2r)(5R + 4r)}{9} > 0, \end{aligned}$$

because $R > 2r$. Hence, $OI > IG$. ∎

Proposition 2. *Let a triangle ABC be not regular. Then the points I, O, G are collinear iff the triangle ABC is isosceles.*

Proof. Assume that a triangle ABC is isosceles and, for example, $AB = AC$. By Proposition 1, the points I, O, G are pairwise distinct and lie on the altitude passing through the vertex A.

Conversely, suppose that the points I, O, G are collinear for a nonregular triangle ABC. By Lemma 2, it means that $IO = IG + GO$. Therefore,

$$\begin{aligned} OI^2 - IG^2 - GO^2 &= R^2 - 2Rr - \frac{9R^2 + 2r^2 + 8Rr - 2p^2 + p^2 + 5r^2 - 16Rr}{9} \\ &= \frac{p^2 - 7r^2 - 10Rr}{9} = 2\, IG \cdot GO. \end{aligned}$$

Hence,

$$\begin{aligned} (p^2 - 7r^2 - 10Rr)^2 &= 4 \cdot 9\, IG^2 \cdot 9\, GO^2 \\ &= 4(p^2 + 5r^2 - 16Rr)(9R^2 + 2r^2 + 8Rr - 2p^2), \\ (p^2 - (2R^2 + 10Rr - r^2))^2 &= 4R(R - 2r)^3, \\ p^2 &= (2R^2 + 10Rr - r^2) \pm 2(R - 2r)\sqrt{R^2 - 2Rr}. \end{aligned}$$

We have arrived to the case of equality in the fundamental triangle inequality. By [3, p. 13], it is proved that the triangle ABC is in this case isosceles. ∎

Remark 1. In [4], it is proved that

$$\begin{aligned} &(1 - \cos(\alpha - \beta))(1 - \cos(\alpha - \gamma))(1 - \cos(\beta - \gamma)) \\ &\qquad = \frac{4R(R - 2r)^3 - (p^2 - 2R^2 - 10Rr + r^2)^2}{8R^4}. \end{aligned}$$

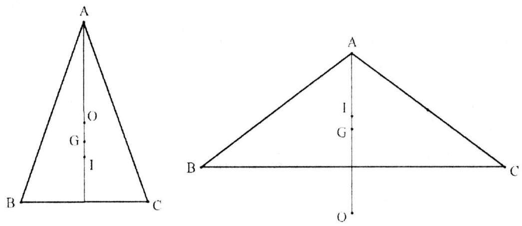

Fig. 2

If $4R(R-2r)^3-(p^2-2R^2-10Rr+r^2)^2=0$ then, for example, $\cos(\alpha-\beta)=1$ and $\alpha=\beta$. So, the point G lies inside the segment OI for an isosceles triangle ABC (Fig. 2).

PROPOSITION 3. *If a triangle ABC is not isosceles then the triangle IGO is obtuse-angled and*

$$\cos\angle IGO=-\frac{p^2-10Rr-7r^2}{2\sqrt{(p^2+5r^2-16Rr)(9R^2+2r^2+8Rr-2p^2)}}<0.$$

Proof. By the cosine theorem, we have

$$\begin{aligned}\cos\angle IGO&=\frac{IG^2+OG^2-IO^2}{2\,IG\cdot OG}\\&=\frac{\frac{p^2+5r^2-16Rr}{9}+\frac{9R^2+2r^2+8Rr-2p^2}{9}-R^2+2Rr}{2\,IG\cdot OG}\\&=-\frac{p^2-10Rr-7r^2}{18\,IG\cdot OG}\\&=-\frac{p^2-10Rr-7r^2}{2\sqrt{(p^2+5r^2-16Rr)(9R^2+2r^2+8Rr-2p^2)}}.\end{aligned}$$

By Proposition 1 and Lemma 1, $p^2>16Rr-5r^2$. It implies that

$$p^2-10Rr-7r^2>(16Rr-5r^2)-(10Rr+7r^2)=6r(R-2r)>0.$$

Thus, $\cos\angle IGO<0$ and $\angle IGO>\pi/2$. ■

PROPOSITION 4. *Let a triangle ABC be not isosceles. Then the triangle IGO is isosceles iff $p^2=3R^2+8Rr-r^2$ and $R\geq\frac{8}{3}r$.*

Proof. Since $\angle IGO>\pi/2$, the triangle IGO is isosceles iff $IG=GO$, i.e.,

$$\frac{p^2+5r^2-16Rr}{9}=\frac{9R^2+2r^2+8Rr-2p^2}{9}.$$

The last equality is equivalent to $p^2 = 3R^2+8Rr-r^2$. Replacing p^2 by $3R^2+8Rr-r^2$ in the fundamental inequality (1), we have

$$(3R^2 + 8Rr - r^2 - 2R^2 - 10Rr + r^2)^2 \leq 4R(R - 2r)^3,$$
$$R^2(R - 2r)^2 \leq 4R(R - 2r)^3, \quad R \geq \frac{8}{3}r,$$

because $R > 2r$. ∎

Corollary 1. *There is a single right-angled triangle ABC (up to similarity transformation) such that the triangle IOG is isosceles; moreover, this triangle ABC is similar to the triangle with sides $3+\sqrt{2}+\sqrt{1+2\sqrt{2}}$, $3+\sqrt{2}-\sqrt{1+2\sqrt{2}}$ and $4+2\sqrt{2}$.*

Proof. Assume that a triangle ABC is right-angled (with right angle at C) and that the respective triangle IOG is isosceles. Then $2R+r = c+\dfrac{a+b-c}{2} = p$. It follows that $p^2 = 4R^2 + 4Rr + r^2 = 3R^2 + 8Rr - r^2$, wherefrom $R = (2 \pm \sqrt{2})r$ (see Proposition 4). Since $R \geq \frac{8}{3}r > 2r$, we have $R = (2 + \sqrt{2})r$. Hence,

$$c = 2R = (4 + 2\sqrt{2})r, \quad p = 2R + r = (5 + 2\sqrt{2})r,$$
$$a + b = 2p - c = (6 + 2\sqrt{2})r, \quad c^2 = a^2 + b^2 = (24 + 16\sqrt{2})r^2.$$

It means that (for $a > b$) $a = (3+\sqrt{2}+\sqrt{1+2\sqrt{2}})r$ and $b = (3+\sqrt{2}-\sqrt{1+2\sqrt{2}})r$. ∎

Corollary 2. *If the triangle IOG is isosceles then all the angles of triangle ABC are different from $\pi/3$.*

Proof. In [3], it is proved that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are roots of the equation

$$4R^2x^3 - 4R(R + r)x^2 + (p^2 + r^2 - 4R^2)x + (2R + r)^2 - p^2 = 0. \tag{2}$$

Let $\alpha = \pi/3$. Then $\cos\alpha = 1/2$ is a root of the equality (2), i.e.,

$$4R^2 \cdot \frac{1}{8} - 4R(R + r) \cdot \frac{1}{4} + (p^2 + r^2 - 4R^2) \cdot \frac{1}{2} + (2R + r)^2 - p^2 = 0.$$

It follows that $p = (R + r)\sqrt{3}$. If the triangle IOG were isosceles, it would follow that $p^2 = 3R^2 + 8Rr - r^2 = 3(R + r)^2$ by Proposition 4. Hence, $R = 2r$ and the triangle ABC would be regular (see Lemma 1), a contradiction. Thus, such triangle ABC does not exist. ∎

The following lemma follows from Heron's formula.

Lemma 3. *Let $S = S(\triangle ABC)$ be the area of a triangle ABC. Then*

$$S^2 = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{16}.$$

3. Main result

Using Lemma 3, we will calculate the area $S(\triangle IOG)$ of the triangle IOG for a given triangle ABC.

THEOREM 1. Let ABC be an arbitrary triangle. Then

$$S^2(\triangle IOG) = \frac{1}{144}\big(4R(R-2r)^3 - (p^2 - 2R^2 - 10Rr + r^2)^2\big).$$

Proof. Only the case when the triangle ABC is not isosceles has to be treated.

Let $IO = c_1$, $IG = a_1$, $OG = b_1$. Then $c_1^2 = R^2 - 2Rr$, $a_1^2 = \dfrac{p^2 + 5r^2 - 16Rr}{9}$, $b_1^2 = \dfrac{9R^2 + 2r^2 + 8Rr - 2p^2}{9}$. By Lemma 3, we have

$$\begin{aligned}
S^2(\triangle IGO) &= \Bigg(\frac{4}{81}(p^2+5r^2-16Rr)(9R^2+2r^2+8Rr-2p^2)\\
&\quad - \Big(\frac{p^2+5r^2-16Rr}{9} + \frac{9R^2+2r^2+8Rr-2p^2}{9} - R^2 + 2Rr\Big)^2\Bigg)\cdot\frac{1}{16}\\
&= \frac{1}{16\cdot 81}\big(4(p^2+5r^2-16Rr)(9R^2+2r^2+8Rr-2p^2) - (-p^2+10Rr+7r^2)^2\big)\\
&= \frac{1}{16\cdot 81}\big(4[(9R^2+2r^2+8Rr)(5r^2-16Rr)\\
&\quad + p^2(9R^2+2r^2+8Rr+(-10r^2+32Rr)) - 2p^4]\\
&\quad - p^4 - (10Rr+7r^2)^2 + 2p^2(10Rr+7r^2)\big)\\
&= \frac{1}{16\cdot 81}\big(-9p^4 + p^2(20Rr+14r^2+36R^2-32r^2+160Rr)\\
&\quad + 4(45R^2r^2 - 144R^3r + 10r^4 - 32Rr^3 + 40Rr^3 - 128R^2r^2)\\
&\quad - 100R^2r^2 - 49r^4 - 140Rr^3\big)\\
&= \frac{1}{16\cdot 81}\big(-9p^4 + p^2(36R^2 - 18r^2 + 180Rr)\\
&\quad + (-576R^3r - 432R^2r^2 - 9r^4 - 108Rr^3)\big)\\
&= \frac{1}{16\cdot 81}\big(-p^4 + p^2(4R^2 - 2r^2 + 20Rr) + (-64R^3r - 48R^2r^2 - r^4 - 12Rr^3)\big)\\
&= \frac{1}{144}\big(4R(R-2r)^3 - (p^2 - (2R^2 + 10Rr - r^2))^2\big). \quad \blacksquare
\end{aligned}$$

COROLLARY 3. *Let ABC be an arbitrary triangle. Then*

$$(p^2 - 2R^2 - 10Rr + r^2)^2 \le 4R(R-2r)^3$$

(the fundamental inequality for a triangle).

Corollary 4. *For any triangle ABC the following holds*

$$S(\triangle IOG) = \frac{2}{3}R^2 \sin\Big(\frac{|\alpha-\beta|}{2}\Big)\sin\Big(\frac{|\beta-\gamma|}{2}\Big)\sin\Big(\frac{|\gamma-\alpha|}{2}\Big).$$

Proof. Let a triangle ABC be isosceles and $\alpha=\beta$. Then $\sin\Big(\frac{|\alpha-\beta|}{2}\Big)=0$, i.e., by Proposition 2, both left-hand and right-hand sides of our equality are equal to zero.

Now, let a triangle ABC be not isosceles. By Remark 1 and Theorem 1,

$$\begin{aligned} S^2(\triangle IOG) &= \frac{1}{144}\big(4R(R-2r)^3-(p^2-2R^2-10Rr+r^2)^2\big)\\ &= \frac{8R^4}{144}(1-\cos(\alpha-\beta))(1-\cos(\beta-\gamma))(1-\cos(\gamma-\alpha))\\ &= \frac{4}{9}R^4\sin^2\Big(\frac{|\alpha-\beta|}{2}\Big)\sin^2\Big(\frac{|\beta-\gamma|}{2}\Big)\sin^2\Big(\frac{|\gamma-\alpha|}{2}\Big), \end{aligned}$$

and the desired formula follows. ∎

In [2, Chapter 1], it is noted that Corollary 5.1° was proved by R. Sondat and E. Lemoine in 1891.

Corollary 5. *Let ABC be an arbitrary triangle. Then*

1° $$S(\triangle OIH) = 2R^2\sin\Big(\frac{|\alpha-\beta|}{2}\Big)\sin\Big(\frac{|\beta-\gamma|}{2}\Big)\sin\Big(\frac{|\gamma-\alpha|}{2}\Big),$$

2° $$S(\triangle GIH) = \frac{4}{3}R^2\sin\Big(\frac{|\alpha-\beta|}{2}\Big)\sin\Big(\frac{|\beta-\gamma|}{2}\Big)\sin\Big(\frac{|\gamma-\alpha|}{2}\Big).$$

Proof. It is known that the points H, G, O lie on the same (Euler) line (see [1])). Moreover, $HO = 3\,OG$ and $HG = 2\,OG$. Hence, $S(\triangle IOH) = 3\,S(\triangle IOG)$, $S(\triangle IGH) = 2\,S(\triangle IOG)$ and the results follow from Corollary 4. ∎

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REFERENCES

[1] S.I. Zetel, *New Geometry in Triangle* (in Russian), Moscow, 1962.

[2] D. Mitrinović, J. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., the Netherlands, 1989.

[3] V. Soltan, S. Maidman, *Identites and inequalities in a triangle* (in Russian), Shtiinca Publ., Kishinev, 1982.

[4] Yu.N. Maltsev, A.S. Monastyreva, *On some remarkable points and line segments in triangles*, Izvestiya AltSU (accepted for publication).

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