DISPLAYING PARAMETRIC CURVES WITH VIRTUAL AND PHYSICAL TOOLS

Attila Körei and Szilvia Szilágyi

Abstract. Parametric curves play an important role in mathematics and engineering education. The purpose of this paper is to introduce the hypotrochoid family of curves and to provide ideas and tools for teaching the topic. We give the general parametric equations of hypotrochoids and show that several special curves can be derived from these equations. To display the curves we use the Hypotrochoid Equation Tracer, which is a dynamic application made by Desmos graphic calculator. In addition we have created a LEGO robot called Spikograph, that models a Spirograph and draws different hypotrochoids providing a physical experience for students. Finally, we have compiled a student project to practice and deepen the knowledge about hypotrochoids.

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1. Introduction

Plane curves are commonly used tools in science and engineering. In mathematics they represent function graphs, in physics they describe the path of a moving particle, and they are essential when designing an object or building. In the rectangular coordinate system, the implicit equation $F(x, y) = 0$ for plane curves is often too complicated and it is worth looking for other definitions. For certain curves, a good solution may be to switch to the polar coordinate system and express the equation in the form $r = r(\phi)$. However, if switching to a different coordinate system is not suitable, we can generate and use the parametric equations for the curve by expressing the coordinates of its points (x, y) as functions of a parameter t within a closed interval $t_1 \le t \le t_2$. For example, a circle centered at the origin, with radius R, is represented by the parametric equations

$$(1) \qquad x = x(t) = R\cos t, \qquad y = y(t) = R\sin t, \qquad t \in [0, 2\pi].$$

In equations (1) parameter t denotes the angle subtended by the vector (x, y) to the positive x-axis. As t increases from 0 to 2π, the point moves once around the circle in the counterclockwise direction starting from the point $(R, 0)$. To make sure that this is indeed the equation of the circle, eliminate the parameter t by squaring and summing the equations:

$$x^2 + y^2 = R^2\cos^2 t + R^2\sin^2 t = R^2(\cos^2 t + \sin^2) = R^2.$$

Note that the simplest equation of the above circle is obtained in the polar coordinate system with $r = R$.

It is often the case in engineering applications that we need to determine the trajectory of a moving point, where the motion is defined by physical and geometrical constraints. Many important curves can be derived as the locus of a point on a plane curve that rolls without slipping along another fixed plane curve. The curves obtained in this way are called roulettes. If both curves in question are circles, then the name of the roulette is trochoid (hypotrochoid/epitrochoid if a circle rolls inside/outside a fixed circle). Trochoids also form a very general family of curves, since almost all of the famous plane curves studied in Calculus courses can be obtained as special cases of them. In Section 2 we deal with hypotrochoids, and show some important curves which can be obtained with a special choice of parameters. In Section 3 we present a dynamic geometric software that can be very useful in the classroom to display different types of trochoids. For students studying the subject of parametric curves, it can be very inspiring to use a graphical software to create spectacular animations that show how the trochoid is formed from point to point as described in its definition. It can be even more inspiring for students if they can hold and use a tool to draw these beautiful curves on a piece of paper. This is possible with a simple toy called Spirograph, which has been around for over 50 years. Engineering students may also be challenged to design and build a Spirograph-like device on their own. This task can be accomplished with a LEGO set that includes a motor to be able to model the moving circle. Non-slip rolling can be ensured with gears, similar to the original Spirograph. We chose the LEGO Spike Prime set to build our drawing robot which is introduced in Section 4. Finally, Section 5 describes a possible student project to solve problems related to hypotrochoids using the tools presented.

2. Hypotrochoids

There is no complete consensus among all authors related the terms roulettes and trochoids. Here we use the categories suggested by Lockwood [7] as follows:

- If a curve rolls, without slipping, along another, fixed curve, any point which moves with the rolling curve describes a *roulette*.

- The term *trochoid* is used more particularly for the roulettes traced by points carried by a circle rolling on a fixed circle.

- A trochoid is called *epitrochoid* or *hypotrochoid* according as the circle rolls on the outside or inside of the fixed circle.

Thus hypotrochoids are parameterized curves that are traced out by a point P attached to a circle that is rolling inside of a fixed circle where both circles are on the same plane. Let d be the distance from the center of the rolling circle to the point that is fixed on this circle. If the center of the fixed circle is at the origin and has radius R, the rolling circle has radius r, and the angle between a line through the center of both circles and the x-axis is t; then a parametric representation for

this hypotrochoid is

$$x(t) = (R - r) \cos t + d \cos \left(\frac{R - r}{r} t \right)$$

(2)

$$y(t) = (R - r) \sin t - d \sin \left(\frac{R - r}{r} t \right),$$

where $t \in \mathbb{R}$ [1]. If the distance from the fixed point on the moving circle to the center is equal to the radius of the moving circle (that is, $d = r$), then the resulting curve is called a *hypocycloid*. If the point is within the radius of the inner circle ($d < r$), the curve is called a *curtate hypotrochoid*. In case of $d > r$, it is called a *prolate hypotrochoid*.

Parametric equations (2) contain three parameters that are fixed numbers (R, r, d) and one that is variable (t). Observe that the parametric equations of the curve allow any choice of parameters excepting $r = 0$. If we allow r and d to be negative in (2) we obtain formally the parametric equations of an epitrochoid. In this paper we focus on hypotrochoids, so in the following we assume that the fixed parameters are positive numbers. Although we have so far considered that a circle of smaller radius rolls inside a fixed circle of larger radius, the situation can be reversed. If r is larger than R, then a large circle rotates around a smaller, fixed circle, but of course they still touch each other from the inside. For example see Fig. 1, where we have a cardioid, which is a special hypotrochoid with $d = r = 2R$.
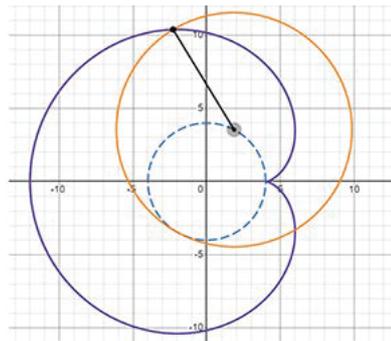

Fig. 1. A hypotrochoid with parameters $d = r = 8$ and $R = 4$ is a cardioid.
The circle with the dashed outline is fixed, the circle with the larger radius
rolls around its circumference.

There are many other famous curves that arise as special cases of hypotrochoid. If $d = 0$ and $R \neq r$, the obtained curve is a circle with radius $R - r$ (replacing $d = 0$ in equation (2) yields equation (1) with $R - r$ instead of R). A hypotrochoid with $R = 2r$ is an ellipse, because in this case equation (2) simplified to

(3) $\qquad x(t) = (r + d) \cos t, \qquad y(t) = (r - d) \sin t,$

which is the parametric equations of the ellipse. In case of hypocycloids (that is $r = d$) the second coordinate is 0 in (3), and the resulting curve is a line segment

called Tusi couple [15]. Tusi couple is a two cusped hypocicloid, where a *cusp* of the hypocycloid is defined as a point where the hypocycloid meets the fixed circle. If R/r is an integer equaling n, we get an n-cusped non-self-intersecting hypocycloid. Specially, a 3-cusped hypocycloid is called a *deltoid* and a 4-cusped hypocycloid is called an *astroid*. If R/r is rational, the curve closes on itself and has R cusps (if R and r are relatively prime numbers). If R/r is irrational, the curve will never return to the initial starting point. Some examples of n-cusped hypocycloids are shown in Fig. 2. For a detailed discussion of the properties of hypocycloids, see the richly illustrated paper by Phan-Yamada and Gwin [9].

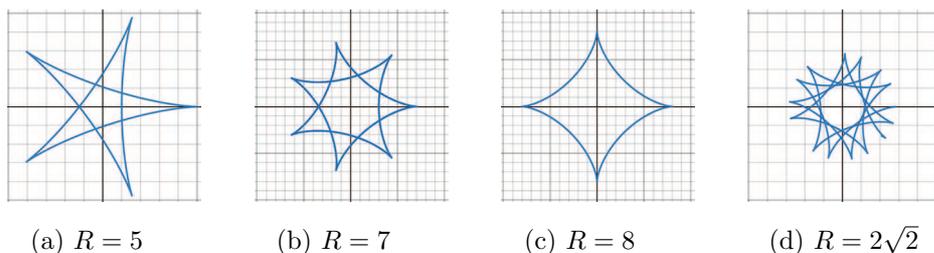(a) $R = 5$ (b) $R = 7$ (c) $R = 8$ (d) $R = 2\sqrt{2}$

Fig. 2. Hypocycloids with different number of cups. The common parameters are $d = r = 2$, the radius of the fixed circle is changing.

Another interesting class of curves is the rhodonea family, which is often called rose or rosette because of its flower-like form. If the rose is given by the polar equation $r = a \sin n\phi$ or $r = a \cos n\phi$, then the hypotrochoid defined by the parameters

$$r = \frac{a(n-1)}{2(n+1)}, \qquad R = \frac{an}{n+1}, \qquad d = \frac{a}{2}$$

is the same curve [17]. Fig. 3 shows an example of the relationship between roses and hypotrochoids.
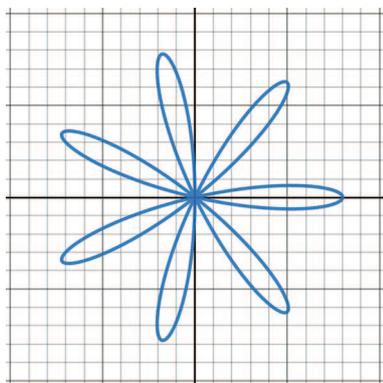

Fig. 3. The rhodonea given by the polar equation $r = 8 \sin 7\phi$ corresponds to the hypotrochoid with $R = 7$, $r = 3$ and $d = 4$.

Trochoids, in particular epi- and hypocycloids, have many applications, mainly in mechanical engineering problems. These curves are fundamental in the design of gears or gear trains and it is well known that the Wankel rotary engine moves in an oval-like epitrochoidal housing. But it is also easy to find applications in recent research. Ravankar et al. have worked on autonomous robot navigation and presented novel smooth hypocycloidal paths for robot motion [10]. Similarly, special hypocycloids are used in studying optimal bus door opening mechanism in [14]. A physical process, the mixing of viscous fluids was investigated in [6], and a device using hypotrochoids was proposed to achieve an effective solution. The topic has also been addressed in abstract mathematical research: in [13], the authors have revealed trochoidal patterns in the geometric characterization of roots of univariant trinomials.

3. Dynamic geometry software

When teaching parametric curves, the use of an appropriately chosen digital graphics tool can be of great help. The range of such tools is very wide, from the graphical display of a pocket calculator to professional mathematical software or CAD systems. In recent years, dynamic geometry softwares (DGSs) have become more advanced and popular and are increasingly used in education. A DGS provides the possibility to interactively create and manipulate geometric constructions. In a dynamic learning environment students can understand better geometrical concepts than if they had dealt with only paper-based tasks. Over the last twenty years, as software has gradually evolved, numerous studies have been published on the effectiveness of DGS at different levels of education, and thorough reviews summarise the results of these studies [2], [4] and [8]. Most of the authors agree that learning using DGS has a high positive effect on students' mathematical abilities, but some limitations should be considered. In their study, Juandi et al. [5] focus on the conditions that make the use of DGS tools more effective and identify some important factors such as the size of the learning group, the student-to-computer ratio and the level of education. One of the conclusions of that paper is that using DGS is more effective in universities and colleges than in junior high schools. In university textbooks, illustrations created with DGS tools and suggested student laboratory projects using graphing calculators are increasingly common [12].

One way to use DGS in the classroom is to enter the equation of the curve, in Cartesian, parametric or polar coordinate form, and the program will plot the curve based on the formula. But a DGS can do much more than that, it can also be used to create moving animations, making the teaching of parametric curves very visual. For the students it is instructive to watch a parametric curve being drawn by the program from point to point as the corresponding parameter values increase. This is particularly true in the case of roulettes, where we trace the path of a point fixed on a moving curve.

In this paper we used Desmos graphing calculator to plot the curves. Desmos is a free application that is particularly suitable for classroom work: it is easy to
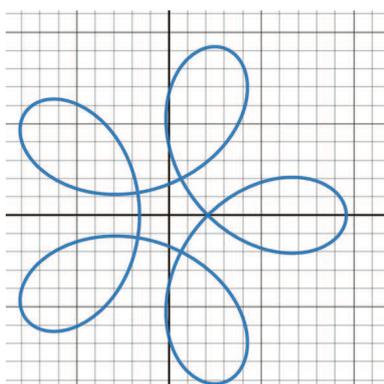
Fig. 4. A hypotrochoid drawn by Desmos graphing calculator.
Its parameters are $R = 7$, $r = 1.4$ and $d = 4$.

use and does not require a computer, as it can be downloaded to a mobile phone. The equation of the curves to be plotted can be given in several ways, including a parametric system of equations. To plot a parametric curve, simply create an ordered pair $(x(t), y(t))$ where one or both coordinates are defined in terms of the parameter t. By default, the domain of the parameters is the interval $[0, 1]$, but it is possible to redefine this set. If the formula contains additional variable values you can change these values with a slider to see the effect of the change on the plotted curve. Fig. 4 presents a general hypotrochoid, drawn by Desmos, where all the three parameters are different numbers. Fig. 1 was also made by Desmos, it is a screenshot from the application Hypotrochoid Equation Tracer (see `https://www.desmos.com/calculator/3plby3pgqv`). This program animates the process of hypotrochoid formation, drawing not only the trajectory of the fixed point but also the moving and fixed circles according to the current parameter value. It is also possible to run the value of one or more other variables in an interval with a given step size, thus performing experiments to discover interesting curves. When you start Hypotrochoid Equation Tracer you see the formation process of a deltoid, which is a 3-cusped hypocycloid with parameters $R = 12$, $R_2 = 4$ and $d = 4$. Note, that the second radius is denoted by R_2 instead of r, because variable r is mainly used in polar equations. Similarly, letter t is also reserved, that is why the running parameter is denoted here by k.

4. Spikograph: A LEGO model for drawing hypotrochoids

A harmonograph is an archaic, scientific device that uses a system of pendulums to create geometric images to illustrate mathematical harmony. A very interesting two-part historical overview of harmonographs was published by Whitaker, the second part of which describes the drawing machines based on circular movements [16]. This article also deals with the Spirograph toy, which can be considered

Fig. 5. An original Spirograph set.

a modern harmonograph. Spirograph was invented by a British engineer Denys Fisher in 1965 and it is suitable for drawing both epitrochoids and hypotrochoids.

The original Spirograph set (Fig. 5) consisted of two differently sized plastic rings (or stators) with gear teeth on both the inside and outside of their circumferences, and several gearwheels (or rotors). The gears ensure that the circles roll on each other without slipping. Once either of the stators were held in place any of the rotors—each having holes for a ballpoint pen—could be spun around the ring to draw geometric shapes. The Spirograph produces graphs of trochoids, but it can be used to draw only curtate ones, because all of the holes for the pen are obviously located inside the moving circle [3].

The Spirograph gave us the idea to create a real model to draw hypotrochoids and make the process of generating these curves even more visual. Our device is called the Spikograph, because it is built from the parts of a LEGO Spike Prime set. More exactly we needed four additional gear pieces from the Spike Prime Expansion set to build a large stator which has teeth inside. As a result, our model is only able to draw hypotrochoids, because the moving circle can only roll inside the other. For hypotrochoids, however, our model can do more than the original Spirograph, because the fixed point whose path we trace can be not only inside the moving circle, but also outside it or on its circumference, so we can plot all the three types of hypotrochoids. Fig. 6 shows a possible configuration of the Spikograph. A motor solves the rolling of the moving circle, to which a pen is attached corresponding to the fixed point. As the circle rolls through the large circle using the gears, the pen draws the trajectory of the fixed point on a sheet of paper.
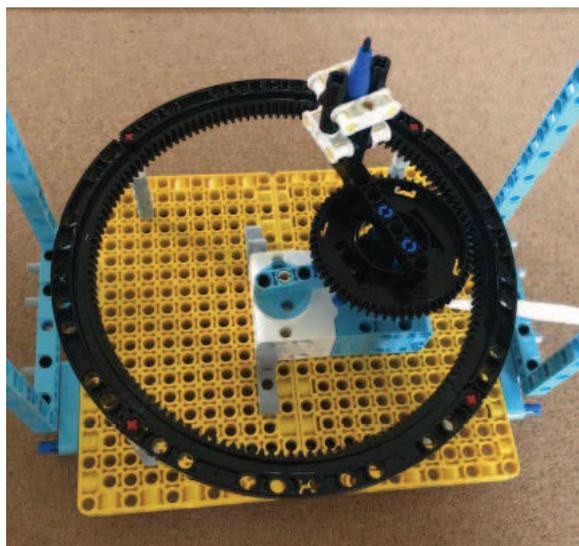
Fig. 6. Bottom view of the the Spikograph. The motor drives the smaller gear, which rolls inside the larger fixed gear. Drawings are made by a pen attached to the moving circle.

Of course, the Spikograph is not capable of drawing any hypotrochoid since the radius of the fixed circle is a given value, 7 cm. We have more freedom in the radius of the moving circle, but the number of possibilities is limited, depending on the size of the gears available in LEGO sets. When we talk about the radius of the gears, we mean the radius of the so-called pitch circle, where the pitch radius represents the distance from the center of each gear such that the meshing gears touch. This pitch radius is proportional to the number of teeth and can be expressed in LEGO units (studs) by the formula

$$r = \frac{\text{number of teeth}}{16},$$

where one stud equals the width of the smallest brick, which is 8 mm wide. The Spike Prime set contains gears in four different size, with 12, 20, 28 and 36 teeth and there is one larger 60-tooth gear in the expansion set. Applying the pitch radius formula we get that the radius of the moving circle can be 6, 10, 14, 18 and 30 mm. In the two sub-figures of Fig. 7, the Spikograph is equipped with different sized moving gears. The variable d in equation (2) is the distance between the fixed point and the center of the rolling circle. In the model we have attached a beam to the moving circle and to one of the holes of this beam the pencil holder is fixed, corresponding to the point that moves with the circle. The holes in the beam are spaced by studs, so the value of d can range from 0 to $(n-1) \times 8$ mm, where n is the number of the beam holes.
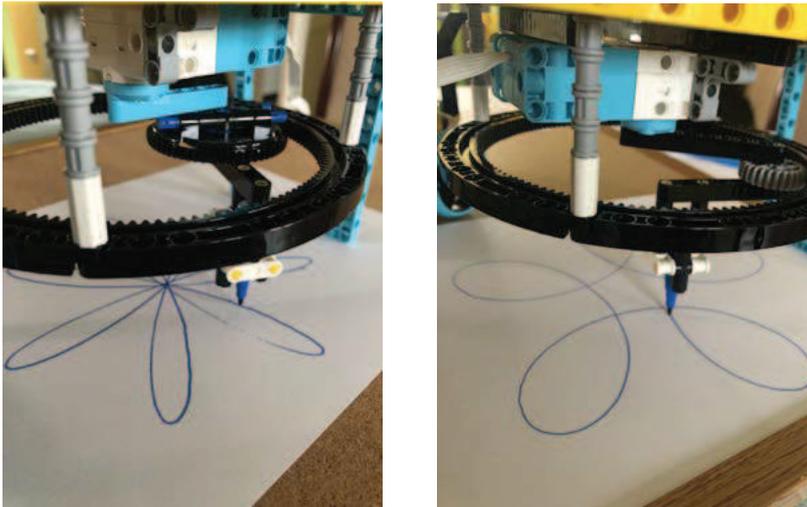
(a) $R = 7$, $r = 3$ and $d = 4$ (b) $R = 7$, $r = 1.4$ and $d = 4$

Fig. 7. The Spikograph is at work, drawing the hypotrochoids shown in Figs. 3 and 4. Only the radius of the inner circle differs, and this change results in significantly different curves.

5. A student project on hypotrochoids

In this chapter, we suggest some exercises that students can practice to better understand hypotrochoids. Some of the problems are of a theoretical nature, while others are more practical, so the Spikograph and the Hypotrochoid Equation Tracer plays an important role in their solution.

PROBLEM 1. Derive the parametric equations of the hypotrochoid given in (2).

Some hints:

- Start the Hypotrochoid Equation Tracer, set the parameters to $R = 12$, $R_2 = 4$, $d = 6$, and see the animation while k is moving in its range. Fig. 8 shows two scenes of this animation, according to $k = 0$ and $k = 1.3$.

- The easiest way to track the position of point P is using the vector equation

$$(4) \qquad \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}.$$

- The center Q of the rolling circle follows a counterclockwise circular route of radius $R - r$, so

$$(5) \qquad \overrightarrow{OQ} = ((R - r)\cos t, (R - r)\sin t).$$

- The point P follows a clockwise circular path of radius d around point Q, thus

$$(6) \qquad \overrightarrow{QP} = (d\cos b, -d\sin b).$$

(a) Initial position   (b) Position at $k = 1.3$

Fig. 8. Two scenes from the formation of the hypotrochoid
with parameters $R = 12$, $r = 4$ and $d = 6$.

- Note that the lengths of $\mathrm{arc}(SV)$ and $\mathrm{arc}(SW)$ are equal since both were created via the same rolling action. Using the relationship between arc length, angle, and radius we have

$$\mathrm{arc}(SV) = Rt = \mathrm{arc}(SW) = r(b + t),$$

which yields

(7) $$b = \frac{R - r}{r}\, t.$$

Replacing (7) into (6) and then adding equations (5) and (6) we get the coordinates of point P which corresponds to the parametric equations of the hypotrochoid.

PROBLEM 2. Drawing curves with the Spikograph.

(a) Build the Spikograph robot (Fig. 9) using the building instructions. You need a LEGO Spike Prime Core set (No. 45678) and four 11x11 LEGO circle gear racks, which are available for example in the LEGO Spike Prime Expansion Set (No. 45680).

(b) Put a pen in the writing head, then make and start a program which moves the motor for one rotation. Give the parametric equations of the drawn curve and check the correctness of the parameters using Hypotrochoid Equation Tracer.

Hint: The parameters of the hypotrochoid can be identified by measuring the characteristics of the built model. The radii of the circles are $R = 7$ and $r = 1.4$, the distance of the tracing point from the center of the moving circle is $d = 4 \cdot 0.8 = 3.2$.
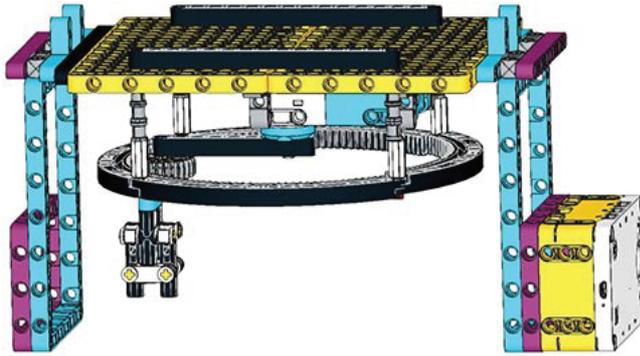
Fig. 9. Spikograph with 28-teeth grey LEGO gear.

The building instructions can be downloaded from

`https://drive.google.com/file/d/1renVjL6DjXJ4-KyamEgpz_HKoyk9VVRI/view?usp=sharing`

(c) Draw the hypotrochoid with parameters $R = 7$, $r = 3$ and $d = 4$ by Hypotrochoid Equation Tracer. Rebuild the Spikograph to be able to display the same curve.

Hint: In this case, the installation of a 60-teeth LEGO gear is required to model the moving circle as we can see in Fig. 7 (a).

(d) Change the moving gear of the Spikograph and the position of the writing head, i.e. modify parameters r and d. Draw at least three new types of hypotrochoids. Give the parametric equations of the drawn curves.

PROBLEM 3. Special hypotrochoids. To solve these problems, trigonometric identities must be used.

(a) Set the parameters of the Hypotrochoid Equation Tracer to $R = 1$, $R_2 = 2$ and $d = 2$. Identify the obtained curve. Verify that the hypotrochoid with parameters $d = r = 2R$ is a cardioid.

(b) Set the parameters of the Hypotrochoid Equation Tracer to $R = 4$, $R_2 = 2$ and $d = 2$. Identify the obtained curve. Verify that the hypotrochoid with parameters $d = r = \frac{R}{4}$ is an astroid.

(c) Set the parameters of the Hypotrochoid Equation Tracer to $R = 7$, $R_2 = 3$ and $d = 4$. The obtained curve is a rose with 7 petals (see Fig. 3). Show that the polar coordinate equation of this curve is $r = 8 \sin 7\phi$.

6. Conclusion

The study of different curves and their properties is very important in university engineering courses. This is a topic that most students find difficult to learn, so teachers should try to make it as clear and understandable as possible [11]. The

curves given by parametric equations can be made very visual using dynamic geometry software. These programs can produce spectacular animations, bringing the definitions of famous curves to life before your eyes.

Trochoids form a family of curves in which, with the right choice of variables, you can obtain a number of special curves. Among these curves, there are many eye-catching ones, and some of them can be drawn by the Spirograph toy, that exploits the mechanism of trochoids. In this article we presented a LEGO model that acts as a Spirograph to draw different hypotrochoids. With this model, the teaching of parametric curves can be made very visual, as students can see the point-by-point formation of hypotrochoids in their physical reality. By replacing the moving gear or relocating the fixed point, the machine draws a different curve. It is very instructive to check the result using a dynamic geometry software. Just measure the parameters of the current layout, enter them into the hypotrochoid formula and the software will show the same curve as the robot drew. By experimenting with the robot and the software together, students can gain knowledge that makes parametric curves more natural and understandable to them.

The idea and the model of the Spikograph can be further developed to draw even more types of trochoid. We used gears from the Spike Prime sets, but of course other LEGO elements can be used or the parts needed can be made by a 3D printer. In our model, the position of the pen is determined by the holes in the fixed beam, but it is also possible to fix the drawing head steplessly, allowing the parameter d to be chosen as desired. We plan to implement these developments with the involvement of engineering students, who will hopefully be inspired by the design and construction project.

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A. K.: University of Miskolc, Department of Applied Mathematics, Miskolc, Egyetemváros, H-3515, Hungary

E-mail: matka@uni-miskolc.hu

Sz. Sz.: University of Miskolc, Department of Analysis, Miskolc, Egyetemváros, H-3515, Hungary

E-mail: szilvia.szilagyi@uni-miskolc.hu