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# REVISITING THE FIRST MEAN VALUE THEOREM FOR INTEGRALS

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**Abstract.** We provide a proof of the first mean-value theorem for integrals using the Cauchy mean-value theorem, and give an interesting application of the mean-value theorem related to a Taylor remainder.

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## 1. Introduction

The textbook proof of the first mean-value theorem for integrals, by which we mean (2), takes into account the intermediate value theorem for continuous functions and the elementary estimates

$$m \int_\alpha^\beta v(t)\,dt \leqslant \int_\alpha^\beta (u \cdot v)(t)\,dt \leqslant M \int_\alpha^\beta v(t)\,dt,$$

where $v, u$ are Riemann integrable function, $v(t) \geqslant 0$, and $m \leqslant u(t) \leqslant M$, $t \in [\alpha, \beta]$. The idea, albeit using sums, already appears in [2], where Cauchy noticed[1] that

$$\sum_{k=0}^{n-1} (x_{k+1} - x_k) u(x_k) v(x_k) = u(\xi) \sum_{k=0}^{n-1} (x_{k+1} - x_k) v(x_k),$$

for any partition $\alpha = x_0 < x_1 < \cdots < x_n = \beta$ and $\alpha \leqslant \xi \leqslant \beta$ from which the first mean-value theorem for integrals is obtained. It is worth pointing out that in [2] both $u$ and $v$ are required to be continuous whereas nowadays the textbook formulation of such a theorem requires one function to be continuous and the other Riemann integrable. As an application of such a theorem, Cauchy gave the following results

$$\int_\alpha^\beta u(t)\,dt = u(\xi)(\beta - \alpha),$$
$$\int_\alpha^\beta u(t)\,dt = \xi u(\xi) \ln \frac{\beta}{\alpha},$$

for $\beta/\alpha > 0$, and

$$\int_\alpha^\beta u(t)\,dt = (\xi - \gamma) u(\xi) \ln \frac{\beta - \gamma}{\alpha - \gamma},$$

[1] in the *vingt-troisième leçon*, appearing in p. 92.

for $(\beta-\gamma)/(\alpha-\gamma)>0$, which follow from the mean value-theorem taking $v(t)=1$, $v(t)=1/t$, and $v(t)=1/(t-\gamma)$, respectively.[2]

In this note, we provide a somewhat different proof, relying on the Cauchy mean-value theorem and giving a small theoretical improvement when both functions are continuous. This approach is not new but, up to our knowledge, does not appear in mainstream mathematical analysis books, even as an application of the Cauchy mean-value theorem, maybe due to some technicality in the proof. We also provide an application of the mean-value theorem related to a Taylor remainder, which allows us to obtain a handful of new Taylor remainders to the best of our knowledge.

## 2. First mean value theorem for integrals

The first mean value theorem for integrals is a well-known result in analysis. Although, in principle, the proof using the Cauchy mean-value theorem is straightforward, a technicality appears if $v$ is allowed to be zero, and we need to use a limiting argument. Even though the complete proof is more involved than the textbook proof, it nevertheless has a pedagogical interest since it exposes the students to a limiting argument, which is a quite recurring topic in higher analysis.

THEOREM 1. *Let $u,v\in C([\alpha,\beta])$. Then there exists a point $\xi\in(\alpha,\beta)$ such that*

$$v(\xi)\int_\alpha^\beta (u\cdot v)(t)\,dt = u(\xi)v(\xi)\int_\alpha^\beta v(t)\,dt. \tag{1}$$

*If, additionally, $v$ is a nonnegative (or nonpositive) function on $[a,b]$, then*

$$\int_\alpha^\beta (u\cdot v)(t)\,dt = u(\xi)\int_\alpha^\beta v(t)\,dt, \tag{2}$$

*with $\xi\in[a,b]$.*

REMARK 1. Observe that:

(i) the point $\xi$ in the equation (1) belongs to the interval $(\alpha,\beta)$, whereas in (2) the point $\xi\in[\alpha,\beta]$. In the case $v(t)>0$, for all $t\in[\alpha,\beta]$, the first part of Theorem 1 gives a slight improvement over the textbook formulation of the first mean-value theorem for continuous functions, which requires that $\xi\in[\alpha,\beta]$. This minor improvement is crucial in the proof of Theorem 2;

(ii) the hypothesis that both $u$ and $v$ are continuous is to ensure that we have antiderivatives for $uv$ and $v$. We would like to recall that being Riemann integrable does not entail having antiderivatives, e.g., Heaviside's function, and having antiderivative does not imply being integrable, e.g., Volterra's function, see [1].

[2] The book [2] has a misprint in the last application, it is written $u(\xi-a)$ instead of the correct one $u(\xi)$.

*Proof of Theorem* 1. Let $\varphi$ be an antiderivative of $uv$ and $\psi$ an antiderivative of $v$. From the Cauchy mean-value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$\psi'(\xi)[\varphi(\beta) - \varphi(\alpha)] = \varphi'(\xi)[\psi(\beta) - \psi(\alpha)], \tag{3}$$

from which formula (1) follows.

To prove (1), let us first suppose that $v(t) > 0$, for all $t \in [\alpha, \beta]$. Then (2) follows immediately from (1). If $v(t) \geqslant 0$ we use a limiting argument. For each $n \in \mathbb{N}$, there exists $\xi_n \in (\alpha, \beta)$ such that

$$\int_\alpha^\beta (u \cdot (v + \tfrac{1}{n}))(t)\, dt = u(\xi_n) \int_\alpha^\beta (v(t) + \tfrac{1}{n})\, dt.$$

Since $\{\xi_n\}_{n\in\mathbb{N}} \subset [\alpha, \beta]$, by Bolzano–Weierstraß theorem, there is a subsequence $\{\xi_{n_k}\}_{k\in\mathbb{N}}$ such that $\xi_{n_k} \to \xi \in [\alpha, \beta]$ as $k \to \infty$. The result now follows by a limiting argument taking into account that the function $u$ is continuous and that $u(v + 1/n) \rightrightarrows uv$ and $v + 1/n \rightrightarrows v$ in $[\alpha, \beta]$, allowing to interchange the integral and taking the limit. ∎

## 3. An application of the first mean value theorem for integrals

Let $f \in C^n(I)$, where $I$ is an open interval of $\mathbb{R}$, and take $a, x \in I$, with $a < x$ for simplicity. Suppose, moreover, that $f^{(n+1)}(a)$ exists. We have

$$f^{(n)}(x) = f^{(n)}(a) + (x - a) q_a(f^{(n)}, x), \tag{4}$$

where $q_a$ is the Newton quotient

$$q_a(\varphi, x) := \frac{\varphi(x) - \varphi(a)}{x - a}, \quad x \neq a. \tag{5}$$

Applying the Cauchy formula for repeated integration[3] to equation (4), we get

$$\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(a)\, dt + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} (t-a) q_a(f^{(n)}, t)\, dt. \tag{6}$$

We have

$$\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(a)\, dt = \frac{(x-a)^n}{n!} f^{(n)}(a). \tag{7}$$

Using (1) and integration by parts, understanding $q_a(f^{(n)}, a) := f^{(n+1)}(a)$ to have the function $q_a(f^{(n)}, t)$ continuous in $[a, x]$, yields

$$\begin{aligned} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} (t-a) q_a(f^{(n)}, t)\, dt &= q_a(f^{(n)}, \xi) \int_a^x (t-a) \frac{d}{dt} \left( -\frac{(x-t)^n}{n!} \right) dt \\ &= \frac{(x-a)^{n+1}}{(n+1)!} q_a(f^{(n)}, \xi), \end{aligned} \tag{8}$$

with $a < \xi < x$, since $v(t) = (x-t)^{n-1}(t-a) \neq 0$ in $(a, x)$.

[3] This already appears in the *trente-cinquième leçon* in [2].

Defining $F_n(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$, we get

$$F_n(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \frac{d}{dt}\Big(f^{(n-1)}(t)\Big)\,dt = -\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + F_{n-1}(x). \tag{9}$$

Taking (6)–(8) and iterating (9), we finally obtain

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$
$$\cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} q_a(f^{(n)}, \xi), \tag{10}$$

with $a < \xi < x$. We have thus proved the following result.

Theorem 2. *Let $f \in C^n(I)$ and $a, x \in I$, where $I$ is an open interval of $\mathbb{R}$. Assume, moreover, that $f^{(n+1)}(a)$ exists. Then*

$$f(x) = T_n[f,a](x) + \frac{(x-a)^{n+1}}{(n+1)!} q_a(f^{(n)}, \xi) \qquad (a \lesseqgtr \xi \gtreqless x), \tag{11}$$

*where*

$$T_n[f,a](x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

*is the Taylor polynomial of order $n$ of the function $f$ centered at the point $a$.*

Remark 2. Several remarks are in place.

(i) The remainder appearing in (11) is known as the *Gonçalves remainder*. The assumption of the existence of $f^{(n+1)}(a)$ is due to our choice of proof since it is possible to prove Theorem 2 without it, see [4]. It would be interesting to lift this restriction in our proof since the proof given for Theorem 2 can be modified to provide several new Taylor remainders.

(ii) The Gonçalves remainder appeared for the first time in the textbook [3]. Since this textbook was written in Portuguese, it did not draw attention and this remainder remains, up to this day, almost unknown. It should be pointed out that this remainder has advantages over other well-known remainders since the function needs to be differentiable up to order $n$, instead of $n+1$. If $f$ has derivatives up to order $n+1$, the Lagrange remainder follows from Gonçalves remainder due to Lagrange's mean-value theorem. For a detailed history about Taylor remainders, see [5].

(iii) The original proof of (11) appearing in [3] is somewhat different, following a convoluted bottom-up approach instead of our top-down approach that gives a very streamlined proof. For completeness of presentation we give the original proof that runs as follows:

From (5), for $x \neq a$, we have $f'(x) = f'(a) + (x-a)q_a(f', x)$, from which, integrating from $a$ to $x$ and by the first mean-value theorem for integrals, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} q_a(f', \xi_1),$$

with $\xi_1$ between $a$ and $x$. Changing $f$ to $f'$ in the previous equation, we obtain

$$f'(x) = f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2!} q_a(f'', \xi_2),$$

from which, after integrating from $a$ to $x$, entails

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} q_a(f'', \xi_2).$$

Iterating this idea $n$ times, we arrive at (11).

Other proofs of the Gonçalves remainder, although more elaborate, are known (see [4] and references therein).

(iv) Instead of the Cauchy formula for repeated integration, we could simply apply $n$-iterated integration, which is more appropriate for a classroom exposition. The usage of the Cauchy formula instead of $n$-iterated integration has some advantages, as can be seen from (13)–(15).

The method of proof of Theorem 2 can be easily adapted to obtain other remainders, which are completely new, to the best of our knowledge. The novelties will appear when estimating the integral

$$\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} (t-a) q_a(f^{(n)}, t)\, dt \tag{12}$$

in different ways in comparison with (8). We give several examples, where the details are left to the reader.

(a) Taking $u(t) = (t-a)q_a(f^{(n)}, t)$ in the first mean-value theorem gives

$$G_n^1 = \frac{(x-a)^n}{n!} [f^{(n)}(\xi) - f^{(n)}(a)], \tag{13}$$

thus

$$T_n[f,a](x) + G_n^1 = T_{n-1}[f,a] + \frac{(x-a)^n}{n!} f^{(n)}(\xi),$$

which is simply Taylor's formula of order $n-1$ with the Lagrange remainder.

(b) Taking $u(t) = (t-a)(x-t)^{n-p} q_a(f^{(n)}, t)$ in the first mean-value theorem applied to (12) gives a remainder with a parameter $p$

$$G_n^{2,p} = \frac{(x-\xi)^{n-p}(x-a)^p}{(n-1)!p} [f^{(n)}(\xi) - f^{(n)}(a)], \tag{14}$$

for all $p > 0$, which is akin to the Schlömilch remainder.

(c) Taking $u(t) = [(x-t)^{n-1}(t-a)q_a(f^{(n)}, t)]/\varphi'(t)$ in the first mean-value theorem, with $\varphi' \in C([\min\{a,x\}, \max\{a,x\}])$ and $\varphi' \neq 0$ in $(\min\{a,x\}, \max\{a,x\})$, we obtain

$$G_n^{3,\varphi} = \frac{\varphi(x) - \varphi(a)}{\varphi'(\xi)} \frac{(x-\xi)^{n-1}}{(n-1)!} [f^{(n)}(\xi) - f^{(n)}(a)]. \tag{15}$$

We now provide several special cases:

$$\begin{aligned} G_n^{3,\varphi}\big|_{\varphi(t)=(x-t)^{n+1}} &= n\frac{(x-a)^{n+1}}{(x-\xi)(n+1)!}[f^{(n)}(\xi)-f^{(n)}(a)], \\ G_n^{3,\varphi}\big|_{\varphi(t)=(x-t)^{n}} &= G_n^1, \\ G_n^{3,\varphi}\big|_{\varphi(t)=x-t} &= (x-a)\frac{(x-\xi)^{n-1}}{(n-1)!}[f^{(n)}(\xi)-f^{(n)}(a)]. \end{aligned}$$

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