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STUDENTS' SUCCESS IN SOLVING MATHEMATICAL PROBLEMS DEPENDING ON DIFFERENT REPRESENTATIONS

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Abstract. In order to solve mathematical problems, students often need to make the transition from one representation of mathematical concepts in those problem formulations to another representation. In this paper we explore the influence of the representations used in the problem formulation (problems with the same mathematical background with regards to solving easier or more complex equations and determining the unknown value of the proportion) on students' success in solving those problems. On a representative sample of 584 8th grade students, we tested whether there were differences in students' success in solving mathematical problems while using symbolic, graphic, or verbal representations in the formulations of problems belonging to different level of complexity. Results of this research indicate that there is significant impact of the representations of mathematical concepts used in problem formulation on students' success. Furthermore, the level of impact of using different representations in problem formulations depends on the level of the problem complexity when it comes to students' success in solving those problems.

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Key words and phrases: Mathematical problem posing; representations; transition between representations.

1. Introduction

Problem solving, problem posing, and real-world mathematics are being impelled to inevitably become an integral part of compulsory mathematical education due to the needs of modern society. For instance, in order to solve a mathematical problem, student need to perform the transition from representation of the mathematical concept used in the problem formulation to another representation. Therefore, students should be trained to perform this kind of transition. As a result, numerous studies have been investigating the problem of transition between the different representations of the same problem [15, 19, 25]. If representations of mathematical objects are planned to be used in the mathematics class, then teachers have to support students to acquire the skill of interpreting different representations, through providing "effective transitional experiences" [7]. Some other authors have already noticed that all types of transitions are not equally included in school practice [21], whereas other studies claim that some transitions between the representations are more difficult than others [6, 9, 21]. Traditionally, in mathematics school practice in Serbia, two transitions have been recorded: verbal → symbolic and graph → symbolic; symbolic → graph transition is much rarer, whereas the transition symbolic → verbal is almost completely avoided. Bearing all that in mind, in this research we focused on transitions of representations in problem-solving process. In addition, we wanted to examine the impact the mathematical complexity of given problem had on the success in solving mathematical problems by students in verbal → symbolic and graph → symbolic transitions.

Theoretical background

Representations of mathematical objects/problems and the use of those representations by students have been the topic of many theoretical and practice researchers. Understanding representations of mathematical objects should be considered as the key factor for students to grasp and express mathematical ideas. In [8] the author provided a theoretical framework presenting three forms (3 modes of representations) of students' interaction with the environment. Those are:

- Enactive representation (action, concrete) consisting of a series of concrete actions, with manipulative teaching aids which help students form awareness of a concept and the properties of that concept;
- Iconic representation implies a visual representation (graph, scheme, image of a geometric figure) of mathematical objects, their properties, as well as interrelationships, regardless of the physical treatment of objects;
- Symbolic representation including logically connected symbols, words and numbers.

This "rule of three" was later expanded by "rule of four" in [20]:

- Graphic representations,
- Numeric representations,
- Symbolic representations, and
- Verbal representations.

Having been influenced by the studies [8] and [25], the author in [27], proposed somewhat different representational system consisting of the following five categories:

- Symbolic representation – representations used in mathematical notation, such as numbers, letters, and symbols,
- Linguistic representation – representations using everyday languages, such as Serbian or English,
- Illustrative representation – representations using illustrations, figures, graphs, and so on,
- Manipulative representation – representations such as teaching aids that add the dynamic operation of objects that have been artificially fabricated or modeled,
- Realistic representation – representations based on actual states and objects.

Due to the fact that while solving mathematical problems students often need to make transitions from one representation of the problem to another, some researchers claim that multiple representation students' skills are the key to successful mathematical problem-solving [1, 4, 22]. Multiple representation skills present the ability of students to encode the given information based on the domain of the given representation, to choose representations appropriate for the given problem, and to identify the relationship between the two representations [1]. In [2] the author claims that since solving problems requires the application of the mathematical knowledge, students need to analyze and find specific patterns and connections within the problem, i.e., students need to convert concrete application problems into mathematical problems.

Students have to possess multiple representation skills in order to understand the same problem posed in various forms. In [15] authors state that every problem can be solved by using various types of representations implying that there is a close relationship between problems and representations. Some studies [10, 11, 14, 26] claim that the need to use different representations in mathematics learning is important, since the use of various representations can help students solve mathematical problems [3, 12, 22, 28, 31, 33]. The author, in [5], argues that different representations of mathematical concepts are significant for learning, since students' ability to translate one representation into another can have impact on their mathematics learning achievement. The multiple representation ability of students employed in solving mathematical problems is closely related to understanding a particular concept as noted in [13]. The study [30] argues that if students can create connections between multiple representations of the same concept or problem, they show conceptual understanding which can later provide a strong foundation for effective solving mathematical problems as concluded in [23]. The findings of the study [23] imply that students are most successful in solving equations represented symbolically, less successful in finding the unknown lengths of line segments (solving problems posed with graphic representations), and the least successful in solving verbal problems. The results in [23] also show that sometimes students can manipulate symbols while simultaneously show a lack of conceptual understanding. The studies [19] and [29] conclude that problems in the form of symbolic representations are easily solved by students whereas, on the other hand, they have difficulties using verbal or graphic representations.

The problem with the same mathematical background could be put in the different context. Solving verbally represented problem placed in the realistic context gives students the opportunity to organize and apply the knowledge they have acquired in the mathematics class as presented in [17]. In [35] the author claims that the context of a problem could increase significance of the given problem. He claims that introducing the problem placed in the given context provides students an opportunity to show his/her abilities. Three main difficulties that students encounter [18] while solving problems placed in context are: not understanding the problem, the lack of mathematical skills and mathematical knowledge deficiency, and the absence of the ability to make transition from real life context to mathematical context.

Word problems can be defined [36] as: “verbal descriptions of problem situations where one or more questions are raised, the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement". Students often fail in bridging the gap between their school mathematical knowledge and representations of real-life situations in word problems as recorded in [5]. Numerous studies on student behavior while solving word problems report various student blockages in the problem-solving process [16, 34, 37].

Recently, in [24], authors investigated effects of representational context of mathematical task (symbolic, mathematical verbal, realistic verbal or visual) on student achievement of a learning outcome and relationship between learning outcomes and assessment tasks. One motive for this study was reformed mathematics curriculum for elementary schools in Serbia, which defines outcomes as a basis for planning instructions. The study focused on particular outcome defined for 8th grade student (fourteen-year-old) and authors instead of the traditional content-based evaluation of students' work, evaluated the main steps that students made of the mathematical modeling process. Findings of the study evidence that representational context of mathematical task largely affect student's achievement, and surprise was that students had the most problem with visual representation (more than with realistic verbal representation). Conclusion was that there is need for more research on this subject and that theory of multiple representations must be seriously considered in development of assessment strategies and that assessment criteria should be aligned with the mathematical modeling cycle.

Standards represent essential knowledge, skills, and abilities that students should possess at the end of a certain education cycle [38]. Standards shape the most important requirements of school learning and teaching and express them as outcomes visible in students' behavior and reasoning. Through standards, educational goals and objectives are translated into much more specific language that describes student achievements, acquired knowledge, skills, and abilities. For the purpose of this research, we analyzed formulations of educational standards of students' achievements given by the Institute for the Advancement of Education (on behalf of Ministry of Education of the Republic of Serbia) to see in what extent they contain representations and/or transition between representations. The mentioned standards are divided into five categories: Numbers and operations with the numbers, Algebra and functions, Geometry, Measurements, and Data analysis. There are also three levels of students' achievements: elementary, medium, and advanced. In the majority of standards, it is not explicit which of the representations of mathematical concepts could be used for problem formulation, but in some standards the representation of the mathematical concept is mentioned. Those educational standards are listed below.

For the elementary level, student should be able to use graphic representation as help while: comparing different numbers or adding them (MA.1.1.3., MA1.1.4.); he/she uses integers and simple expressions helping himself/herself with the image (MA.1.1.6.); determining the value of functions given with table or with formula (M.A.1.2.4.). When it comes to the context of a problem, student should know and understand concepts such as: line, line segment, plane and angle, triangle, quadrilateral, square and rectangle, circle, circular line, prism, cube, conus, cylinder, sphere; he/she can recognize their models and can draw them (M.A.1.3.1., M.A.1.3.2., M.A.1.3.3., M.A.1.3.4., M.A.1.3.5.). As far as the Data analysis standard is concerned, student should be able: to read and understand data from charts, diagrams or from the tables; to determine minimum or maximum of the dependent variables (M.A.1.5.2.); to represent data from a table with a graph and vice versa (M.A.1.5.4.).

For the medium level, student should be able to: use numbers and numerical expressions in simple real-life situations (M.A.2.1.4.); notice connection among the variables, knows the function $y = ax$ and graphically interpret properties of the function (M.A.2.2.4.); use equations in simple textual problems (M.A.2.2.4.); read data from diagrams and tables and analyze data by criteria (M.A.2.5.2); analyze collected data and express them with diagrams or graphically (M.A.2.5.3).

For the advanced level, student should be able to: use numbers and numerical expressions in real-life situations (M.A.3.1.3); distinguish direct and inverse proportion and express appropriate connections, understand linear function and graphically interpret its properties (M.A.3.2.4); use equations, inequalities and systems of equations and solve more complex textual assignments (M.A.3.2.5.); interpret diagrams and tables (M.A.3.5.2); collect and process data using a diagram or a table and draw a graph representing the dependence of variables (M.A.3.5.3.).

Given that these educational standards were created based on a methodology involving real data collected on a significant number of students in Serbia, they should be able to confirm the already observed differences in transitions from one representation to another. By analyzing these educational standards of students' achievements, we can see that using graphic representation by a student should not pose a problem for all three levels of students knowledge. Of course, complexity of the demands in problems considerably varies within the problems for the different levels of students' achievements. On the other hand, when it comes to verbal representations in the formulations of problems placed in real-life situations context, those problems are recognized as medium and advanced level problems.

Purpose of the study

Many authors have discussed how various formulation (symbolic, table, graph, and description) of mathematical problems affect the students' success in solving those problems [19, 23, 24, 29]. Their studies suggest the significant impact that the type of representation has on students' success in solving problems. In the present study we conducted a similar, but more thorough analysis considering the complexity of the problem. The main goal of our analysis was to determine whether there are differences in the degree of the impact different formulations of the problems have, i.e., the impact of different representations used for problem posing on the success in solving the mathematical problems of different levels of complexity. Namely, we analyzed the students' success in solving one quite simple mathematical problem (Appendix I, Problem A) and two more complex problems (Appendix I, Problems B and C), all three of them posed using different representations: symbolic, graphic, and verbal (with real-life situation context). We consider two types of transition in the process of solving the problems mentioned above:

- from graphic to symbolic,
- from verbal to symbolic.

The difference between two more complex problems was specified: for solving Problem B, the transition from the verbal or graphic representation to symbolic representation (algebraic equation) is obligatory (it cannot be bypassed), whereas solving the Problem C is a bit different. In case of Problem C one can find solution by some naive technique, i.e., students can find solution by trying cases because there are not so many possibilities, but for that strategy they do not provide any explanation and they do not prove that solution is unique. Therefore, this kind of solutions are not mathematically complete. To give complete solution for problem C students must solve more complex linear equation than for problem B. So, if the complexity of the problems was evaluated only by the complexity of the mathematical formulas used in the solving process, we would conclude that the complexity of aforementioned three problems was in increasing order from Problem A to Problem C.

Methodology

Hypotheses for our research question are:

(H1) There is a significant impact of problem formulation (more precisely, the impact of representations of mathematical concepts used in problems formulation) in all three considered problems.

(H2) The level of impact of different problem formulations on the students' success depends on the level of the problem complexity.

Participants and procedure

Data were collected from 584 8th grade students (14.5–15.5 years old, the senior class of the primary school in Serbia (compulsory education) in 12 Serbian primary schools from 8 cities during the spring 2019. Let us emphasize that:

- cities and schools were chosen randomly, but in such a way that each and every part of the country was represented,
- the observed sample represents approximately 0.8% of the selected student population in the country.
- The sample was selected to demographically well represent the whole country.

Data were collected in schools, during the regular mathematics classes in the presence of mathematics teachers. Participants were briefed on the purpose of the study and informed that their participation was voluntary and anonymous. Students filled in questionnaires consisting of some basic information (gender, average mark in Mathematics during the current school year), three mathematical problems and two questions about their opinions on the given problems. The questionnaires were administered in Serbian and it took one school class for the participants to complete them.

Design

This study included 3 mathematical problems of 3 different levels of complexity, and for each of them we designed 3 different formulations, using different representations in the problem formulation:

- symbolic representation (algebraic expression or equation),
- graphic representation (geometric figure, diagram),
- verbal representation (situational description, word mathematical problem placed in some realistic context).

By using that method, we formulated 9 problems (Appendix I). Consequently, we formed 6 different tests, each of them consisting of 3 mathematical problems, formulated in three representations (each representation was used only once, for each different problem). In order to be more concise, we introduced labels for each of 9 problems and the same labels were used for the groups of students that were solving them (Table 1).

Table 1. Labels of 9 problems in the tests

	Problem A	Problem B	Problem C
Symbolic representation	AS	BS	CS
Graphic representation	AG	BG	CG
Verbal represntation	AV	BV	CV

These 9 problems were distributed in 6 different tests for students, as it is shown in Table 2.

Table 2. Distribution of the problems in the tests

Test	1.	2.	3.	4.	5.	6.
Problems	AS,BG,CV	AS,BV,CG	AG,BS,CV	AG,BV,CS	AV,BS,CG	AV,BG,CS

This procedure ensured that each student had to solve one problem of each level of complexity (according to the complexity of corresponding mathematical formulas) and one problem of each type of representation (used in problem formulation).

Mathematical knowledge necessary for solving the given problems included:

- calculating expression $4.68 : 0.12$ for Problem A,
- solving equation $35 : 2 = x : 0.8$ for Problem B, and
- solving equation $4(24 - x) = 31 + x$ for Problem C.

Let us remark that Problem C given with graphic representation or with verbal representation (placed in the realistic context) could be solved by trying cases, i.e., without solving the mentioned equation. For each of the problems, authors divided the students' answers into four categories:

- correct answer (2 points),
- partially correct answer, which usually means that the student made minor calculation mistakes or that the solution was not complete, but everything that the student did was correct and led to solution (1 point),
- no answer (0 points), and
- incorrect answer (−2 points).

Data analysis

Collected data were analyzed using the Statistical Package for the Social Sciences (SPSS 20.0 for Windows). To determine whether there is a significant relation between categorical variables Chi-square test of independence was used, and when necessary Yates' correction for continuity and Fisher's exact test. Mann-Whitney U Test and Kruskal Wallis Test were used for comparison between distributions of values of the same continuous variable in different groups.

Results

Since the students' participation was voluntary and anonymous, in order to be convinced of the reliability of conclusions made about our hypotheses, first we checked the consistency between students' results in the test and students' success in school (average mark in Mathematics at the time of test solving). As it was expected, the results showed that there was significant positive correlation (Spearman's rho = 0.451, Sig. < 0.05) between students' results on the test (number of points students scored on the test) and their school marks. We also inspected whether there were significant differences in mathematics knowledge of the students belonging to different groups. Kruskal-Wallis tests, conducted on each problem, showed that there were no statistically significant differences in the students' Mathematics marks (students' mathematics knowledge) regarding the representations used in the formulation of the problem the students were solving (Table 3).

Table 3. Statistical results of the Kruskal-Wallis Test for the students' marks in Mathematics among the different groups of students

Problem	Kruskal-Wallis Test			Descriptive statistics					
	Value	df	Sig.	Median			N		
				S	G	V	S	G	V
A	1.067	2	0.587	3	3	3	194	197	183
B	0.650	2	0.723	3	3	3.415	193	187	194
C	3.527	2	0.171	3.585	3	3	188	189	197

The statistics of students' marks are given in the Table 3, for all the three problems formulated with symbolic (labelled by S), graphic (labelled by G), and verbal representations in the realistic context (labelled by V). There was no statistically significant difference in mathematical knowledge (based on students' marks) of the students who had solved Problem A posed using symbolic (S), graphical (G) or verbal (V) representation ($p = 0.587$ for the groups AS, AG and AV). The same conclusion stands for Problem B ($p = 0.723$ for the groups BS, BG and BV) and for the problem C ($p = 0.171$ for the groups CS, CG and CV).

As far as the problems in symbolic representation (algebraic expression or equation) are concerned, the students' results are in decreasing order from Problem A to Problem C, which corresponds to the increase of mathematics complexity observed from Problem A to Problem C (Table 4, columns Symbolic). However, when the problems in the form of graphic representation or with verbal representation (problems in the realistic context) are in question, the situation is slightly different. While analyzing the students' success, we consider two forms of qualifying results related to Problem C:

1) We accepted as correct students' answers obtained by guessing, without using the adequate procedure (students usually proposed only final solution with no reasoning method used in reaching that solution, nor possible listing or checking whether the given numbers satisfy the conditions of the problem).
2) The students' answers obtained by guessing were categorized as no answer. Note that in Tables 4 and 5 students' results given in row C* refer to the problem C posed using graphic and verbal representation, but in cases when students' answers were a product of guessing (solving problem without transitions to mathematical formula), they were categorized as no answer.

In both considered cases, Problem A is the best solved one. In the case 1) the students' results proved more successful for Problem C than for Problem B (Table 4, columns Graphic and Verbal, rows A, B, and C), while in the case 2) the students' results for Problem C proved to be the least successful ones (Table 4, columns Graphic and Verbal, rows A, B, and C*). This shows that most students avoided the transition into the symbolic representation whenever it was possible. Namely, in the case of graphic representation in formulation of Problem C, 66.1% of students attempted to solve it by trying, and only 8.5% by transition into the corresponding equation (symbolic representation). When it comes to the situational description formulation (verbal representation) of Problem C, 53.3% of students tried to solve the problem by trying, and 16.8% by transition into the corresponding equation (symbolic representation).

The conducted χ^2 tests of independence, for all three problems, showed that students' success in solving problems depended on the type of representations used in problem formulation (Table 5 and Figure 1); namely, there is a statistically significant difference in distribution of students' answers between different groups of students (one group was formed from students that had solved the same problem in the same formulation). This confirms hypothesis H1.

Table 4. Students' success in solving assigned problems

Problem	Answer	Type of representation		
		Symbolic	Graphic	Verbal
A	Correct	83.3%	59.3%	84%
	Partially correct	2%	21.1%	3.2%
	No answer	0.5%	7.5%	6.4%
	Incorrect	14.1%	12.1%	6.4%
B	Correct	71.0%	41.2%	20.8%
	Partially correct	12.4%	6.2%	22.3%
	No answer	8.3%	24.7%	32.0%
	Incorrect	8.3%	27.8%	24.9%
C	Correct	66.0%	55.3%	47.5%
	Partially correct	7.7%	0%	4.0%
	No answer	14.9%	25.3%	30.5%
	Incorrect	11.3%	19.5%	18.0%
C*	Correct		4.2%	7.5%
	Partially correct		0%	3.5%
	No answer		91.1%	83.5%
	Incorrect		4.7%	5.5%

Table 5. χ^2 test for the students' success in solving the problems posed by using different representations

Problem	Chi-Square test of independence			Cramer's V	
	Value	Df	Sig.	Value	Sig.
A	78.262	6	.000	0.259	.000
B	122.508	6	.000	0.324	.000
C	35.464	6	.000	0.174	.000
C*	311.460	6	.000	0.516	.000

In order to evaluate the impact of representation used in the problem formulation onto students' success in solving considered mathematical problems, we should look at Cramer's V value. It is obvious that the impact is not the same for each of the three problems. It is indicative that the impact was the lowest for Problem

C when it was allowed to avoid transition to a mathematical formula. In contrast, when we insisted on students' making the transition from one representation to another in order to solve the problem, we recorded a significant increase of the impact as a consequence of the increase of complexity, which confirms hypothesis H2.

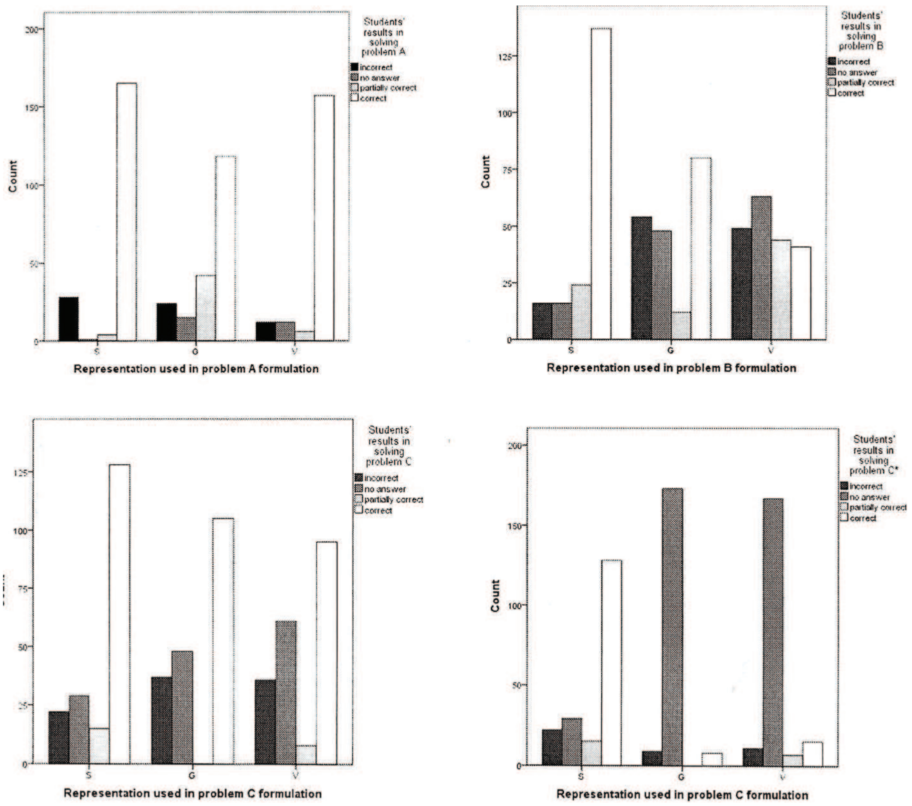

Fig. 1. Students' results for each problem depending on the representation used in formulation

Additionally, points scored by students and analyzed by using the Mann-Whitney Test imply that:

- there is a significant difference in the students' success between students from groups AS and AG ($Z = -3.104$, $p = 0.002 < 0.05$), as well as between students from groups AG and AV ($Z = -2.404$, $p = 0.016 < 0.05$) in solving Problem A;
- there is a significant difference in the students' success between students from groups BS and BG ($Z = -3.530$, $p < 0.0005$), as well as between groups BS and BV ($Z = -2.172$, $p = 0.03 < 0.05$) in solving Problem B;
- there is a significant difference in the students' success between students from groups CS and CG ($Z = -2.608$, $p = 0.009 < 0.05$), as well as between groups CS and CV ($Z = -4.020$, $p < 0.0005$) in solving Problem C.

The presented results were compared with students' subjective evaluation of the complexity of the problems in the test. In the questionnaire students had the opportunity to give their feedback on the difficulty of problems. The results of their feedback (Table 6 and Figure 2 (on the next page)) illustrate students' attitude that Problem A was perceived as the easiest one in each representation, and that for each problem the easiest version was when the problem was formulated by using symbolic representation. It is also important to note that the difference between the students' opinion about the problem difficulty (using different representations in the problem formulation) is more noticeable for Problems B and C (problems of medium and advanced levels of difficulty).

Table 6. Students' attitudes toward problems' difficulty, on scale from 1 to 10

Problem	Kruskal-Wallis Test			Descriptives					
	Value	Df	Sig.	Median			N		
				S	G	V	S	G	V
A	10.807	2	0.005	1	2	1	195	191	182
B	86.272	2	0.000	2	6	6	189	186	187
C	50.167	2	0.000	3	5	7	184	184	188

Discussion

The results showing better students' achievement in solving mathematical problems when the problem was posed by using symbolic representations unlike in the case when the problem was posed by using verbal or graph representation indicate that students may have instrumental, but not relational knowledge [32]. Namely, the students have learned and practiced the proper techniques, but they successfully use them in a satisfactory percentage only when the instruction in the problem formulation is direct (when symbolic representation is used in the problem formulation). When solving the same problem posed by using the other two types of representation (graphic and verbal), students insufficiently recognize which mathematical tools should be used for solving problems. We may even conclude that students in those situations perform regression in quality of mathematical thinking, while applying methods for solving problems typical for the first cycle of their mathematics education (up to 11 years of age).

Besides the evident influence of the representations used in the problem posing and problems of transition from one to another type of representation onto the students' success in solving mathematical problems which has already been examined in numerous studies as in [19, 24, 25], the presented results indicate that within the framework of such analysis, the complexity of problems themselves must be considered. Namely, as the problems are more difficult, the difference in students' success in solving the problems posed using different representation in the problem

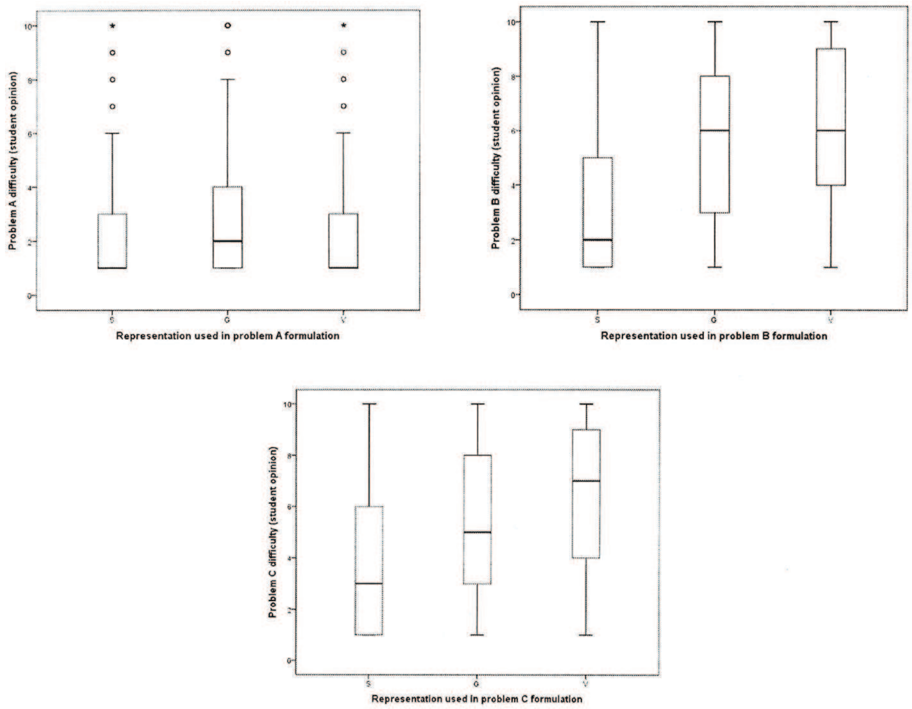

Fig. 2. Distribution of the students' attitudes toward problems' difficulty for all three problems depending on the representations used in problem formulation

formulation is more prominent, whereas students show best results in solving the problems posed by using the symbolic representation.

Thorough analysis of the students' success in individual solving each of the three problems suggests the following conclusions.

Problem A. The analysis of the correct answers to the simplest problem based on students' responses to both symbolically and verbally represented formulations given in realistic context implies practically same results. Namely, it can be noticed that the students have no problem recognizing the most typical application of division of two decimal numbers (seen many times in textbooks and met in real life by the students). However, even in the case of the simplest problem, the transition to the graphically represented geometry problem decreases their success rate (40.7% of students failed to calculate the length of one side of the rectangle when its area and the length of the second side were given). Nevertheless, taking into consideration the number of students who gave no answers to Problem A, it seems obvious that symbolic representation in the formulation stands out. Namely, only 0.5% of students (only 1 of them) from the group AS did not try to solve the problem, whereas the percentage of students who did not try to solve Problem A in the group AG is 7.5% (15 students) and 6.4% (12 students) in the group AV.

Problem B. As far as Problem B (mid-level difficulty problem) is concerned, students' answers for different representations used in formulation vary much more than it is the case with the simpler Problem A. A clearly decreasing order of number of correct answers from symbolically represented problems (71.0%, 137 students) in comparison to both graphically represented problems (41.2%, 80 students) and verbally represented problems given in, situational context (20.8%, 41 students) was observed. On the other hand, the number of students who did not try to solve Problem B is in increasing order from group BS (8.3%, 16 students) over group BG (24.7%, 48 students) to group BV (32.0%, 63 students).

Problem C. Only the students' responses based on the transition into symbolic representation in formulation (C^*), i.e., only students' results including complete understanding of the problem, proper reasoning and making adequate calculations were considered. The review of the number of students who correctly solved this most difficult problem and the number of those who did not try to solve it implies that the students' success in solving the problem given with the symbolic representation in formulation differs considerably from the success in solving the problem given in the other two types of representations. Namely, 66,0% of students (128 students) from the group CS correctly solved Problem C while 14,9% of students (29 students) did not try to solve it. In the other two types of the formulation, the percentages are much more different: 4.2% of students (8 students) from the group CG and 7.5% of students (15 students) from the group CV correctly solved the problem, whereas 91.1% of students from the group CG (173 students) and 83.5% of students (167 students) from the group CV did not try to solve Problem C.

In case of conducting similar research, instrument of research could be improved. Namely, results about problem C indicate that authors should avoid problems that can be solve by trying cases. Also, problems with graphic representation should have minimal verbal description (like in considered problems A and B).

Conclusion

Among other things, the goals of mathematics education in the Republic of Serbia (as well as worldwide) include the acquisition of language and mathematical literacy of students, enabling students to solve problems in unknown situations, enabling students to apply acquired mathematical knowledge in solving various life problems, etc. Of course, all this would not be possible if students do not master the understanding of the problem posed in different ways, using different representations for given concepts, objects, variables and with transitions from one type of representation to another in the process of solving mathematical problems. Having in mind the experience of a significant number of colleagues and the habits of mathematics teachers (in terms of choosing problems that teachers themselves solve in mathematics class, problems that students solve in the class, and problems that students should solve for their homework), we came up with the idea to compare students' success in solving problems posed in different ways, by using different representations in problems formulations.

Therefore, previously mentioned research results of this study can lead to certain conclusions. Firstly, if the opportunity allows, regardless of their maturity (assuming students' approach to each problem is thorough, with the appropriate level of reasoning) students are still inclined to solve the problem using the principle of trial and error, if the problem allows such an approach, without transition from one type of representation to another. Secondly, when it comes to the impact of the problem formulation, i.e., the impact of representations of the mathematical concepts used in problems formulation in all three considered problems, results showed that there was a significant difference in students' success while solving the problems posed using different representations. More precisely, the students were proved to achieve different results when solving the problems posed by using symbolic, graphical, or verbal representations, no matter how easy or difficult the process of solution went (from verbal and graphic to symbolic representations). Moreover, this conclusion can also be applied to the problems with different level of complexity – we made the same conclusion for the students' results in solving all three problems from the study (which confirms the hyphothesis H1).

Based on statistical results in the analysis of all three problems, it can be concluded that the largest differences in students' success were observed in solving problems posed using symbolic and graphic representations. Differences in the students' success in solving problems posed by using symbolic and verbal representations are noticeable in more difficult problems (Problems B and C), while in the simplest problem the difference in the students' success is significant when solving the problem posed with graphic and verbal representations. Thirdly, when the impact of the complexity of the problem to the differences in students' achievements in solving the problem posed in different ways is concerned, it can be concluded that the more difficult the problems are, the greater the differences in students' success in solving these problems are (which confirms the hyphothesis H2).

These results could be interpreted in the context of the types of the problems that the students are inclined to. The role of the teacher is also very important for achieving students' success in solving problems posed in different ways. If the mathematics teacher when in class exclusively asks students to solve problems posed in one way (using one type of representations), or if the representations of concepts in the problem formulation in most cases are the same, it can be expected that students will achieve some success in solving problems posed in the same manner. However, if the problem is posed in the way being unknown or insufficiently known to the students, we cannot have high expectations from the students in that situation. Some strategies the teacher uses while solving problems can also be transferred from the teacher to the student, e.g. if the teacher nurtures an analytical approach to solving problems and rarely chooses graphical approach to the problem, or if he/she does not use the transition from one type of representation to another to a large extent in the process of solving mathematical problems, students could be expected to act in the similar manner. That is why it is important for the teacher to give students the opportunity to solve a certain unknown problem, using experiences related to a certain type of representation of mathematical concepts.

It is also most desirable to develop students' reflective and critical thinking about different ways of solving problems, to use the best approach for them, or the strategy for solving particular mathematical problem. For that reason, in classroom learning, the teachers need to interpret the meaning of the transition of the word problems and the problems posed using images into other (primarily symbolic) representations so that students can easily understand the purpose of the problem and can solve the problem properly. Teaching the use of various representations in mathematics learning should be delivered by teachers in elementary school and further in order to provide students with extensive knowledge about various representations of problems and with advanced multiple representation skills for solving.

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Appendix

Problem AS. Calculate $4.68 : 0.12$.

Problem AG. Based on the data in Figure 3, calculate the length of the segment AB.

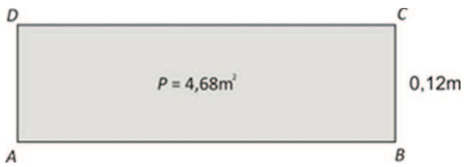

Fig. 3

Problem AV. The price of one chewing gum is 12 cents. How much of this chewing gum can Ljubica buy if she has 4.68 euros? One euro has 100 cents.

Problem BS. If $35 : 2 = x : 0.8$, determine the number x.

Problem BG. Based on the elements given in Fig. 4, determine the length of the segment CD.

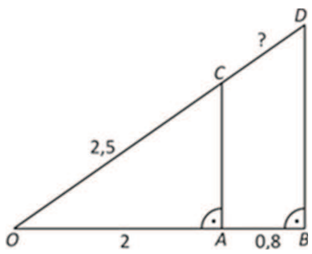

Fig. 4

Problem BV. Marko set off from his house to a nearby park 2.8 km away. He is moving at the same speed all the time and after 35 minutes he has covered 2 km. How much time does he need to arrive to the park?

Problem CS. Solve the equation: $4(24 - x) = 31 + x$.

Problem CG. Fig. 5 shows sets A and B. Which two numbers from set A should be transferred to set B so that the sum of numbers from set A is four times smaller than the sum of numbers from set B?

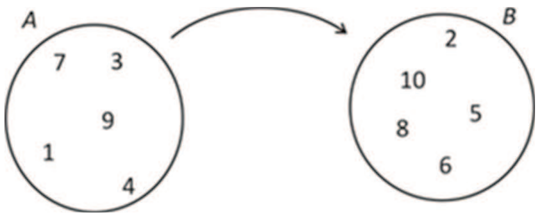

Fig. 5

Problem CV. Milica has a bag with balls marked with numbers 3, 7, 4, 1 and 9, and Jovan has a bag with balls marked with numbers 10, 8, 5, 2 and 6. Which two of her balls should Milica transfer into Jovan's bag so that the sum of the numbers on the balls she has left in her own bag is four times smaller than the sum of the numbers on the balls placed in Jovan's bag?

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