THE TEACHING OF MATHEMATICS
2022, Vol. XXV, 1, pp. 21–29

TWO HIDDEN PROPERTIES OF HEX NUMBERS

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Abstract. In this paper, we prove that the n-th hex number is exactly the sum of the number of pieces and the number of triple points associated with an 'n-balanced' partition of a triangle obtained by $n-1$ cevians from each vertex. Moreover, we see via hex numbers an extension of a Feynman's result: the $(k+1)$-th hex number is the ratio of the area of a triangle $\mathcal{T}$ and the area of central triangle associated with a regular partition of $\mathcal{T}$ of order $2k+1$.

MathEduc Subject Classification: G34

AMS Subject Classification: 97G30

Key words and phrases: Hex numbers; cevians; Balanced partitions of triangles; Regular partitions of triangles; Feynman's triangle.

1. Introduction

Among many fascinating relationships of numbers are those that suggest (or were derived from) the arrangement of points representing numbers into a series of geometrical figures. Such numbers, known as figurate numbers, appeared in 15th-century books and were probably known to the ancient Chinese, but they were of particular interest to ancient Greek mathematicians (see [14]). In the Didactics of Mathematics, figurate numbers are objects of relevant interest (see [1–4, 8–10]; in particular, as highlighted in the article of Kempen and Biehler [11], the use of figurate numbers can be considered "a heuristic in the field of problem solving or proving, which involves interesting perspectives of the semiotic theory of Peirce ('diagrammatic reasoning' and 'collateral knowledge') and cognitive psychology ('schema theory' and 'Gestalt psychology')".

Polygonal numbers and centred polygonal numbers are special kinds of figurate numbers (see Figure 1). For an exhaustive introduction to figurate numbers, see [5].

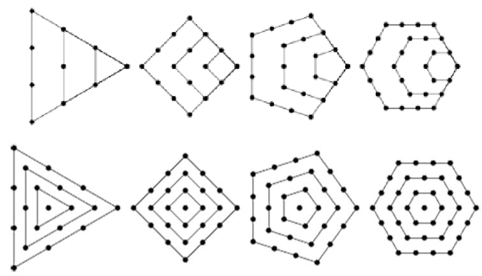

Fig 1. Examples of polygonal numbers and centred polygonal numbers (see [15])

Particular centred polygonal numbers are centred hexagonal numbers, also called *hex-numbers* (see [5, p. 41]). They are $Hex(1) = 1$, $Hex(2) = 7$, $Hex(3) = 19$, $Hex(4) = 37$, $\ldots$ (see Figure 1) and have many interesting properties (see A003215 in OEIS, [13]). In this article, we will see two elementary properties referring to the partition of a triangle which do not seem to be highlighted in the literature.

Definition. Let $\mathcal{T}$ be a triangle. An *n-balanced partition* of $\mathcal{T}$, $\mathfrak{F}_n$, is a dissection of $\mathcal{T}$ obtained by dividing each side in n parts by arbitrary $(n-1)$ points and joining them to the opposite vertex by $(n-1)$ cevians. Every intersection of such cevians will be called *repartition point*. A repartition point will be called *triple* (or *cevian*) if it is the intersection of three cevians (see Figure 2).

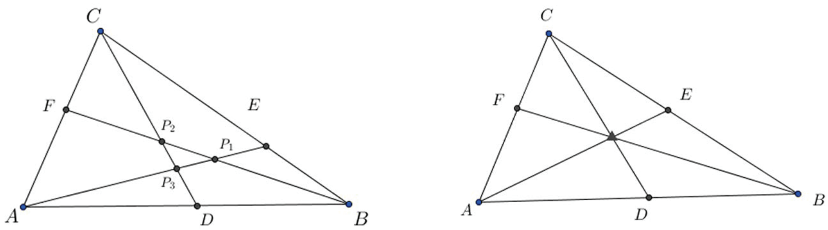

Fig. 2. Two examples of 2-balanced partitions $\mathfrak{F}_2$ and $\mathfrak{F}'_2$.
Left: in $\mathfrak{F}_2$, there are three distinct repartition points.
Right: the cevians of $\mathfrak{F}'_2$ are concurrent in a repartition triple point.

Clearly, referring to Figure 2, in $\mathfrak{F}_2$, we find 7 pieces, while, in $\mathfrak{F}'_2$, we find 6 pieces and 1 triple point. Thus, it appears that in every 2-balanced partition, the sum of the number of pieces and triple points is in any case 7.

A first question then arises:

Does the invariance property expressed for the configuration of order 2, that is, that the sum of the number of pieces and of the triple points of any partition $\mathfrak{F}_2$ of a triangle is constant, hold for partitions of order greater than 2?

In this article, we will prove that the answer is positive, and that in general the sum of the pieces and triple points in any balanced partition of order n is the *n-th centred hexagonal number*, $Hex(n)$ (see Theorem 2.1).

Hex numbers appear again in another context referred to as special partitions of a triangle.

An n-balanced partition is said to be *regular* if each side is divided in n equal parts. We will denote these partitions as $\mathfrak{R}_n$. A well-known example is the 3-regular partition of a triangle (see Figure 3, where the area of the inner triangle formed by these lines is exactly one-seventh of the area of the initial triangle (for example, see [6]).

We will see that for every regular partition of odd order $n = 2k + 1$ we can define a central triangle $\mathcal{F}_n$ associated with a regular partition $\mathfrak{R}_n$, and that the

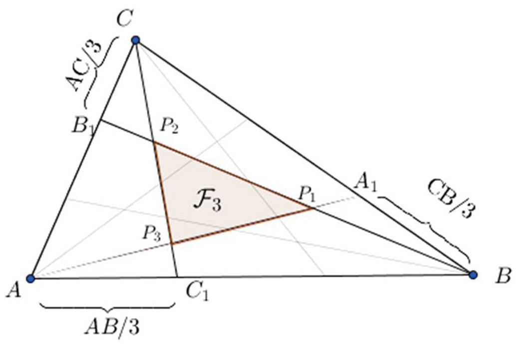

Fig. 3. The area of $\mathcal{F}_3 = P_1P_2P_3$ is one-seventh of ABC

ratio between the area of the triangle ABC and $\mathcal{F}_n$ is exactly $Hex(k+1)$ (Theorem 3.1).

2. Hex numbers as the sum of the number of pieces and triple points in balanced partitions

Hex numbers are connected to triangular numbers $T(n)$ by an elementary relation:

(1) $\quad Hex(n) = 6T(n-1) + 1;\quad$ moreover, it is easy to see that

(2) $$Hex(n+1) = Hex(n) + 6n.$$

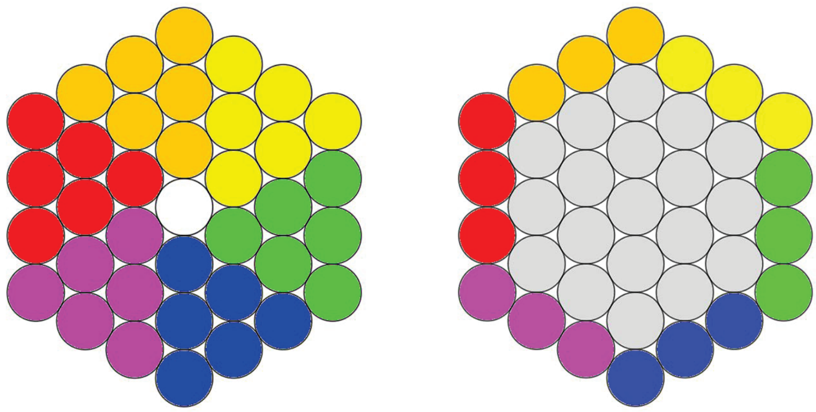

Fig. 4. Left: a visual proof for hex numbers: $Hex(n) = 6T(n-1) + 1$, where $T(n-1)$ is the $(n-1)$-th triangular number. Note that $Hex(2) = 7$, $Hex(3) = 19$, $Hex(4) = 37$.
Right: a property of Hex numbers.

Remark 1. The reader can easily check by visual proof that for any n-centred polygonal number $C_l(n)$ (l vertices), the above relations (1) and (2) become the following:

$$C_l(n) = l\,T(n-1) + 1, \tag{3}$$
$$C_l(n+1) = C_l(n) + l\,n. \tag{4}$$

We also highlight that easy induction shows the following:

$$Hex(n) = 3n^2 - 3n + 1.$$

The following result shows that hex numbers are strongly related to the balanced partitions of a triangle.

THEOREM 2.1. *Let $\mathfrak{F}_n$ be an n-balanced partition of a triangle $\mathcal{T}$. Then the sum of the number of pieces and the number of triple points of $\mathfrak{F}_n$ is equal to $Hex(n)$.*

Proof. In the following, for every positive integer, we will denote by $\#(\mathfrak{F}_n)$ the sum of the number of pieces and the number of triple points of $\mathfrak{F}_n$.

The statement is trivial if $n = 1$. Indeed the 1-balanced partition $\mathfrak{F}_1$ has just one piece and no triple point, so $\#(\mathfrak{F}_1) = 1$; on the other hand $Hex(1) = 1$.

We have already observed that $\#(\mathfrak{F}_2) = 7$; therefore, since $Hex(2) = 7$ the statement also holds for $n = 2$. Let now $n > 2$, and proceeding by induction assume that for every $(n-1)$-balanced partition $\mathfrak{F}_{n-1}$, the relation $\#(\mathfrak{F}_{n-1}) = Hex(n-1)$ holds (see Figure 5).

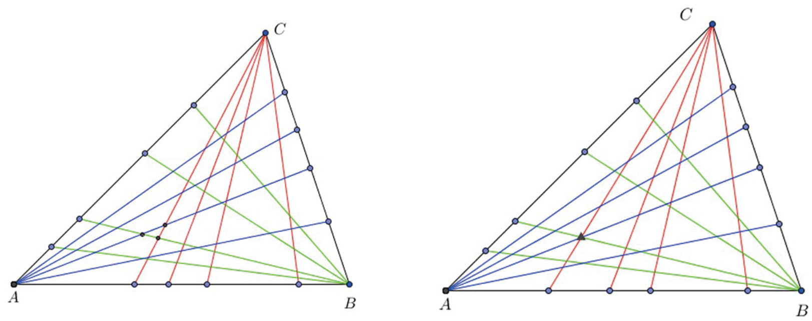

Fig. 5. Left: $(n-1)$-balanced partition $\mathfrak{F}_{n-1}$ without triple points.
Right: an $(n-1)$-balanced partition $\mathfrak{F}'_{n-1}$ with a triple point.
In both the cases, $\#(\mathfrak{F}_{n-1}) = \#(\mathfrak{F}'_{n-1}) = Hex(n-1)$.

Starting from $\mathfrak{F}_{n-1}$, we will show that for every n-balanced partition of $\mathcal{T}$, we have $\#(F_n) = Hex(n)$.

For, let $\mathbf{r} := CD$ be a new cevian from the vertex C. In this way, $\mathcal{T}$ will be dissected into more pieces and other possible triple points could appear. Clearly, $\mathbf{r}$ will cross only those pieces just lying between two consecutive C-cevians of $\mathfrak{F}_{n-1}$

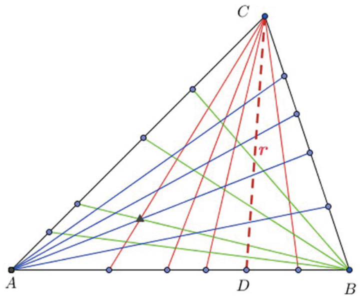

Fig. 6. Adjoining to a balanced partition $\mathfrak{F}_{n-1}$ one more cevian $\mathbf{r}$, the number of pieces increases, and some more triple points might appear.

(see Figure 6), so that the other parts of the new dissection coincide with those of $\mathfrak{F}_{n-1}$.

Then, to determine the variation in the number of pieces and the number of triple points in the transition from $\mathfrak{F}_{n-1}$ to the new dissection $\mathfrak{F}_{n-1} \cup \mathbf{r}$, we have to count the number of pieces of $\mathfrak{F}_{n-1}$ that are crossed and hence spit into two pieces by $\mathbf{r}$, and the number of repartition points crossed by $\mathbf{r}$. To show that this variation is $2(n-1)-1$, we consider two possible cases:

1) The new cevian $\mathbf{r}$ does not pass through repartition points of $\mathfrak{F}_{n-1}$: in this case, $\mathbf{r}$ does not generate new triple points, and intersects $n-2$ A-cevians and $n-2$ B-cevians. Thus, also considering the extremes of $\mathbf{r}$ we have $2(n-2)+2 = 2(n-1)$ points dividing $\mathbf{r}$ into $2(n-1)-1$ segments, each of which divides a piece of $\mathfrak{F}_{n-1}$ (crossed by $\mathbf{r}$). In this case the number of pieces increases by $2(n-1)-1$ (see Figure 7, left).
2) The new cevian $\mathbf{r}$ passes through some repartition points of $\mathfrak{F}_{n-1}$: in this case, if t is the number of points crossed by $\mathbf{r}$, we have t new triple points; moreover, $\mathbf{r}$ intersects the cevians of $\mathfrak{F}_{n-1}$ in $2(n-2)-t$ distinct points. Thus, also considering the extremes of $\mathbf{r}$, we have $2(n-2)-t+2 = 2(n-1)-t$ points dividing $\mathbf{r}$ in $2(n-1)-t-1$ segments, each of which divides a piece of $\mathfrak{F}_{n-1}$ crossed by $\mathbf{r}$. In this case the number of pieces increases by $2(n-1)-t-1$. In total we have an increment of $2(n-1)-t-1$ pieces and t triple points, the sum of which is $2(n-1)-1$ (see Figure 7, right).

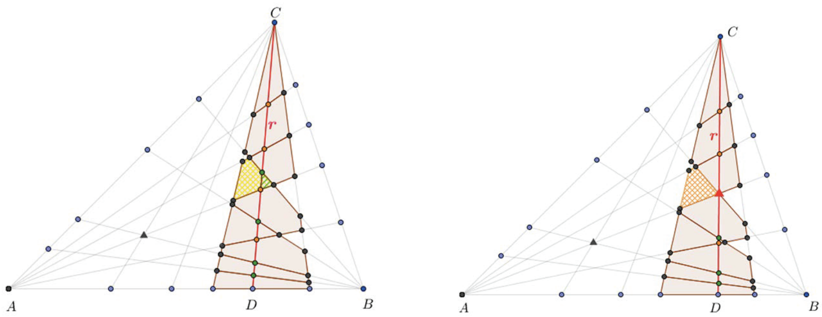

Fig. 7. Left: the cevian $\mathbf{r} := CD$ does not cross any repartition points.
In the new dissection $\mathfrak{F}_{n-1} \cup \mathbf{r}$ of ABC the number of pieces increases by $2(n-1) - 1$.
Right: the cevian $\mathbf{r} := CD$ crosses through a repartition point of $\mathfrak{F}_{n-1}$ and generates a new triple point. The number of pieces increases by $2(n-1) - 2$, one less than in the first case.

Starting from the new configuration $\mathfrak{F}_{n-1} \cup \mathbf{r}$, we may consider an A-cevian $\mathbf{s} := AE$ (see Figure 8). Taking into account that $\mathfrak{F}_{n-1} \cup \mathbf{r}$ has an additional C-cevian, $(\mathbf{r})$, repeating the above argument we have that the number of pieces and the number of triple points determined by $\mathbf{s}$ increases by $2(n-1)-1+1 = 2(n-1)$.

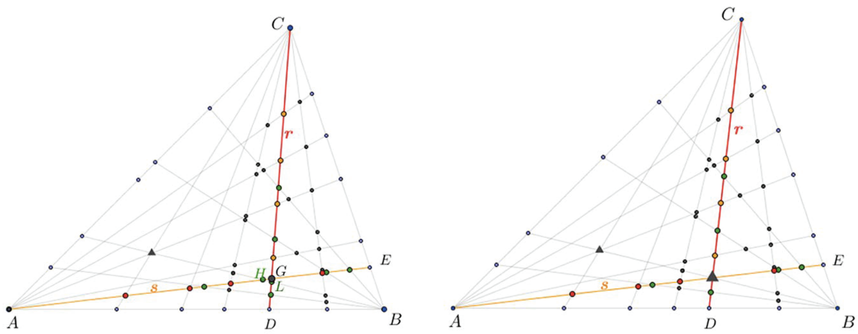

Fig. 8. Left: the cevian $\mathbf{s}$ does not determine triple points.
Right: a possible configuration in which $\mathbf{s}$ determines a new triple point.

Repeating the same reasoning again with a B-cevian $\mathbf{u} := BF$ (outgoing from B) compared to the configuration $\mathfrak{F}_{n-1} \cup \mathbf{r} \cup \mathbf{s}$, we have that the increment of the number of pieces plus the number of triple points is $2(n-1)+1$ (see Figure 9).

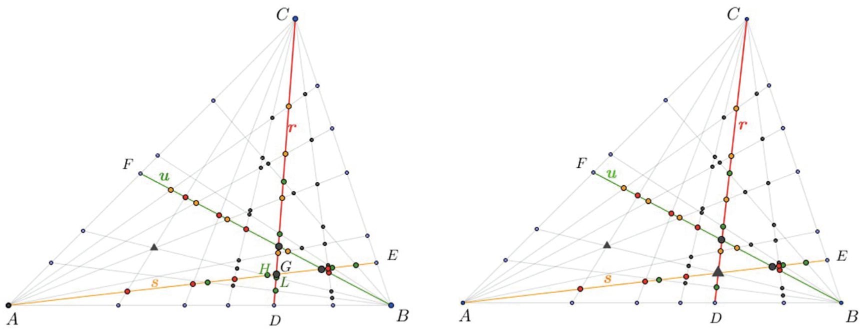

Fig. 9. Two examples of balanced partitions. Slightly modifying the position of D, we see that triangle GHL collapses at triple point. It turns out that the right partition has one more triple point and one less piece than the left partition.

Therefore, the sum of the number of pieces and the number of triple points in the partition $\mathfrak{F}_n = \mathfrak{F}_{n-1} \cup \mathbf{r} \cup \mathbf{s} \cup \mathbf{u}$, is incremented with respect to those of the partition $\mathfrak{F}_{n-1}$ by $[2(n-1)-1] + 2(n-1) + [2(n-1)+1] = 6(n-1)$. It follows from Eq. (2) and by induction hypothesis that

$$\#(\mathfrak{F}_n) = \#(\mathfrak{F}_{n-1}) + 6(n-1) = Hex(n-1) + 6(n-1) = Hex(n).$$

The theorem is proved. ■

3. Hex numbers in generalized Feynman's triangles

The result shown in Figure 3 in Introduction, is often referred to as *Feynman's theorem* and the central triangle is also called *Feynman's triangle*. It appears that the great physicist tried to show the theorem at the end of a dinner with a guest, Prof. Kai Li Chung of Stamford University during a visit to Cornell University. Feynman proved the theorem for equilateral triangles, and in the more general case, there are different proofs for this theorem (see [6]). Indeed, extensions of this theorem were already known.

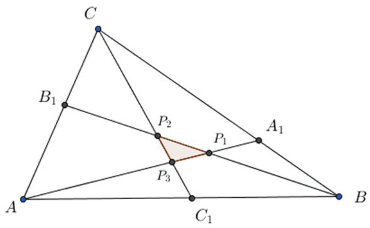

Fig. 10. Routh's theorem: The area of $P_1P_2P_3$ can be given in terms of the ratios $\dfrac{CA_1}{A_1B}$, $\dfrac{BC_1}{C_1A}$, and $\dfrac{AB_1}{B_1C}$.

One of these theorem, as suggested in [12], is Routh's theorem, which can be found for example, in H. S. M. Coxeter, (see [7, Equation 13.55]) in the following form:
(5)

$$\text{Routh's formula} \qquad \mathcal{T}(\lambda,\mu,\nu) = \frac{(\lambda\mu\nu-1)^2}{(\lambda\mu+\lambda+1)(\mu\nu+\mu+1)(\nu\lambda+\nu+1)}ABC,$$

where $\dfrac{CA_1}{A_1B} = \lambda$, $\dfrac{BC_1}{C_1A} = \mu$, $\dfrac{AB_1}{B_1C} = \nu$, and $\mathcal{T}(\lambda,\mu,\nu)$ both denotes the triangle $P_1P_2P_3$ and its area.

Let now $n = 2k+1$ be an odd integer, and let $\mathfrak{R}_n$ be a regular partition of order n of a triangle $\mathcal{T} := (ABC)$. Then every side of $\mathcal{T}$ is divided into n parts by $2k$ cevians. If we consider the k-th cevian (counterclockwise) from each vertex, a_k, b_k, and c_k, we obtain a central triangle $\mathcal{F}_n := \mathcal{T}\left(\frac{k+1}{k}, \frac{k+1}{k}, \frac{k+1}{k}\right)$ (see Figure 11), that we call the *n-th Feynman's triangle* associated with the regular partition $\mathfrak{R}_n$. Clearly, the triangle shown in Figure 3 is the third-Feynman's triangle associated with the regular partition $\mathfrak{R}_3$.

Remark 2. Let $\mathcal{T}$ be a triangle. We have seen that $\dfrac{\mathcal{T}}{\mathcal{F}_3} = 7 = Hex(2)$. Applying the Routh's formula, one may check that $\dfrac{\mathcal{T}}{\mathcal{F}_5} = 19$ and $\dfrac{\mathcal{T}}{\mathcal{F}_7} = 37$, which coincide with $Hex(3)$ and $Hex(4)$, respectively.

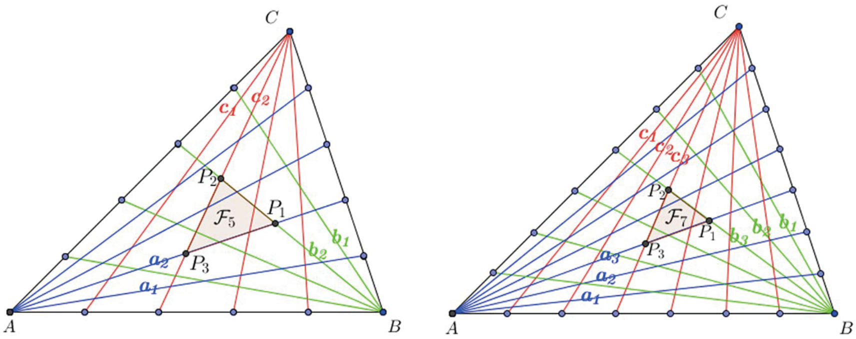

Fig. 11. Left, the central triangle $\mathcal{F}_5$ associated with the partition $\mathfrak{R}_5$: its sides lie on the cevians a_2, b_2 and c_2. Right, the central triangle $\mathcal{F}_7$ associated with the partition $\mathfrak{R}_7$: its sides lie on the cevians a_3, b_3 and c_3.

REMARK 3. First we note that the area of every n-th Feynman's triangle can be obtained by Eq. (5):

$$\mathcal{F}_n = \mathcal{T}\left(\frac{k+1}{k}, \frac{k+1}{k}, \frac{k+1}{k}\right) = \frac{((\frac{k+1}{k})^3 - 1)^2}{[(\frac{k+1}{k})^2 + \frac{k+1}{k} + 1]^3} ABC.$$

On the other hand, for every integer k, the number $((\frac{k+1}{k})^3 - 1)^2$ is different from 0, which implies that the area of $\mathcal{F}_n$ is not zero, and in particular, the intersection points of a_k, b_k and c_k are three distinct points.

The relations given in Remark 2 are particular cases of a general result, which shows another property of hex numbers.

THEOREM 3.1. *Let $n = 2k+1$ be an odd integer, and let $\mathfrak{R}_n$ be a regular partition of order n of a triangle $\mathcal{T} := (ABC)$. Then, the ratio of the area of $\mathcal{T}$ and the area of the central triangle $\mathcal{F}_n$ associated with $\mathfrak{R}_n$ is the $k+1$-th hex number:*

$$\frac{\mathcal{T}}{\mathcal{F}_n} = Hex(k+1)$$

Proof. The m-th Hex number is

$Hex(m) = 3m(m-1)+1 = 3m^2-3m+1,$ (see A003215 in the OEIS, or [5, p. 41]),

so that

$Hex(k+1) = 3(k+1)^2 - 3(k+1) + 1 = 3k^2 + 6k + 3 - 3k - 3 + 1 = 3k^2 + 3k + 1.$

By definition, $\mathcal{F}_n := \mathcal{T}\left(\frac{k+1}{k}, \frac{k+1}{k}, \frac{k+1}{k}\right)$, and by the Eq. (3) we have:

$$\begin{aligned}\frac{ABC}{\mathcal{F}_n} &= \frac{[(\frac{k+1}{k})^2 + \frac{k+1}{k} + 1]^3}{((\frac{k+1}{k})^3 - 1)^2} = \frac{(\frac{k+1}{k})^2 + \frac{k+1}{k} + 1}{(\frac{k+1}{k} - 1)^2} = \frac{\frac{(k+1)^2}{k^2} + \frac{k+1}{k} + 1}{\frac{1}{k^2}} \\ &= (k+1)^2 + k(k+1) + k^2 = 3k^2 + 3k + 1 = Hex(k+1). \quad \blacksquare\end{aligned}$$

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