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PAGES DEDICATED TO PROFESSOR MILOSAV (MILO) MARJANOVIĆ ON THE OCCASION OF HIS 90-TH BIRTHDAY

Brief biography. Milo was born on August 24, 1931 in Nikšić, Montenegro, where he completed his primary and secondary education. He received a diploma in mathematics from the Faculty of Natural Sciences, University of Belgrade in 1955. He was employed at the same faculty as a teaching assistant in 1957 and at that faculty his entire career ran until 1997, when he was retired. (Some complementary biographical data can be found in The Teaching of Mathematics, XIV, 2, 2011).

Milo wrote text-books and research papers on education at all levels, from primary school to university. A number of papers that he finds to be particularly representative for his work on education are included here in this material.

Fig. 1. Milo's mountain retreat, a spot at 950 m above the sea level.

SELECTED PAPERS ON EDUCATION
of Milosav M. Marjanović

I (Milosav M. Marjanović) use the following Notices to present a selection of my papers on education. Had I enough skill to do it? Well, only the interested reader will have the right judgment about it.

Notice 1. Concepts as the building blocks of reflexive thinking

A course of didactics mathematics for both, primary and math teachers, should be designed so as to help them understand deeper the contents of school mathematics that they teach. This understanding requires the acquaintance with some specific elements of cognitive psychology which, according to our opinion, such course have to contain. Here, we present briefly the elements that we select, laying stress on primary mathematics.

From the very beginning of teaching and learning mathematics concepts are gradually built i.e., their meaning is communicated by long series of corresponding examples. We find it to be quite convenient to comprehend mathematical concepts to be tripartite entities consisting of *corresponding examples*, *mental image* and *name*—a word or a phrase from the natural language being the denotation (and in mathematics also a denoting symbol exists).Let us also add that the psychologist L. S. Vigotsky calls a class of mutually related concepts the *system of concepts*. In mathematics such a class is called the (concrete) *mathematical structure* whose reflection in the mind is called the *mental scheme*.

Let us notice that a concept P is taken to be *more general* or *more abstract* than the concept Q, when Q is an example for P. In classical logic (see, for example, Kant's book on logic) a concept that is more general than a whole class of concepts is called *conceptus summum*. For example, the concept of set is the conceptus summum with respect to all concepts of classical mathematics. Let us notice that the concept of set has its examples at all level of abstractness. Among these examples the lowest degree of abstractness the collections of visible objects in the surrounding space have, which are called sets at *sensory level* and they are the basis upon which school arithmetic is built.

At the early stages of learning school mathematics, particularly important is *iconic representation*. Icons are conveyors of meaning and when they are representing a reality which is not immediately experienced in the form of pictures or drawings, they are called *pictograms*. When they represent abstract concepts, then they are called *ideograms*. The examples of ideograms are number images, which represent numbers by collections of uniform signs (usually dots) which project cardinality at the first glance. Ideograms are also geometric drawings which represent visually geometric concepts.

The natural dependence of conception of number on perception of sets motivated Georg Cantor to describe this way of cognition as a process of abstracting (ignoring) the nature of elements of sets and the way they are arranged. Modifying slightly this formulation, we express *Cantor principle of invariance of number* in the following way:

Starting with observation of a set of visible objects and abstracting (forgetting)

(i) the nature of these elements

and

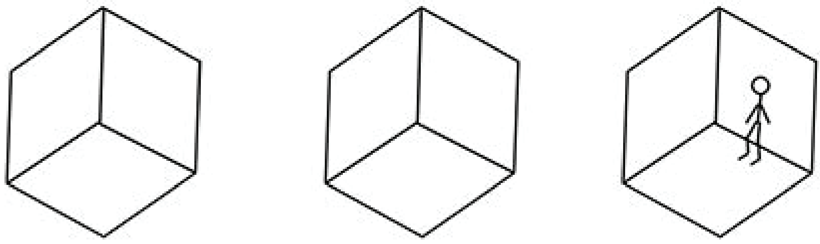
(ii) any kind of their organization (grouping, arranging, etc.)

an abstract idea of number results.

Let us note that G. Cantor denotes cardinal number of the set S writing $\overline{\overline{S}}$, where two overbars indicate two above abstractions (two above ways of ignoring), (See G. Cantor, Beitrage zur Begrundung der transfinite Mengenlehre, Band 46, Num. 4, 1895). A property of a concept that all its examples have is called essential and all those which some its example have and some others do not are called collectively noise. Let us notice here that a more realistic way of abstracting is a systematic suppress of the noise than selection of essential properties.

The elements of cognitive psychology are distributed in the series of our papers [1] where they are selected for and applied on elaboration of elementary school arithmetic.

Addendum. Reviewing a paper, this author "discovered" that an arrangement of cubes can also be seen as the same arrangement of solid angles. Thus he came upon his favorite example, when one and the same sensory input has three different interpretations: a regular hexagon with three radii of its circumscribed circle, a cube and a solid angle. (To help you see this angle, imagine the corner of the room with Pedestrian's feet on its flour).



a) cube b) hexagon with three radii of its circumscribed circle c) solid angle

Fig. 2

REFERENCES

1. M. M. Marjanović, *A Broader Way through Themas of Elementary School Mathematics II–VII*, The Teaching of Mathematics, Vols. II, 2, (1999), 81–103; III, 1 (2000), 41–51; V, 1 (2002), 47–55; VI, 2 (2003), 113–120; VII, 1 (2004), 35–52; VII, 2 (2004), 71–93.

Notice 2. School arithmetic elaborated in the form of number blocks

A particularly subtle period of learning and teaching school arithmetic are first two years of primary school, when concepts are communicated in long series of corresponding examples and acquired by repeated doing of exercises. In this period the main didactical task is the establishing of permanent meaning of the four arithmetic operations within the initial blocks of numbers up to 20 and 100.

A well-organized exposition of the themes of school arithmetic in the period of first three years can be found in the paper [1]. We also express here our conviction that children learn arithmetic with lightness and ease only when it is properly structured and worked out with care and nicety of detail. Let us also note that for realization of different tasks of school arithmetic the technique of place holders is inevitable. Children acquire quickly and easily the information that these graphical signs transfer without converting it into words and avoiding so possible rigmarole of words.

Now we present the content of the paper cited above, in the condensed form, indicating didactical tasks within blocks of numbers up to 10, 20 and 100.

1. Counting as an introductory theme of arithmetic

Elaborating this theme, the following didactical tasks have to be achieved:

– Children learn to recite the number names from one to ten, going onwards and backwards.

– Associating objects with number names, children practice to count the elements of concrete sets (including their pictorial representations as well).

– In case of two sets of different objects, children count their elements, taking each set separately as well as when they are taken together. They understand the meaning of questions “How many?”, “How many altogether?”.

– Following suitable situations when some objects are taken away, children count and answer questions “How many where there?”, “How many remain?” etc.

– At this stage, children give the answers orally and, by means of visual representations, they start to form abstract ideas of numbers (up to 10) and become familiar with the addition and subtraction tasks.

2. Block of numbers up to 10

We describe here what is done when this block is built and which didactical tasks are accomplished.

– The meaning of numbers 1, 2, . . . , 10 and 0 is established by relating each of them with the corresponding sets of various objects.

– Each of these numbers is elaborated as an individual concept, denoted by its digit and associated with its number image whose shape projects at first glance the meaning of the number that it represents.

– A somewhat more delicate is the case of zero which is related with the materialization of the empty set as being an “empty place”. For example in a sequence of boxes containing less and less marbles the last empty one is an example of the empty place.

– Each digit is written in a number of moves of the hand that are made in a fixed order. Exercising, these moves tend to be more and more regular and the handwriting becomes nicer. It is important that, from the very beginning, children learn to write digits correctly.

– Children are given examples of situations which can be modeled as the idea of two disjoint sets. We call such an idea an *additive scheme* and when the numbers of elements of these sets are given and the number of their union is to be found, we say that this scheme is followed by the *addition task*. As soon as the children are aware of an additive scheme followed by the addition task they write a sum to denote the sought number. This meaning of addition is permanent and should not be confused with a more technical one, when sums are calculated to find the unique decimal notation for their value.

– When an additive scheme is imagined or pictorially represented and when the numbers of elements of the union and of one of these sets are given and the number of elements of the other set is to be found, then we say that this scheme is followed by the *subtraction task*. Examples of situations of this kind provide a permanent meaning for subtraction.

Let us note that, in the classroom practice, additive schemes are mostly given in the form of simple pictures, the sets are "small"' (their number of elements does not exceed 10) and the sought numbers are found by counting, but that is not the reason for the teacher not to recognize the importance of the activities when the meaning of these two arithmetic operations starts to be formed.

– Children write two different sums when they denote the number of elements of the union of two disjoint sets taken in two different orders. By equating these sums they discover the rule of *interchange of summands*. Teachers should understand this rule is one of the principles of arithmetic which is intuitively acceptable, but not being proved in that way.

– A sum (a difference) denotes a number which is called its value. To calculate a sum (a difference) means to express its value by its unique decimal notation. By doing sufficient number of exercises, children are trained to calculate quickly sums and differences when their value is within this block. As already said, calculation at this stage reduces to counting and the imagery that children develop in contact with the number images often helps them perform these simple calculations.

– Numbers up to 10, together with the operations of addition and subtraction and the relation "to be larger than" constitute a system of concepts (a mathematical structure) that we denote by $(\mathbb{N}_{10}, +, -, <)$.

3. Block of numbers up to 20

The majority of children can count up to 20, reciting the number names in order. But counting is not taken any longer to be a base upon which the number blocks are built. A sum as, for example, $10+7$ already has its meaning established:

it denotes the number of elements of the union of two sets, one having 10 elements and the other one 7. Decimal notation for that number is not included in the material of which block of numbers up to 10 is built. But the use of such notations is the way how the block of numbers up to 10 is extended to the block of numbers up to 20. Namely, the sums: $10 + 1$, $10 + 2$, … , $10 + 10$ are denoted shortly as 11, 12, … , 20 respectively and these numbers are read: eleven, twelve, … , twenty. By equating two notations for the same number, the equalities $10+1 = 11$, $10 + 2 = 12$, … , $10 + 10 = 20$ can be written.

– Further properties of addition and subtraction are derived: the rule of association of summands, the rule of subtracting a number from a sum, the rule of subtracting a sum from a number, the rule of interchange of the subtrahend and the difference, etc. Application of these rules is not a very simple task for children. Thus, the exercises of this kind have to be programmed using techniques of place holders.

– Methods of adding and subtracting when the 10-line is crossed are worked out in detail. By practicing these methods, children learn to calculate quickly the results of all entries of addition table. Therefore, the block of numbers up to 20 is a natural framework within which the addition (and subtraction) table is formed.

4. Block of numbers up to 100

The sums $20 + 1$, $20 + 2$, … , $20 + 10$, and their decimal notations 21, 22, … , 30 make the first step in extension. Then $30 + 1$, $30 + 2$, … , $30 + 10$ and their decimal notations 31, 32, … , 40, … $90 + 1$, $90 + 2$, … , $90 + 10$ and their decimal notations 91, 92, … , 100 are the numbers of this block. The reader will notice that addition, not the counting, is the base of this extension

– Technique of vertical addition (including the cases when one ten is carried over) and vertical subtraction (including the cases when one ten is borrowed) are presented in detail. Number images are employed to illustrate these procedures and to supply them with the meaning.

– These procedures are also described in terms which underline decimal structure of the manipulated numbers but as soon as these manipulations become more automatic, such descriptions are simplified obtaining the abbreviated form of the inner speech.

– Up to this stage the block of numbers up to 100 is an additive structure which we denote by writing $(\mathbb{N}_{100}, +, -, <)$. But this block is also a natural range of numbers within which the meaning of multiplication and division is established.

– A situation which can be modeled as a finite family of finite equipotent sets is called a *multiplication scheme*. When the number m of the members of this family and the number n of the elements of these sets are given and the number p of the elements of their union is to be found, we say that this is a *multiplicative task* that follows this scheme. When p and n are given and m is to be found, we say that this is a *division task* that follows this scheme and which is called *partition* and when p and m are given and n is to be found, we also say that this is a *division task* that follows this scheme and which is called *quotation*.

– Equivalent ways of representing a multiplicative scheme as a rectangular arrangement or a direct product of two sets are also considered.

– All products that enter multiplication table have their iconic representations which are used to suggest quick calculations of their value. Thus, children take an active part in the building of this table.

– Properties of multiplication are established and expressed: the rule of interchange of factors, the rule of association of factors, the rule of multiplication of sums and differences by a number, etc. A collection of m boxes each containing n marbles is a suitable model of the multiplicative scheme. Similarly, k packages, each containing m boxes and each box containing n marbles is a suitable model for the triple product. Teachers should be aware of the role of the rule of association of factors (the associative law for multiplication) which reduces triple and multiple products to the product of two numbers. Thus, it is enough to form the multiplication table in the case of products of two numbers or to express properties of multiplication in the same case as above.

– The following properties of division are established and formulated: the rule of interchange of the divisor and the quotient, the rule of division of a sum by a number, the rule of division of a difference by a number, etc.

– By doing exercises, children learn the idea of division with remainder and particularly in the cases of division: by 2 up to 20, by 3 up to 30, . . . , by 9 up to 90 they do these calculations mentally. Together with addition and multiplication tables, these cases of division are also the material which enters the long term memory store.

With multiplication and division, the block of numbers up to 100 becomes a still richer structure which we denote by writing $(\mathbb{N}_{100}, +, -, \cdot, :, <)$.

The author of the above cited paper wrote math books for primary grades and he finds that art to be much more delicate than in the case when mathematical content is already more formalized.

REFERENCES

1. M. M. Marjanović, *Structuring the Content of School Arithmetic*, pp. 11–123 (in Serbian), Serbian Academy of Sciences and Arts, Monographs, vol. DCLXXXVIII, book 5, 2015.

Notice 3. Dividing natural numbers

When we speak of natural numbers, here we also include zero. Let us recall this arithmetical fact: Given two natural numbers m and n, $n > 0$, there exist two unique natural numbers q and r, such that $m = qn + r$, where $r < n$. The number q is called the quotient and r the remainder of $m : n$. The case of division, where $m < n$, when $q = 0$, we exclude as trivial and we concentrate our considerations on the case when $m \geq n$. The case when $m = n$ and $q = 1$, $r = 0$ can also be excluded as being trivial.

Here, we summarize briefly the content of our papers [1], [2].

Let us call a case of division $A : B$ standard, when the dividend A is less than tenfold divisor B. And as it is well known each long division splits into a series of standard divisions. In the first of the above papers a systematic search for true digits is sketched and we think that it deserves to be studied carefully. But we proceed further with presentation of the content of the second of the above papers, where an algorithm is established and where it is proved that such algorithm produces true digits. We have already excluded from our consideration trivial cases of division $A : B$, when $A \leq B$. We consider a standard division to be easy when its divisor is at most two digit number. In the second of the above papers, an algorithm is established and it is proved that such algorithm reduces each standard division $A : B$ whose divisor has more than two digits to the case of easy division. This way of reducing consists of rounding up B by increasing its second digit for 1 and replacing all its digits that follow by 0's and rounding down A by replacing the same number of its final digits by 0's as it has been done in the case of B. Omitting the same number of final 0's of the reduced A and B, one easy case of division is obtained, whose quotient represents the trial number. This procedure is continued when B is multiplied by the trial number and this product subtracted from A. When this difference is less than B the trial number represents the true digit but when it is equal or greater than B, the trial number plus 1 represents the true digit.

The third of the above papers presents clearly the content of the previous two. We suggest to the interested reader to start reading the third paper. Though elementary, we hope these results are a nice contribution to the Mathematics of Al-Khwarizmi.

REFERENCES

1. M. M. Marjanović, *Division – A Systematic Search for True Digits I–II*, The Teaching of Mathematics, Vols. VIII, 2 (2005), 89–101; XVIII, 2 (2015), 84–92.
2. M. M. Marjanović, *Elaboration of Division – A Systematic Search for True Digits*, Open Access Library Journal, Vol. 4 (2017).

Notice 4. Calculating Euler-Poincaré characteristic inductively

Traditionally Euler characteristic is defined and calculated for topological spaces which are homeomorphic to (finite) symplicial complexes. Namely, for a topological space X its Euler characteristic is taken to be $\chi(X) = \sum f_i$, where f_i is the number of i-dimensional faces of the corresponding symplicial complex. In our paper [1], the calculation of this characteristic proceeds without triangulation and, saying it figuratively, just by decomposing lines into points, surfaces into lines, bodies into surfaces, etc. This decomposition is precisely described in the paper cited above, where it is illustrated by examples. For instance in Example 8 we derive the formula $\chi(X) = \sum n_i$ for the Euler characteristic of a finite CW-complex K, where n_i is the number of i-cells of K.

In our second paper [2] which is also devoted to the inductive calculation of Euler-Poincaré characteristic, somewhat relaxed conditions are given under which this calculation is performed. (See two addenda: 4. Addendum and 7. Addendum). But what makes this paper particularly interesting is a pictorial representation of 2-surfaces in $\mathbb{R}^3$ where they are decomposed into "lines" (circles, self-intersecting circles ∞, and points).

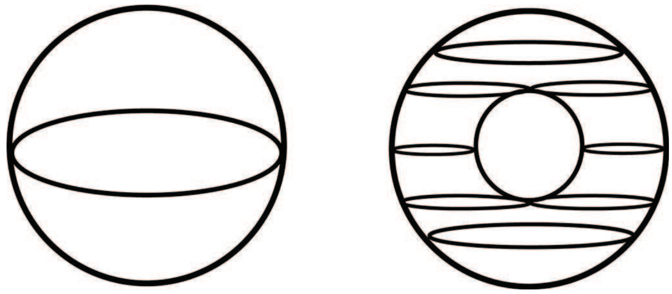


Fig. 3. The sphere M_0 and the torus M_1

(a) Orientable surfaces M_g (spheres with g holes). The 2-sphere and the torus ($\chi(M_0) = 2, \chi(M_1) = 0$) are exhibited in Figure 3. The general case ($\chi(M_g) = 2 - 2g$) is depicted in Figure 4.

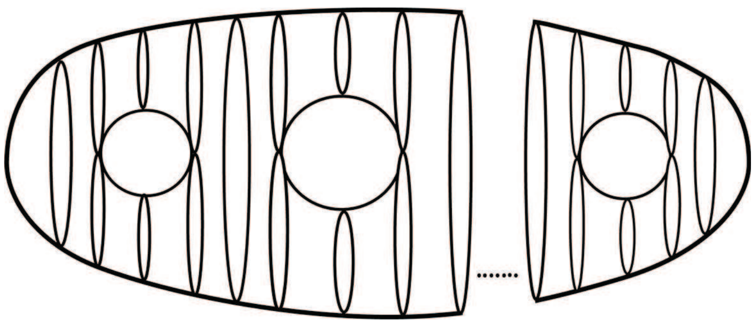


Fig. 4. The 2-sphere with g-holes

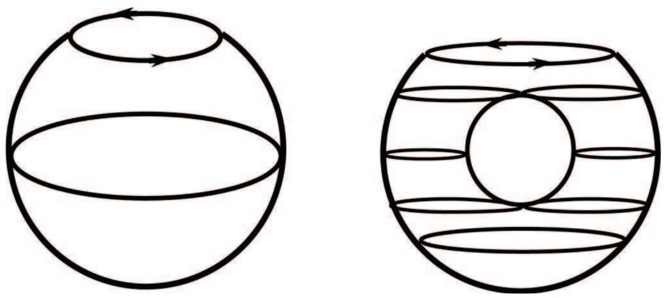


Fig. 5. Projective plane M_0' and the non-orientable surface M_1'

(b) Non-orientable surfaces M'_g arise when a spherical cap is cut off along a circle whose diametrally opposite points are identified. The projective plane M'_0 and the non-orientable surface M'_1 ($\chi(M'_0) = 1, \chi(M'_1) = -1$) are exhibited in Figure 5. The general case ($\chi(M'_g) = 1 - 2g$) is depicted in Figure 6.

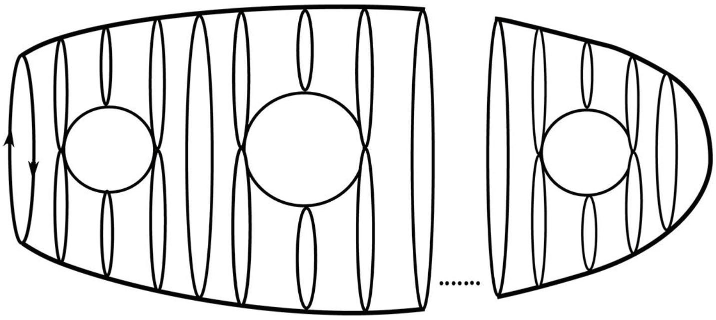


Fig. 6. Non-orientable surface M'_g with g-holes

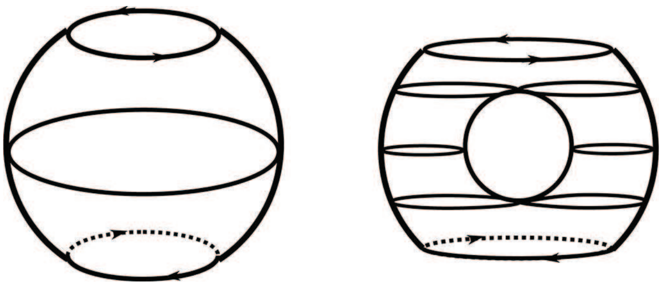


Fig. 7. Klein bottle M''_0 and the non-orientable surface M''_1

(c) Non-orientable surfaces M''_g arise when two opposite spherical caps are removed and the corresponding boundary circles are identified. The Klein bottle M''_0 and the non-orientable surface M''_1 ($\chi(M''_0) = 0, \chi(M''_1) = -2$) are exhibited in Figure 7. The general case ($\chi(M''_g) = -2g$) is depicted in Figure 8.

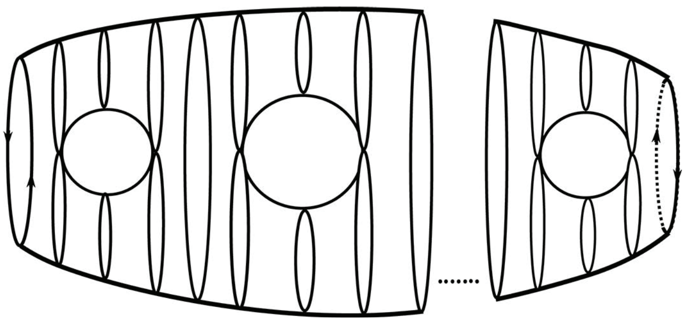


Fig. 8. Non-orientable surface M''_g with g-holes

These models of 2-surfaces, which represent their decompositions into lines, serve for easy calculation of the Euler-Poincaré characteristic and at the same time they enrich our geometric imagination.

REFERENCES

1. M. M. Marjanović, *Calculating Euler-Poincaré Characteristic Inductively*, arXiv:1212.0154 [math.GT], December 1, 2012.
2. M. M. Marjanović, *Euler-Poincaré Characteristic – A Case of Topological Self-convincing*, The Teaching of Mathematics, Vol. XVII, 1 (2014), 21–33.

Notice 5. Interchanging two limits

Several statements in mathematical analysis, where double limit is discerned, have one and the same form—one of the two limits exists and the other one exists uniformly. We summarize briefly the content of such statements emphasizing some particularly interesting examples, following the paper [1].

Due to different terms, the following three types of limit are present in the classical analysis

$$\lim a_n, \qquad \lim f(x), \qquad \lim \sigma(f, P),$$

being respectively: the limit of a sequence, the limit of a function at a point and the limit of integral sums. Our idea of integrating these different types of limit is based on the procedure of introducing metric on the set $\mathbf{N}$ of natural numbers together with an ideal point and on the set of all partitions of an interval together with an ideal point. Then, the model of the limit at a point of a function mapping a metric space into another one embraces all three types of limit present in the classical analysis. The effects of such an approach are also seen in the fact that an apparently varied set of conditions associated with these particular statements, when the problem of interchanging limits is concerned, translates uniquely into one requirement.

Let A be a nonempty subset of a metric space (M_1, d_1) and B a nonempty subset of a metric space (M_2, d_2). Let $f\colon A \times B \to M$ be a mapping into a complete metric space (M, d). Denote by X' the set of accumulation points of a subset X of a metric space. Let $x_0 \in A'$ and $y_0 \in B'$. A limit $\lim\limits_{y \to y_0} f(x, y) = \varphi(x)$ exists for each $x \in A$ if the following condition holds:

$$(\forall x \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B)(0 < d_2(y, y_0) < \delta \implies d(f(x, y), \varphi(x)) < \varepsilon);$$

and one can formulate similarly the condition for the existence of $\lim\limits_{x \to x_0} f(x, y) = \psi(y)$. The limit $\lim\limits_{y \to y_0} f(x, y) = \varphi(x)$ exists *uniformly in* $x \in A$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(\forall y \in B)(0 < d_2(y, y_0) < \delta \implies d(f(x, y), \varphi(x)) < \varepsilon);$$

and the condition for the existence of $\lim\limits_{x \to x_0} f(x, y) = \psi(y)$, uniformly in $y \in B$ is formulated similarly.

Writing formally,

$$\lim_{x\to x_0}\varphi(x)=\lim_{x\to x_0}(\lim_{y\to y_0}f(x,y)),$$
$$\lim_{y\to y_0}\psi(x)=\lim_{y\to y_0}(\lim_{x\to x_0}f(x,y)),$$

we call these expressions *iterated limits* (which, of course need not exist). In the sequel, we shall omit additional parentheses. In this context, the expression

$$\lim_{(x_0,y_0)}f(x,y)$$

is called *double limit* (and it may equally be non-existent).

The main theorem of the mentioned article is the following.

THEOREM 1. (The theorem on interchange of two limits) *Let $f\colon A\times B\to M$ be a mapping into a complete metric space, where A and B are subsets of metric spaces M_1 and M_2, respectively, and let $x_0\in A'\setminus A$, $y_0\in B'\setminus B$. If*

(i) $\lim\limits_{x\to x_0}f(x,y)=\psi(y)$ *exists for each* $y\in B$;

(ii) $\lim\limits_{y\to y_0}f(x,y)=\varphi(x)$ *exists uniformly in* $x\in A$,

then the three limits

$$\lim_{x\to x_0}\lim_{y\to y_0}f(x,y),\qquad \lim_{y\to y_0}\lim_{x\to x_0}f(x,y),\qquad \textit{and}\qquad \lim_{(x_0,y_0)}f(x,y)$$

all exist and are equal.

The first particular case of this theorem is concerned with a sequence (f_n) of functions with domain A which is a subset of a metric space M_0 and codomain which is a metric space M. Then, we can consider $f_n(x)$ as a function of two variables, taking

$$\varphi(n,x)=f_n(x).$$

Since $\mathbf{N}$ can be understood as a subset of the metric space $\mathbf{N}^*=\mathbf{N}\cup\{\infty\}$ with the metric

$$d(m,n)=\left|\frac{1}{m}-\frac{1}{n}\right|,\qquad \left(\frac{1}{\infty}=0\right)$$

and since $\infty\in\mathbf{N}'$, for the function $\varphi\colon\mathbf{N}\times A\to M$, the fact that for each $x\in A$, $\lim\limits_{n\to\infty}\varphi(n,x)=f(x)$ exists, simply means that the sequence (f_n) converges to the function f on the set A. Similarly, the fact that $\lim\limits_{n\to\infty}\varphi(n,x)=f(x)$ exists uniformly in $x\in A$, is equivalent to the fact that the sequence (f_n) converges uniformly on A.

As a corollary, we obtain the well-known theorems of continuity of the limit function for a uniformly convergent sequence of continuous functions from a metric space into another complete metric space, as well as the assertion on conditions of termwise differentiation.

The next two corollaries are concerned with the problem of interchange of the order of summation for double sequences, and with the known Toeplitz limit theorem.

THEOREM 2. *Let* (a_{ij}) *be a double sequence such that:*

(i) $(\forall i)\ \sum_{j=1}^{\infty} |a_{ij}| = b_i < +\infty$, (ii) $\sum_{i=1}^{\infty} b_i$ *is a convergent series.*

Then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

THEOREM 3. *The coefficients of the matrix*

$$\begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & \dots & a_{1n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{k0} & a_{k1} & \dots & a_{kn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

are assumed to satisfy the following two conditions:

(a) *for each fixed* n, $a_{kn} \to 0$ *as* $k \to \infty$,

(b) *there exists a constant* K *such that for each fixed* k *and any* n,

$$|a_{k0}| + |a_{k1}| + \cdots + |a_{kn}| < K.$$

Then, for every null sequence $(z_0, z_1, \dots, z_n, \dots)$, *the numbers*

$$z'_k = a_{k0}z_0 + a_{k1}z_1 + \cdots + a_{kn}z_n + \cdots$$

also form a null sequence.

In order to apply our result to the problems where (Riemann) integral is involved, we introduce the following notions.

Denote by Π the set of all partitions P of a segment $[a, b]$. For the partition P given by $a = x_0 < x_1 < \cdots < x_n = b$, the number

$$\|P\| = \max\{\, x_i - x_{i-1} \mid i = 1, 2, \dots, n \,\}$$

is called the *norm of partition* P. Let ∞ be an element which does not belong to Π and let $\|\infty\| = 0$. The set $\Pi^* = \Pi \cup \{\infty\}$, together with the metric

$$d_\Pi(P_1, P_2) = \begin{cases} \|P_1\| + \|P_2\|, & P_1 \neq P_2, \\ 0, & P_1 = P_2, \end{cases}$$

is a metric space which will be called the *partition space* and will be denoted by (Π^*, d_Π).

Let $f\colon [a, b] \to \mathbf{R}$ be a bounded function, $s(f, P)$ and $S(f, P)$ be its lower and its upper sum, with respect to the partition P. These sums can be considered as functions

$$s\colon \Pi \to \mathbf{R}, \qquad S\colon \Pi \to \mathbf{R}$$

from the space of partitions into $\mathbf{R}$. The limit $\lim\limits_{\infty} s(f,P)=\alpha$ exists if the condition

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall P\in\Pi)\ 0<d_{\Pi}(P,\infty)<\delta \implies |\alpha-s(f,P)|<\varepsilon$$

holds, or, equivalently, if the condition

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall P\in\Pi)\ \|P\|<\delta \implies |\alpha-s(f,P)|<\varepsilon$$

holds. Similarly, the limit $\lim\limits_{\infty} s(f,P)$ is defined. The function f is integrable on $[a,b]$ if

$$\lim_{\infty} s(f,P)=\lim_{\infty} S(f,P)=I,$$

and in this case we denote $I=\displaystyle\int_a^b f(x)\,dx$.

As a consequence of our main result we obtain

THEOREM 4. *Let (f_n) be a sequence of functions, integrable on the segment $[a,b]$, and let (f_n) converges uniformly to a function f. Then, the function f is integrable and the equality*

$$(*)\qquad \lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)\,dx$$

is valid.

Finally, we derive the Lebesgue theorem on bounded convergence for the Riemann integral in the following form.

THEOREM 5. *Let (f_n) be a sequence of functions, integrable on $[a,b]$, converging to a function f. If*

(i) *the function f is integrable,*

(ii) $(\exists M\in\mathbf{R})(\forall n\in\mathbf{N})(\forall x\in[a,b])\ |f_n(x)|\leqslant M$,

(iii) *the set*

$$A=\{\,x\in[a,b]\mid (f_n)\ \textit{does not converge uniformly around}\ x\,\}$$

is a closed set of Lebesgue measure 0,

then the relation $(*)$ holds.

REFERENCES

1. M. M. Marjanović, Z. Kadelburg, *Interchanging Two Limits*, The Teaching of Mathematics, Vol. VIII, 1 (2005), 15–29.

Notice 6. Structuring of number systems

The papers [1–4] of this author and his younger colleague Zoran Kadelburg sketch the way how number systems are structured and their operative properties displayed.

Let us remark that numbers have always been the foundation stone of mathematics. Considered for centuries to be a paragon of rigor and logical purity, Euclid's Elements of Geometry (See, for example, Kill Patrick's edition) have out of thirteen their books, six of them (books 5, 7, 8, 9, 10 and 12) primarily concerned with Eudoxus' conception of (positive) real numbers, their properties expressed in terms of proportions and their applications. According to Eudoxus, real number is conceived of as a class of mutually equivalent ratios of quantities (magnitudes) of the same kind. What made this theory additionally complicated was the fact that operations were performed on magnitudes instead of directly on numbers.

A significant step towards simplification and clarification of this theory made François Viète (1540–1603), who created literal algebra (which he called *logistica speciosa*), where the letters were used to denote species of numbers and when the fundamental concept of variable entered mathematics. Viéte applied algebra to geometry promoting so the coordinate method, but Viète's algebra was still rhetoric and operations were performed on quantities instead of on numbers. The development of symbolic algebra is mainly due to René Descartes (1596–1650). It is often said that where Viète stopped Descartes continued. Descartes' particularly significant invention was the number axis being a geometric model representing the system of (positive) real numbers. Namely, a half line with origin O is that model and each closed interval OA represents uniquely a real number. Simple geometric constructions performed on these intervals produce sums, differences, products and quotients of numbers. The point A of the interval OA can also be considered to represent the number which is corresponded to that interval and the unit interval represents number 1. Descartes also adopted symbols for operations and relations so that his notation is already modern. Descartes is the founder of analytic geometry whose methods had a profound impact on development of analytic areas of mathematics.

In the first half of nineteenth century a number of outstanding English mathematicians: George Peacock, Augustus de Morgan, Duncan Gregory et al., tackled the problem of logical justification of operations with literal expressions. By the middle of nineteenth century, a list of axioms of algebra was generally accepted: commutative and associative laws for addition and multiplication, distributive law, agreement of addition with equality and the order relation, etc. As a method of derivation of valid algebraic relations, the Peacock's principle of permanence was also accepted: Whatever algebraic forms are equivalent when the symbols are general in form but specific in value will be equivalent likewise when the symbols are general in value as well in form. Simplifying the formulation of this principle, the following could be said: Whenever a property of natural numbers is formulated in terms of literal expressions, it continues to be valid when interpreted as a property of extended systems of (positive) rational numbers, integers, rational and real numbers.

As the history of mathematics demonstrates it clearly, the development of the number idea starts with natural numbers, goes via rational numbers and terminates with real numbers, what is a path followed from antiquity to the modern time. What is noticeable in the contemporary following of this path is the use of algebra when the properties of operation and the order relation are formulated and established. For the sake of clarity, we make a difference between school algebra and abstract algebra. In the former case the range of values of the involved variables is a fixed number system, while in the later case is any non-empty set on which an algebraic structure is defined. It is always a dilemma how to treat natural numbers and what is the degree up to which intuition is abandoned in favor of logical rigor. This is the right place to cite H. Freudenthal (see his China Lectures, Kluwer Academic Press) who says: " ... as popular as axioms and axiomatising might be in modern mathematics, they are only the highlights and finishing touches in the course of an activity, where the stress is on the form rather than on the content".

The way how the system of natural numbers is comprehended divides mathematicians into two groups (leaving out many of us whose stand is not very essential). On one hand there exist mathematicians, called Naturalists (among them the great classical mathematician Henri Poincaré (1854–1912), German mathematician Leopold Kronecker (1823–1891), et al., who consider natural numbers to be direct product of human mind. A very well-known Kronecker's saying: God made the integers, all else is the work of man, expresses this point of view in a nice symbolic way. On the other hand, the so called Formalists consider the natural numbers to be constructions executed on logical basis. The typical example is the case of construction of natural numbers which starts ab ovo, with Peano's axioms. We take here the system of natural numbers in the way it is established throughout learning in school, avoiding all formal constructions.

In search for basic operative properties (properties of operations and the order relation) of the system $\mathbb{N}_0$ of natural numbers with 0, the list that follows has been formed:

(i) $(\forall k)(\forall l)\, k+l=l+k$

(ii) $(\forall k)(\forall l)(\forall m)\, (k+l)+m=k+(l+m)$

(iii) $(\exists 0)(\forall k)\, k+0=k$

(iv) $(\forall k)(\forall l)\, kl=lk$

(v) $(\forall k)(\forall l)(\forall m)\, (kl)m=k(lm)$

(vi) $(\exists 1)(0<1 \textit{ and } (\forall k)\, k\cdot 1=k)$

(vii) $(\forall k)(\forall l)(\forall m)\, k(l+m)=kl+km$

(viii) $(\forall k)(\forall l)\, (k<l \iff (\exists m>0)\, k+m=l)$

(ix) $(\forall k)(\forall l)\, (k<l \textit{ or } k=l \textit{ or } l<k)$

(x) $(\forall k)(\forall l)(\forall m)$
$(k<l \iff k+m<l+m)$

(xi) $(\forall k)(\forall l)(\forall m>0)$
$(k<l \iff km<lm)$

Transcribing the properties of the system $\mathbb{N}_0$ of natural numbers with 0 that are listed above, by replacing $\mathbb{N}_0$ with a non-empty set S, the letters k,l,m with

a, b, c; $0, 1$, and $<$ by $0_S, 1_S$ and $<_S$ then S together with the transcribed conditions (i)–(xi) is an abstract structure which we call the *ordered semifield*.

Let us suppose that $(S, +, \cdot, <)$ is an arbitrary ordered semifield. We define a mapping $f\colon \mathbb{N}_0 \to S$ taking $f(0) = 0_S = a_0$, $f(1) = 1_S = a_1$ and supposing that $f(n) = a_n$ has already been defined we take $f(n+1) = a_n + 1_S = a_{n+1}$. Then, it can be proved that f is an isomorphism of $\mathbb{N}_0$ into S. So we prove that each ordered semi-field contains an isomorphic copy of the system $\mathbb{N}_0$. In other words the system $\mathbb{N}_0$ of natural numbers is the smallest ordered semifield.

Adding to the transcribed list of conditions, also the condition $(\forall a)(\exists b)\, a + b = 0_S$, an abstract structure is obtained that we call the *ordered semifield with additive inverse*. A specific example of the ordered semifield with additive inverse is the system of integers and each ordered semifield with additive inverse contains an isomorphic copy of the system of integers. In other words, the system of integers is the smallest ordered semifield with additive inverse.

In a similar way, when to the transcribed list of conditions, also the condition $(\forall a \neq 0_S)(\exists b)\, a \cdot b = 1_S$ is added, an abstract structure is obtained which we call the *ordered semifield with multiplicative inverse*. A specific example of an ordered semifield with multiplicative inverse is the system of positive rational numbers with 0 and each ordered semifield with multiplicative inverse contains an isomorphic copy of positive rational numbers with 0. In other words, the system of positive rational numbers with 0 is the smallest ordered semifield with multiplicative inverse.

When to the transcribed list of conditions, also the conditions $(\forall a)(\exists b)\, a + b = 0_S$ and $(\forall a \neq 0_S)(\exists b)\, a \cdot b = 1_S$ are added, then such an ordered semifield is standardly called the *ordered field*. A specific example of an ordered field is the system of rational numbers and, as it is well-known, each ordered field contains an isomorphic copy of the system of rational numbers. In other words, the field of rational numbers is the smallest ordered field.

This author considers the structuring of the number systems to be his most significant single contribution to education.

REFERENCES

1. M. M. Marjanović, Z. Kadelburg, *Structuring the Number Systems of Natural, Positive Rational and Rational Numbers*, The Teaching of Mathematics, Vol. XXII, 1 (2019), 1–16.
2. M. M. Marjanović, Z. Kadelburg,, *Number Systems Characterized by Their Operative Properties*, The Teaching of Mathematics, Vol. XXII, 2 (2019), 52–57.
3. M. M. Marjanović, *Generalized Commutative and Associative Laws*, The Teaching of Mathematics, Vol. XXII, 2 (2019), 71–76.
4. M. M. Marjanović, Z. Kadelburg, *A Reconsideration of Number Systems Characterized by Their Operative Properties*, The Teaching of Mathematics, Vol. XXIII, 2 (2020), 81–89.

RECOLLECTIONS
of Professor M. Marjanović

1. I started my study of mathematics in 1951. The study program (curriculum) that we had was hopelessly out of date. Once fixed in 1905, it was hardly been changed for those fifty years. A number of our professors had already been retired in the period between two World Wars and reactivated in 1945. The oldest among them was Professor Nikola Saltikov, born in 1864, who left his Russia in 1921. He taught analytic geometry and for the students of the fourth year an optional course on partial differential equations. When asked to suggest a book on mathematical analysis, he would suggest E. Goursat, Course d'analyse mathèmatique (a book from Professor Saltikov's younger days). From the very beginning I was his favorite student. But when I was at my third year of study and when he heard that I read Professor Djuro Kurepa's book on set theory, I lost all his inclination. In that 1951, Professor Jovan Karamata left the country and found a place at the University of Genève. One so much refined intellectual as Miloš Radojčić was, he was deeply concentrated on his synthetic Euclidean geometry and the axiomatisation of the theory of relativity. I hope I will not be wrong if I say that all others of our professors rejected set theory without being much acquainted with it. These professors used each occasion to manifest their anti-Cantorism pretending that it was a sign of purity.

Due to the sanctions imposed by Resolution of Information Bureau, once available very cheap mathematical books in Russian disappeared from our book shops. To illustrate that degree of emptiness, I would mention that the only math book in foreign language that I bought in one of our book shops during four years of my study of mathematics was a collection of math problems for students of engineering. Was it the worst time to study at Belgrade University? Yes, I think yes.

It was in 1953, when I happened to know for the possibility to order scientific books in French using Cultural Center of the French Embassy in Belgrade. The first two books that I ordered were Goursat's Course d'analyse mathèmatique and Sierpinski's Leçons sur les nombres transfinis. This Sierpinski's book together with already mentioned book of Kurepa, Set Theory (in Serbo-Croation) made a solid base for me to get acquainted with transfinite ordinal numbers. In 1955 as a student of the fourth year I wrote a paper on transfinite operations with ordinal numbers. Using this paper as my graduation work and protecting it before a commission, I obtained diploma in mathematics in June 1955.

Probably it was due to some obstacles that this paper was published three years later: M. Marjanović, Une suite transfinie d'opèration avec le nombre ordinaux et son application á la representation de ces nombres, Весник Друштва математичара и физичара 10 (1958), 17–34. What was even worse, the paper was intentially retained not to be reported in Mathematical Reviews. As I learnt it later, it was the way to stop me not to doctorate before than a somewhat older assistant in our Department who was particularly influential as a party member. So, in the course of time, I had also forgotten to think of that my work.

In 1960, the postgraduate study started at Belgrade University. In our Department, the postgraduate students had to take the following four courses: Measure theory, Functional analysis, Topology (theory of neighborhood spaces) and Algebra (theory of groups). It was Professor Slobodan Aljančić who taught Measure theory and Functional analysis. Professor Zlatko Mamuzić taught Topology and Professor Dragoljub Marković taught Algebra. The subject "Exercises" which follow each course at our University, assistants Milosav Marjanović (Measure theory, Functional analysis and Topology) and Slaviša Prešić (Algebra) were engaged to realize. After two years these courses were scheduled for the third year students and it was the end for the nineteen century mathematics in our Mathematical Department.

Fig. 9. Flowers on a steep piece of the lawn

Fig. 10. A spruce trimmed for years to preserve this same cyllindrical shape

2. This recollection goes into the late 1960's, when Professor Dragoslav Mitrinović was preparing the manuscript for his book

1) Dragoslav Mitrinović, Analytic Inequalities. Springer, 1970.

At that time a number of us was engaged to translate into English some material previously published in Serbian. Our translations were sent to an American mathematician who was preparing the final version of the manuscript. So I happened to visit Professor's home a number of times, when Mrs. Mitrinović would serve coffee with cake and then it was proper to leave the host's home. But the visit that I am going to talk about lasted longer than usually and some details from conversation that ran that evening will stay forever in my memory. What particularly surprised me was the fact that Professor mentioned that it was expected from a country as ex-Yugoslavia to practice the branches of mathematics which are more elementary. Two examples that, at that moment ran through my mind were graph theory and inequalities. And if the state of affairs was such really, then the question is what was the way of its establishing. Well, many of us should have to think about it.

But what was the way Professor new some facts that were hidden for many of us? Let us recall that at that time Yugoslavia was a country between The East and The West. Several outstanding mathematicians from both The East and The West visited the country. Let us mention only some of them: from America Solomon Lefschitz and Marshall Stone, from France Jean Leray, from Russia Andrey Kolmogorov and Pavel Alexandrov and a number of others. A number of them also visited the home of Professor Mitrinović. From one of them or from some of them Professor must had known which role was assigned to Yugoslav mathematics.

That evening I came with my review of the following Professor's fine paper:

2) Dragoslav S. Mitrinović, Steffensen Inequality, Publ. de la Fac. elektotechn., 1969.

The reason why I was the reviewer is due to my paper

3) M. Marjanović, Some Inequalities with Convex Functions. Publ. Math., tome 22, 1968.

In this paper I use a basic inequality to derive proofs of tree classical inequalities, one of which is Steffensen inequality. I am thankful to Professor Mitrinović for including this my result in his excellent book cited above under 1), which takes over and continues to have the function which classical book Hardy-Litllewood-Polya, Inequalities had.

3. This recollection is more extended in time and I hope that the exposed details will contribute to the feeling of a whole. Teaching Mathematics II, a course for the second year students of physics and Analysis I, a course for the first year students of mathematics, I practiced to introduce in these courses the elements of metric spaces and in particular Banach contraction principle. As an interesting extension of this principle, I cited the paper of Israeli mathematician M. Edelstein, An extension of Banach contraction principle, Proc. of Amer. Math, Soc. 12, (1961), 7–10. Some time later I wrote a paper, extending further this Edelstein's result: M, Marjanović, A further extension of Banach contraction principle. Proc. of Amer. Math. Soc., (1968), 411–414.

It was one of the early 1970's when I formed a seminar on contraction type theorems to study conditions under which such statements hold true. The most active participant of that seminar was Ljubomir Ćirić, then an assistant at the Faculty of Mechanical Engineering, Belgrade University. A theorem was thought out for him which would be a still further extension of the results cited above. He was supposed to supply with a proof that theorem, what would be the basic result for his doctoral thesis. But it was quite a surprise when one day he informed me that he will continue to be preparing his thesis under the guidance of Professor Djuro Kurepa. I can remember that two of us argued about some details that he was including in his dissertation if they were my observations from the seminar or not. But that all was not of any greater significance and that was the way how our friendship ended.

Fig. 11. Milo's studio under a tall beech

Fig. 12. Globosa with two hornbeams trimmed to preserve their bushy shapes

I supplied with a proof the theorem that was assigned to Lj. Ćirić and I sent that paper to Proc. of Amer. Math. Soc. At that time, American mathematician Ernest Michael was responsible for the section general topology. The reviewer of the paper was a mathematician whose surname was Fraser and it is all that I still remember. Professor Michael informed me that the reviewer has some complementary results and that he suggests a joint paper with me. I wrote a letter to the reviewer, saying that I agree. Replying he formulated a problem that he was thinking over. In a week or so I solved the problem and I sent him the solution. Professor Fraser replied, saying that in the fall that year he will be at another university but that he will prepare our joint paper until Christmas. I waited for Christmas that year and Easter following year and then I decided to write to Professor Michael. His answer was very short: "I have nothing in my files"! Being somewhat discouraged, I published that paper a number of years later: M. Marjanović, Fixed points of local contractions, Publ. de l'Inst. Math., 21, (1976), 187–190. The review of the paper in Mathematical Reviews was quite cynical – short text saying that a theorem of Banach contraction type is proved, without the name of reviewer.

Evidently, I have been the first at our university to write a couple of variations on Banach contraction principle. And knowing for sure that nothing significant can be done dealing further with contraction principle I stopped thinking of this matter. But a number of Serbian mathematicians, led by Professor Ćirić continued to write papers varying endlessly contraction conditions. Their activity was in full blossom for a couple of years in the middle of the second decade of this century. Due to this group of mathematicians, Serbian silence was highly ranked at the Shanghai list (about four hundredth place). I leave it to my younger colleagues to think of the function of Mathematical Reviews, of the function of our highest scientific institutions (Serbian Academy of Sciences, Belgrade University and all other institutions of that rank). And were two first decades of this century the period of progress or of stagnation as I am inclined to think.

Fig. 13. Marigolds and a plate of stone

4. When some events have happened recently, then they are remembered vividly, with all details. Therefore, it is only the past time in which they happened, which includes them in recollections. Thus, just recently, a couple of years ago, my colleague Zoran Kadelburg and I, we started to study operative properties of number systems. Let us recall that operative properties (properties of operation and the order relation) are those which are carried over from one number system to its extensions. That study led us to characterize structurally number systems as the ordered semifields (and what is exposed in Notice 6 of the previous paper with somewhat more details).

There is, as we know it well, an extensive literature on algebra in school. There is also a lot of related theorizing, what makes us to say that a letter that denotes a variable stands for a species of numbers, as Vieta teaches us. And for a child in school a letter should be any of the numbers that he or she has learnt up to that moment.

It is also a firm conviction of this author that for a researcher on education, a good knowledge of the subject matter is very important. Thus, let us say it in the context of this paper that a researcher oriented towards the study of school algebra has to be well acquainted with structural properties of number systems. Which of these properties, when and how, should be elaborated and included in classroom activities is a task that stands before us and we think that we have to be dedicated to this task.

CHATS

A chat means, of course, a friendly informal talk. And we can imagine two persons talking face to face, by phone or each siting at a table with a computer on it, exchanging by e-mail their thoughts and ideas. And when any of these activities is done in an easy, leisure way then these persons chat, don't they? Also, when one person writes something expecting that it will entertain the person which reads it, then we can say figuratively that it is a chat. Let us also say, joking only 99 per cent, that chats are the genre of literature that we start to create.

Fig. 14. Two hornbeams trimmed for more than twenty years.
One of them growing from a vertical wall covered in ivy.

When you see the photographs scattered all over the pieces of this writing, then you know that they deserve a place in these chats. And when they evidently illustrate an ambience and an activity, then they have their right place in these writings. The activity that we are going to talk about is the trimming of different kinds of trees to make them preserve for years more or less the same bushy shape. If it would be an exaggeration to say for such an activity to be an art, but it is surely a serious skill. And you would guess easily that this skill is in the hands and the mind of Professor Milo Marjanović.

All species of trees that Milo trims are domestic, growing all around. In the most cases they are spruce, hornbeam and oak. What is particularly interesting

Fig. 15. From season to season colors change but shapes don't

for these two deciduous species is the fact that their leaves do not fall off in the autumn, but they stay on the trees, withered and pale, until late April the following year. Such their winter appearance is still attractive, especially with a snow-drift on them.

Fig. 16. These mushroom like bushes have been trimmed for years.

The only piece of working tool that Milo used was a pair of scissors for cutting wool of sheep. As there were tens of trees to be trimmed each year and most of them more than once, then it is easy to imagine how such an activity was quite an effort which required days and days to be carried out. Well, now a verity of electrical trimmers is available and the shaping of trees is much easier. But as this Milo's dwelling spot is steep and overgrown with trees, now it is hard for him to move over it and to trim.

There exist some species of trees that are particularly suitable for shaping. We see them in the parks of the cities all over the world and in some exclusive places where they decorate those spots. But how are shaped the trees that grow here and there, all around us in our gardens, our yards, etc.? Milo has no right suggestion how to do it and he directs you to learn it by yourself, trying and trying. In the spring when trees are already in leaf you chose a smaller tree and you cut off its top part and some of its small branches. Maybe you already discern some shape. In any case you leave the tree until the spring following year and when it is into full leaf, you trim it carefully. If a shape discerns sharper, then let you go on trimming. If not, please, let the tree grow freely. As you see it, Milo has no pretension to teach you how to shape trees that grow on your dwelling spot. In fact, try by yourself is all what he suggests and if you don't, then you probably have another equally absorbing activity.