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OUTCOMES FROM AN EXPERIMENTAL TEACHING—ON TWO TYPES OF ISOPTICS FOR CURVES WITH ASYMPTOTES

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Abstract. The paper presents the outcomes of a long-term individual educational trajectory for a team of secondary school students. The topic isoptic of a curve with asymptotes appears naturally when two viewpoints on isoptic are compared. The result about hyperbola hints that the curve with asymptote needs special attention when considering the locus of visibility, but not just the locus of points from which the tangents meet at a constant angle. The particular cases for the exponent and logarithm are studied. The proofs are based on calculations by a computer algebra system (CAS), and the illustrations are done by a dynamic geometry system (DGS).

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1. The experimental teaching

The results in the article were obtained during a long-term educational project that started in 2017. The initial goals of this project were to introduce some concepts from advanced mathematics in extracurricular activities following [4]. The platform for this experimental teaching was the agreement for collaboration between 125th Secondary School (Sofia, Bulgaria) and the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences [6]. The inquiry-based approach of teaching-learning provided an opportunity to go beyond the initially stated problems and to enter a real research area.

The modern computer algebra (CAS) and dynamic geometry (DGS) systems allow investigating some issues, for which we did not find information in the sources we studied. However, the implementation of CAS and DGS by secondary school students requires solid math background and competence of synthetic type (synthetic competence), which is developed along an individual educational trajectory [5]. The final stage of this experimental teaching is involving students in a real research process. Below we present the mathematical content of our activity.

2. Defining the topic area

There are a lot of results about isoptic curves and a rather comprehensive review is given in [1]. The isoptic curve (or simply isoptic) of a figure could be defined in two ways:

Definition 1. The locus k_α of all points where two different tangents of a planar curve k meet at a given constant angle α is called the *isoptic curve of k* (corresponding to α). [8]

Definition 2. To any given plane curve k the locus k_α of points from which k is seen under a given fixed angle α is called the *isoptic curve of k* (corresponding to α). [7]

Further, we consider $\alpha < \pi$. Following [9], the smallest angle with vertex P containing the figure is the angle under which the figure is seen from P. To distinguish the objects that appear as a result of both definitions, let us introduce the notations $\alpha T(k)$ and $\alpha V(k)$ for the loci defined by the Definitions 1 and 2, respectively.

The first definition is preferred by most of the authors. In confirmation to this, all mentioned isoptics in [12] are defined in αT mode. The second definition of isoptic curve seems to be more general. In fact, both definitions are independent. Let us clarify this by two examples.

Example 1. Case that escapes from the scope of Definition 1 is Problem 7 in [9]:

Given a square, find the locus of points from which the square is seen under an angle: (a) of 90°; (b) of 45° (Figure 1)[1].

In this case, Definition 1 is not applicable (there are no tangents to the square).

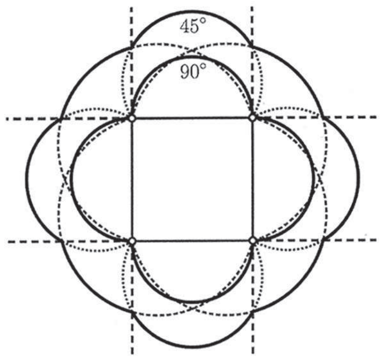

Fig. 1

Example 2. Case in which Definition 2 does not provide interesting object is given by Remark 1.2.3 in [1]:

Let K be an outer point of a hyperbola H. If K is in a focal domain, then the tangent lines are tangent to the same branch of the hyperbola, else they touch both branches. In these cases, the isoptic angles are complementary to each other, i.e. they sum up to π. Therefore, we take the square of the equation and thus obtain both types of isoptic curves.

Having in mind that $\alpha < \pi$, we conclude that $\alpha V(\mathrm{H})$ does not exist for the entire hyperbola H. According to the Example 2, the isoptic does not distinguish $\alpha T(\mathrm{H})$ from $(\pi - \alpha)T(\mathrm{H})$.

3. Orthoptics of a hyperbola

In this section, we are going to pay attention to isoptics when α is a right angle. In these cases, we speak about *orthoptic curves* and we use the notations oT and oV for αT and αV respectively.

[1]The figures are done by GeoGebra [3].

Given the hyperbola H: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the point $P(x_0; y_0)$, the following result is well known:

THEOREM 1. *If $a > b$, then $oT(\mathrm{H})$ is the circle Ω: $x^2+y^2 = a^2-b^2$ (Figure 2). If $a{=}b$, then $oT(\mathrm{H})$ is the point $O(0;0)$. If $a < b$, then $oT(\mathrm{H})$ does not exist.*

A detailed CAS-based reasoning of Theorem 1 is done in [2].

In fact, Theorem 1 needs a clarification: either the points which are on the asymptotes $y = \pm\frac{b}{a}x$ should be excluded from Ω or the asymptotes should be declared as "tangents at infinity". In any case, the asymptotes need special attention when dealing with isoptics.

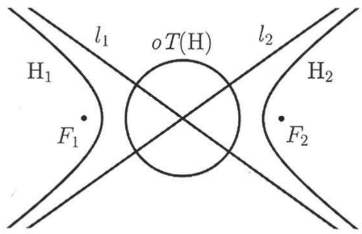

Fig. 2. $oT(\mathrm{H})$ when $a > b$

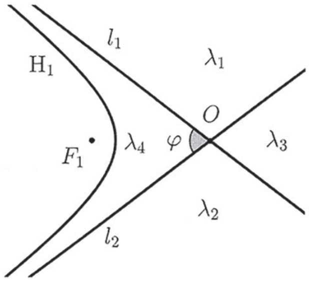

Fig. 3. Additional notations

Initially, Apollonius of Perga used the word hyperbola for one-branched (i.e. connected) curve [10]. Further in this section, H_1 stands for the $(x < 0)$-branch of H. Some other notations are shown in Figure 3: F_1 is the focus of H_1; l_1 and l_2 are the asymptotes; φ is the angle between l_1 and l_2 containing H_1; λ_1, λ_2, λ_3 and λ_4 are the quarters in which l_1 and l_2 split the plane.

Here are some important observations. Let P be a point outside of H_1. H_1 is seen from P under an angle which is formed by:

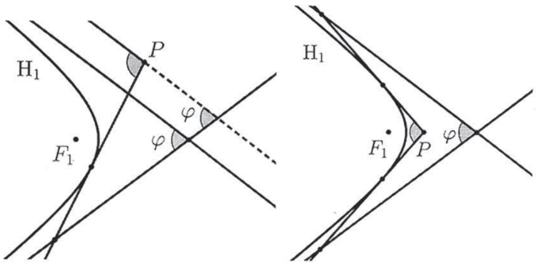

Fig. 4. Visibility of H_1 from P when $P \in \lambda_1$ (left) and $P \in \lambda_4$ (right)

1. the tangent line through P to H_1 and the line through P which is parallel to l_1 when $P \in \lambda_1$ (Figure 4 – left);
2. the tangent line through P to H_1 and the line through P which is parallel to l_2 when $P \in \lambda_2$;

3. the two lines through P which are parallel to l_1 and l_2 when $P \in \lambda_3$;
4. the two tangent lines through P to H_1 when $P \in \lambda_4$ (Figure 4 – right).

Having in mind these observations, we obtain

THEOREM 2. *$oT(\mathrm{H}_1) \subset oV(\mathrm{H}_1)$ and:*

1. If $a > b$, i.e. $\varphi < \pi/2$, then:

1.1. $oT(\mathrm{H}_1)$ is the arc $\overset{\frown}{\Omega}$ of the circle Ω: $x^2 + y^2 = a^2 - b^2$ which is located in λ_4;

1.2. $oV(\mathrm{H}_1)$ consists of the same arc $\overset{\frown}{\Omega}$ and two rays (Figure 5):

$r_1^{\rightarrow} = t_1 \cap \lambda_1$ where t_1: $y = \frac{ax}{b} + \frac{\sqrt{a^4-b^4}}{b}$;

$r_1^{\leftarrow} = t_2 \cap \lambda_2$ where t_2: $y = -\frac{ax}{b} - \frac{\sqrt{a^4-b^4}}{b}$;

2. If $a = b$, i.e. $\varphi = \pi/2$, then:

2.1. $oT(\mathrm{H}_1)$ is the point $O(0;0)$;

2.2. $oV(\mathrm{H}_1)$ coincides with λ_3.

3. If $a < b$, i.e. $\varphi > \pi/2$, then $oT(\mathrm{H}_1) = oV(\mathrm{H}_1) = \varnothing$.

Proof. We are going to consider only the case $\varphi < \pi/2$. Theorem 1 yields that $oT(\mathrm{H}_1)$ is the arc $\overset{\frown}{\Omega}$ of the circle Ω: $x^2 + y^2 = a^2 - b^2$ which is located in λ_4. Denote by $P_1(x_1; y_1) \in \lambda_1$ the common point of Ω and the asymptote l_1 which is in λ_1. Solving the simultaneous equations

$$y = -\frac{b}{a}x, \quad x^2 + y^2 = a^2 - b^2,$$

and taking into account that $P_1 \in \lambda_1$, we obtain

$$x_1 = -a\sqrt{\frac{a^2 - b^2}{a^2 + b^2}}, \quad y_1 = b\sqrt{\frac{a^2 - b^2}{a^2 + b^2}}.$$

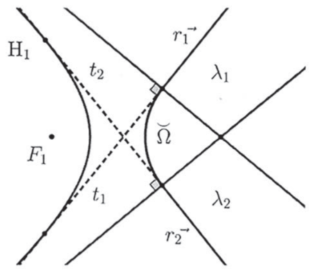

Fig. 5. $oV(\mathrm{H}_1)$ when $a > b$

Since the tangent through P_1 to H_1 is perpendicular to the asymptote, its equation is

$$t_1: \quad y - b\sqrt{\frac{a^2 - b^2}{a^2 + b^2}} = \frac{a}{b}\left(x + a\sqrt{\frac{a^2 - b^2}{a^2 + b^2}}\right),$$

which is equivalent to

$$t_1: \quad y = \frac{ax}{b} + \frac{\sqrt{a^4 - b^4}}{b}.$$

Let $r_1^{\rightarrow} = t_1 \cap \lambda_1$. According to the observation 1, H_1 is seen from any point on $r_1^{\rightarrow}$ under a right angle. Analogously, the second "moustache" is another ray $r_2^{\rightarrow}$ of

$$t_2: \quad y = -\frac{ax}{b} - \frac{\sqrt{a^4 - b^4}}{b}.$$

Thus, $oV(\mathrm{H}_1) = \overset{\frown}{\Omega} \cup r_1^{\rightarrow} \cup r_2^{\rightarrow}$.

4. Isoptic construction algorithm

In order to implement CAS for constructing the isoptic of a smooth convex curve F: $y = f(x)$, we perform the following simple algorithm:

1. Take points $A(a; f(a))$ and $B(b; f(b))$ on F.
2. Let $k_a = f'(a)$ and $k_b = f'(b)$. Solve the equation
$$\left|\frac{k_a - k_b}{1 + k_a k_b}\right| = |\tan\alpha|$$
for b in terms of a.
3. Solving the simultaneous equations
$$\left|\begin{array}{l} y - f(a) = k_a(x - a) \\ y - f(b) = k_b(x - b) \end{array}\right.$$
for x and y in terms of a and b, we get the point $P(x; y)$ from which F is seen under angle α.

Since $b = b(a)$ (by Step 2), Step 3 produces the parametric equations $x = x(a)$, $y = y(a)$ of $\alpha T(F)$.

If F has asymptotes, then finding $\alpha V(F)$ requires some additional reasoning as in the case of the hyperbola-branch. In the next sections, we refer to this algorithm as ICA.

5. Isoptics of exponent and logarithm curves in a particular case

Let us apply the ICA for the particular case E: $y = e^x$, $\alpha = 3\pi/4$.

1. Take $A(a; e^a)$ and $B(b; e^b)$.
2. For $k_a = e^a$ and $k_b = e^b$ by CAS[2] we obtain
$$\text{In}[] := \text{Solve}\left[\left\{\frac{e^a - e^b}{1 + e^a e^b} == -1\right\}, \{b\}\right]$$
$$\text{Out}[] = \left\{\left\{b \to \text{Log}\left[\frac{e^a + 1}{1 - e^a}\right]\right\}\right\},$$
i.e. $b = \ln\dfrac{e^a + 1}{1 - e^a}$.
3. Solving the system of equations
$$\left|\begin{array}{l} y - e^a = e^a(x - a) \\ y - e^b = e^b(x - b) \end{array}\right.$$
in terms of x and y, we get the coordinates of point from which E is seen under angle $3\pi/4$,
$$\text{In}[] := \text{Solve}[\{y - e^a == (x - a)e^a,\ y - e^b == (x - b)e^b\},\ \{x, y\}],$$
$$\text{Out}[] = \left\{\left\{x \to \frac{-e^a + ae^a + e^b - be^b}{e^a - e^b},\ y \to \frac{(a - b)e^(a + b)}{e^a - e^b}\right\}\right\}.$$

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[2]The CAS-calculations are done by MATHEMATICA [11].

Replacing $b = \ln \frac{e^a+1}{1-e^a}$, we obtain the parametric equations of $\frac{3\pi}{4}\,T(E)$:

$$\frac{3\pi}{4}\,T(E): \quad \left|\begin{array}{l} x = -1 + a + \dfrac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}} \\ y = e^a\,\dfrac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}}, \end{array}\right.$$

which holds for $a \in (-\infty; 0)$.

We cannot recognize any famous curve here. However, we can say something about the asymptotic behavior of $\frac{3\pi}{4}T(E)$. First, by CAS (MATHEMATICA) we obtain

$$\text{In}[] := \text{Limit}\left[-1 + a + \frac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}},\ a \to -\infty\right]$$

$$\text{Out}[] = -1,$$

and

$$\text{In}[] := \text{Limit}\left[e^a\,\frac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}},\ a \to -\infty\right]$$

$$\text{Out}[] = 0.$$

This means that $\lim\limits_{a\to-\infty} x = -1$, $\lim\limits_{a\to-\infty} y = 0$, i.e. $\frac{3\pi}{4}T(E)$ starts at the point $(-1; 0)$.

Heuristic. The domain of a and the way we generate points $P(x; y)$ in Step 3 of the ICA hint at the existence of asymptote of $\frac{3\pi}{4}T(E)$. Indeed, the tangent t_a to E at $A(a; e^a)$ tends to the tangent t_0 to E at $A_0(0; 1)$ when $a \to 0$. Perhaps $P(x; y)$ will approach t_0 when $a \to 0$ (from the left).

Let us verify this speculation. First,

$$\text{In}[] := \text{Limit}\left[\frac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}},\ a \to 0\right]$$

$$\text{Out}[] = \infty.$$

Further, eliminating the expression $\dfrac{(1+e^a)\left(\ln \frac{1+e^a}{1-e^a} - a\right)}{1+e^{2a}}$ from the parametric equations of $\frac{3\pi}{4}T(E)$, we obtain

$$y = e^a(x - a + 1).$$

Hence, $\lim\limits_{a\to 0-0} \frac{y}{x} = 1$, $\lim\limits_{a\to 0-0}(y - x) = 1$, which yields taht the line $y = x + 1$ is the asymptote to E in $+\infty$. ■

To get $\frac{3\pi}{4}V(E)$, we paste to $\frac{3\pi}{4}T(E)$ the ray $y = x + 1$, $x \leq -1$ (Figure 6).

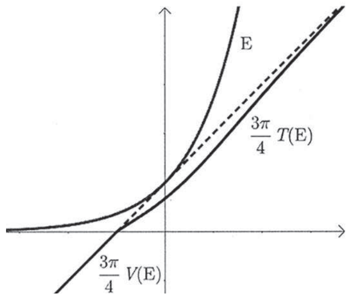

Fig. 6. $\frac{3\pi}{4}T(E)$ and $\frac{3\pi}{4}V(E)$

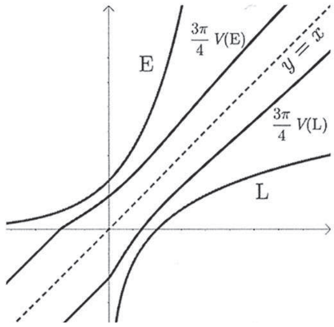

Fig. 7. $\frac{3\pi}{4}V(E)$ and $\frac{3\pi}{4}V(L)$

Since the logarithm is the inverse function of the exponential function, we obtain by reflection about $y = x$ the corresponding loci $\frac{3\pi}{4}T(L)$ and $\frac{3\pi}{4}V(L)$ for the logarithm curve L: $y = \ln x$ (Figure 7).

6. Isoptics of exponent and logarithm curves in a more general case

The isoptics in more general cases of exponent and logarithm curves with an arbitrary base are similar to $\frac{3\pi}{4}T(E)$ and $\frac{3\pi}{4}V(E)$. The differences are in details: domain and asymptote. Performing the ICA, we obtain for F: $y = c^x$, $c > 1$, $\alpha > \pi/2$, the following parametric equations:

$$\alpha T(F): \left| \begin{aligned} x &= -\frac{1}{\ln c} + a + \frac{(1 - c^a \cot\alpha \ln c)\ln \frac{c^a \ln c - \tan\alpha}{c^a(\ln c + c^a \tan\alpha \ln^2 c)}}{\ln c(1 + c^{2a}\ln^2 c)} \\ y &= c^a \frac{(1 - c^a \cot\alpha \ln c)\ln \dfrac{c^a \ln c - \tan\alpha}{c^a(\ln c + c^a \tan\alpha \ln^2 c)}}{1 + c^{2a}\ln^2 c} \end{aligned} \right.$$

where $a \in (-\infty; -\log_c(-\tan\alpha \ln c))$.

Example. Figure 8 illustrates three particular cases for the isoptics of $y = 2^x$.

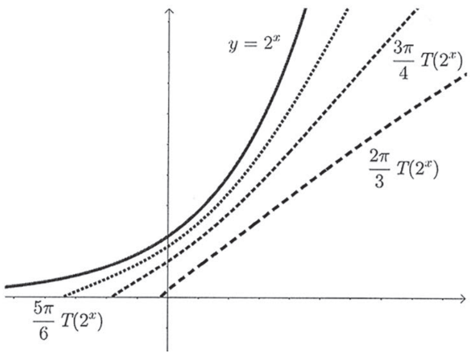

Fig. 8. $\alpha T(2^x)$ for $\alpha \in \{\frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\}$

The ray

$$y = -\tan\alpha\left(x - \log_c\left(-\frac{\tan\alpha}{e\ln c}\right)\right), \quad x \le \log_c\left(-\frac{\tan\alpha}{e\ln c}\right)$$

completes $\alpha T(F)$ to $\alpha V(F)$.

Theorem 3. *The line*

$$y = -x\cot\alpha - \cot\alpha\log_c(-e\tan\alpha\ln c)$$

is an asymptote of $\alpha T(F)$ *at* $+\infty$.

Reasoning. Let $a_0 = -\log_c(-\tan\alpha\ln c)$. The ICA says that when $a \to a_0$ from the left, then the tangent t_a "approaches" the tangent t_{a_0}, whose equation is stated in Theorem 3. Thus, the corresponding point from the locus, which is the intersection point of the tangents t_a and t_b, "approaches" t_{a_0}.

Formal calculations by CAS confirm the above speculation:

$$\text{In}[] := \text{Limit}\left[\frac{y(a)}{x(a)}, a \to -\log_c(-\tan\alpha\ \ln c)\right]$$

$$\text{Out}[] = -\cot\alpha$$

$$\text{In}[] := \text{Limit}\left[y(a) - (-\cot\alpha)x(a), a \to -\log_c(-\tan\alpha\ \ln c)\right]$$

$$\text{Out}[] = -\frac{\cot\alpha(1+\ln(-\tan\alpha\ \ln c))}{\ln c}.$$

where $x(a)$ and $y(a)$ are predefined as the parametric equations of $\alpha T(F)$. ∎

Analogously, about G: $y = \log_c x$, $c > 1$, $\alpha > \pi/2$, the following parametric equations for $\alpha T(G)$ hold:

$$\alpha T(G): \quad \left| \begin{array}{l} x = a\dfrac{(1 - a\cot\alpha\ln c)\ln\frac{a\ln c - \tan\alpha}{a(\ln c + a\ln^2 c\tan\alpha)}}{1 + a^2\ln^2 c} \\ y = \dfrac{1}{\ln c}\left(-1 + \ln a + \dfrac{(1 - a\cot\alpha\ln c)\ln\frac{a\ln c - \tan\alpha}{a(\ln c + a\ln^2 c\tan\alpha)}}{1 + a^2\ln^2 c}\right), \end{array}\right.$$

where $a \in \left(0; -\frac{1}{\ln c\tan\alpha}\right)$.

The ray $y = -x\cot\alpha + \log_c\left(-\dfrac{\tan\alpha}{e\ln c}\right)$, $x \le 0$ completes $\alpha T(G)$ to $\alpha V(G)$.

The line $y = -x\tan\alpha - \log_c(-e\tan\alpha\ln c)$ is the asymptote of $\alpha T(G)$ at $+\infty$.

7. Conclusions

The mathematical results obtained by CAS have become regular practice in modern times. This fact motivates the first two co-authors to design an individual educational trajectory directed to form synthetic competence in a team of secondary school students – the last two co-authors. This trajectory adopted the main points of the didactical model Taipei [5], modified for a team of students, who are interested in research activities in mathematics. One can see that the application of CAS and DGS in our research requires students' high order math thinking that synthesizes computer skills with solid math background. Developing such abilities, the students need a proper area to apply them. The topic under consideration in this article provides an opportunity for collaborative work of educators and students bringing satisfaction to all members of our authors' crew.

Authors contributions. Borislav Lazarov: conceptual framework and research plan; Dimitar Dimitrov: project management and students math-background training, Martin Dimitrov and Gergana Peeva: CAS and DGS processing.

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