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APPLICATIONS OF LOGISTIC AND GENERALIZED LOGISTIC DIFFERENCE EQUATIONS IN ECONOMICS: AK MODEL

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Abstract. In this paper, we demonstrate how logistic equation and generalized logistic equation can be applied in teaching economics with the aim that students easily grasp the AK model. To this end, the AK model is firstly modified into three versions, after which the first and the second model are reduced to the generalized logistic equations. The purpose of this paper is to present how through applying the mathematical modifications of mentioned versions of AK models, they become more cognizable to students, which should result by more deeply understanding of well-known economic models.

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1. Introduction

Production function, which represents a specification of the relationship between the output and the inputs in production, such as capital and labour, is present in the syllabus of both Macroeconomics and Microeconomics courses for students of economic and non-economic majors. The following equation gives the general form of the AKL model (Solow model):

$$Y_t = A_t K_t^{\alpha} L_t^{\beta}, \tag{1}$$

where Y_t is the real income in period t, K_t is the stock of capital in period t, A_t is the level of technology or aggregate efficiency in period t, L_t stands for the labour force in period t, while α and β are output elasticities of capital and labour, respectively.

Model (1) presents a particular part of the mathematical theories of homogeneous functions. If production function can be written as

$$Y = f(sK, sL) = s^{\alpha+\beta} f(K, L), \quad s > 0, \tag{2}$$

then it is a homogeneous functions of the degree of $\alpha + \beta$. Symbol s denotes the change in the inputs, meaning that if $s > 1$, the inputs (and production as well) are increasing; if $s < 1$, the inputs are decreasing, and when $s = 1$, there is no change in inputs nor in production. Homogeneous functions are interesting since Euler's theorem can be applied to them [1]. By differentiating function (2) with respect to s, and for $s = 1$, we obtain:

$$f_K K + f_L L = (\alpha + \beta)Y,$$

where f_K and f_L are partial derivatives of production function with respect to capital and labour, respectively. With further modifications (a division of both sides of the equation by Y) the following equation is obtained:

$$f_K \frac{K}{Y} + f_L \frac{L}{Y} = \epsilon_K + \epsilon_L = \alpha + \beta,$$

where ϵ_K and ϵ_L are partial elasticities of production with respect to inputs. According to the Euler's theorem, the property of the homogeneous production function is that the sum of partial elasticities of production with respect to inputs equals the degree of the returns to scale. For production function with the degree of homogeneity $\alpha+\beta$, the degree of homogeneity of marginal products of the inputs is $\alpha + \beta - 1$.

Cobb and Douglas introduced production function in microeconomics (theory of firm) in 1928 [6]. All symbols from equation (1) are at a firm level. If $\alpha+\beta = 1$, there are constant returns to scale. With $\alpha+\beta > 1$, returns to scale are increasing, while $\alpha + \beta < 1$ means that returns to scale are decreasing [17].

In macroeconomics, specifically in economic growth theory, the aggregate production function, which shows the relationship between aggregate output and aggregate inputs, was developed by neoclassical economists Solow [26] and Swan [30], which provided the analytical framework for studying the determinants of the aggregate output.

In this paper, we focus on aggregate production function (i.e., production function from a macroeconomic aspect) and its mathematical modifications to show how mathematics can be applied in teaching economic courses, both for students with Economics as a major and for students of other majors. Also, in Mathematics courses, students can be presented with Euler's theorem and difference equations on the example of a production function. The methodological-didactic aspects of the paper are as follows: 1. We give modified versions of the AK model and their reduction to the generalized logistic map, which is more cognizable to students since it is more straightforward and more realistic than the general AK model or general AKL model; 2. Students become conscious of the importance of learning mathematics since they see its application in a relatively simple form in economics; 3. Students learn how mathematical and statistical knowledge can be applied to empirical economic data, obtaining parameter estimates with smaller standard errors and narrower confidence intervals when the AK model is used in the form of the generalized logistic map.

We start from the general AKL model presented by equation (1). By setting $\alpha = 1$ and $\beta = 0$, equation (1) becomes

$$Y_t = A_t K_t. \tag{3}$$

Further, by assuming that $A_t = A$, we get the AK model:

$$Y_t = AK_t, A > 0. \tag{4}$$

The rest of the paper is organized as follows. Section 2 briefly analyzes the aggregate AK model and related literature. Section 3 describes the non-linear dynamic and generalized logistic map methodology. In section 4, we present three modifications of the AK model: 1. modified AK model, 2. AK model and the government, and 3. simple chaotic AK model with increasing returns, and after that, we reduced the first and second modification to the generalized logistic equation, showing relevant theorems and proofs. Section 5 presents model validation on the empirical data, where better AK model parameter estimations, with smaller standard errors and narrower confidence intervals, are obtained. Finally, concluding remarks are given in Section 6.

2. The development of growth theory models

All growth theory models can be classified into two generations: exogenous and endogenous. For example, one of the first neoclassical growth theory models is independently developed by Solow [26] and Swan [30]. The model is exogenous in nature, meaning that the long-term growth rate is based on external sources, such as population growth rate, labour force structure, and technological progress. On the other hand, the AK model, as a simple modification of the Solow model, belongs to the generation of the endogenous growth models. The endogenous growth models claim that the long-term growth rate depends on internal sources, such as education and knowledge.

The exogenous neoclassical growth theories could not explain the growth rate changes through time and significant differences in GDP per capita between countries and regions. Therefore, the space for the development of endogenous growth theories emerged. Although Solow [27] made attempts to explain the nonstationary feature of economic growth through time, he did not analyze the factors related to technological changes [19]. Unlike, the endogenous growth theory tries to explain the mechanism of economic growth by incorporating the influence of technological changes over time. The exogenous growth models claim that the only policies that can lead to economic growth in the long term are those that can increase population growth rate or labour force productivity, while conventional macroeconomic policies do not have any influence on long-term economic growth rate. In the 1980s, the neoclassical growth theory was replaced by the endogenous growth theory, introduced by Romer [22] and Lucas [15]. Romer states that knowledge is an essential growth factor. Mankiw et al. [16] include education in the AK model, emphasizing the role of human capital as a factor of economic growth. Education increases the level of human capital and, finally, the aggregate production level. While exogenous models state that economic growth cannot be achieved without technological advancement, endogenous models hold that it is possible to have economic growth even without technological advancement but with a significant increase in human capital. In exogenous models, the critical factors of permanent economic growth are the level of capital per worker and labour force productivity. On the other hand, in endogenous models, the key factors of economic growth are human capital and knowledge, while economic growth can be achieved even without the increase in capital per worker or labour force productivity.

The main advantages of the endogenous growth theory over the exogenous one are (1) attempt to explain the aspects of data not explained in the traditional neoclassical macroeconomic model; (2) better explanation of international differences in economic growth rates; (3) more critical role of knowledge accumulation; (4) more significant role of macroeconomic policy instruments in explaining the long-term economic growth process.

The economic growth system, as well as most economic systems, is complex and stochastic. Therefore, applying mathematical chaos models in economics is becoming widely accepted to explain the sources of economic fluctuations [7]. Logistic and generalized logistic equations are frequently used in the mathematical modeling of non-linear dynamics of complex economic systems with the aim to understand and predict their behavior. The stability analysis of the AK model is necessary since small changes in factors can induce significant changes in the income level. The increase in inputs, such as the level of technology, could lead to an explosive output path in the long run. Therefore, it is crucial to determine the levels of output per worker and capital per worker for which the economy converges in the long run.

3. Methodology: non-linear dynamics, logistic and generalized logistic maps

With the inflow of the elements of mathematical chaos theory into economics, it became possible to explain and predict the behavior of complex and dynamic economic systems and to analyze the stability of the steady state. The importance of applying this part of mathematics in economic courses is even higher, considering the enormous consequences of the crises that hit the world economy from time to time. Therefore, it is crucial to model economic processes so that these unexpected crashes can be explained and, if possible, predicted and mitigated. Non-linear models in which a slight change in one factor can cause complex behavior and chaos in the economic system are often used for mentioned purposes [11, 12]. Since most economic variables, such as prices, consumption, investment, and unemployment, are volatile, the need for implementing complexity into economics is obvious.

With f^n we denote discrete iterations of a function $f : I \to I$, $f^n = f \circ f^{n-1}$, $f^0 = id$, $n = 1, 2, \dots$. The fixed points are the solutions of the equation $x^* = f(x^*)$. The sequence (x_n), such that $x_n = f^n(x_0)$, is the sequence of iterates of a function f generated by point, and the fixed point x^* is locally stable if there exists neighborhood U such that all sequences of iterates are convergent to x^*, for $x_0 \in U$. We say that point s generates a cycle of order k if $f^k(s) = s$ and $f^j(s) \neq s$ for all $j < k$. The famous Li and York result from 1975 [14] tells that the function is chaotic if it has a cycle of order different from 2^k or a cycle of order 3. The main characteristic of the chaotic system is a great sensitivity to initial conditions.

The logistic equation, introduced by Pierre Verhulst in 1838 [2], is one of the most often used first-order non-linear ordinary difference equations in mathematical modeling. It is used for system stability analysis and can be applied to many sciences, such as economics, biology, engineering, demography and chemistry. The following formula gives the logistic equation:

$$x_{t+1} = rx_t(1 - x_t), \tag{5}$$

where x_t denotes a value of the variable x in the period t.

The stability analysis of equation (5) can be briefly summarized as follows:

- $0 < r < 1$: the only stable fixed point is $x_1^* = 0$;
- $1 < r < 3$: the one fixed point x_1^* becomes unstable, while the other fixed point $x_2^* = 1 - \frac{1}{r}$ is stable;
- $3 < r < 4$: both fixed points x_1^* and x_2^* are unstable, while the system firstly behaves cyclically and then chaotically. The first cycle has a period of 2, followed by periods $2^2, 2^3, \ldots$ Finally, for $r \approx 1 + 2\sqrt{2}$, the cycle of period 3 and chaotic behavior appears.

Numerous models in economics can be reduced to the logistic map. Some of the examples are the supply-demand model based on non-linear supply and demand functions [9] and Samuelson's multiplier-accelerator model [24]. Moreover, by examining the system's stability, it can be concluded which kind of behavior it performs, depending on the parameter value.

The following generalization of the logistic map appeared recently in literature [20, 21]:

$$x_{t+1} = rx_t^p(1 - x_t^q), \quad x_0 \in [0, 1], \quad p > 0, q > 0. \tag{6}$$

Some applications of (6) in economics are presented in [13] and [29]. The generalized logistic map (6) has arbitrary powers p and q, which can be chosen to fit the observed data best. We opted for this generalized logistic map to show students how the system's flexibility can be increased.

Analysis of (6) for parameters $p = 1$, $q = 2$ is done in [21] and more advanced analysis of the equation (6) for cases $(p, q) = (\alpha, \alpha)$, $(p, q) = (1, \alpha)$, $(p, q) = (\alpha, 1)$ is presented in [20]. Since the model we propose is reduced to (6) with $(p, q) = (1, \alpha)$ we briefly present stability analysis for that case. Denote left side of (6) as a function:

$$f(x, r, \alpha) = rx(1 - x^\alpha). \tag{7}$$

In order to achieve that $f : [0, 1] \to [0, 1]$ and also $f^n : [0, 1] \to [0, 1]$ for any iteration, where $f^n(x) = f(f^{n-1}(x))$, the maximal value of the function must be less than 1. Therefore, we get first constrain for parameters values. Consider (7) as a function of x, and denote $f_{(r,\alpha)}(x) = rx(1 - x^\alpha)$. The maximum of the function is reached for

$$x_{max} = \left(\frac{1}{1 + \alpha}\right)^{\frac{1}{\alpha}}. \tag{8}$$

When $\alpha \to 0$, $x_{max} \to \frac{1}{e}$. Also, when $\alpha \to \infty$, $x_{max} \to 1$. From the request that $f_{(r,\alpha)}(x_{max}) \leq 1$, we get: 1) for the fixed value of the parameter α, the value of parameter r depends on α, and 2) the range for r is variable:

$$r \in [0, r_{max}], \quad r_{max} = \frac{(1+\alpha)^{1+\frac{1}{\alpha}}}{\alpha}.$$

Again, for limit values of α, we obtain:

- if $\alpha \to 0^+$ then $r_{max} \to \infty$,
- if $\alpha \to \infty$ then $r_{max} \to 1$.

4. Reduction of the modifications of the AK model to the generalized logistic map

This section analyzes three modifications of the AK model, of which two can be reduced to the generalized logistic map.

Consider a closed economy where the population growth rate, the saving rate, and the capital stock depreciation rate are all constant. The equations of the model are:

$$\begin{aligned} Y_t &= AK_t, \\ C_t &= \alpha Y_t^2, \quad \alpha > 0, \\ Y_t &= C_t + I_t, \\ \Delta K_t &= I_t - \delta K_t, \quad \delta > 0, \end{aligned} \tag{9}$$

where Y_t is real income, C_t is private household consumption, K_t is the stock of capital, I_t is investment, $A > 0$ is a positive constant, α is the consumption rate, δ is the rate of depreciation, ΔK is the change in capital stock, while index t denotes time period. Finally, this model is reduced to the standard logistic equation in [10]. Namely, after some calculations, we obtain:

$$Y_{t+1} = (A + 1 - \delta)Y_t - A\alpha Y_t^2, \tag{10}$$

which may be reduced to the logistic equation, by a simple change of variables[1].

The modification of the model (9), with $C_t = \alpha Y_t^q$ instead of $C_t = \alpha Y_t^2$ reduces the model to the generalized logistic equation.

Theorem 1. *The model*

$$\begin{aligned} Y_t &= AK_t, \quad A > 0, \\ C_t &= \alpha Y_t^q, \quad \alpha > 0, q > 1, \\ Y_t &= C_t + I_t, \\ \Delta K_t &= I_t - \delta K_t, \quad \delta > 0, \end{aligned} \tag{11}$$

[1]The proof is analogous to the proof of Theorem 1 presented in this section.

can be reduced to the generalized logistic equation if $A+1-\delta>0$ *and* $\left(\frac{A\alpha Y_{max}^{q-1}}{A+1-\delta}\right)^{\frac{1}{q-1}}\leq 1$.

Proof. From (11), by substitution, we get:

$$\begin{aligned}Y_{t+1}&=AK_{t+1}=A(I_t-\delta K_t+K_t)=A(Y_t-C_t+(1-\delta)K_t)\\&=A(Y_t-\alpha Y_t^q)+(1-\delta)Y_t=(A+1-\delta)Y_t-A\alpha Y_t^q.\end{aligned}$$

Finally, in this case, we obtain the equation

$$Y_{t+1}=(A+1-\delta)Y_t-A\alpha Y_t^q, \tag{12}$$

which can be reduced to (7) by a simple change of variables, as follows.

Starting from the relation

$$y_{t+1}=ay_t-by_t^c=ay_t(1-\frac{b}{a}y_t^{c-1}), \tag{13}$$

after introducing the change of variables

$$y_t=z_t\left(\frac{a}{b}\right)^{\frac{1}{c-1}}, \tag{14}$$

we obtain the generalized logistic equation:

$$z_{t+1}=az_t(1-z_t^{c-1}). \tag{15}$$

Since $y_t\in[0,1]$, we obtain that $z_t\in[0,\left(\frac{b}{a}\right)^{\frac{1}{c-1}}]$. The parameters a, b and c must satisfy $a,b>0$, $c>1$ and $\left(\frac{b}{a}\right)^{\frac{1}{c-1}}\leq 1$.

Now, starting from (12), we first divide it by Y_{max}, the maximal value of real income Y in the observed time series. After denoting $y_t=\frac{Y_t}{Y_{max}}$, we obtain

$$y_{t+1}=(A+1-\delta)y_t-A\alpha Y_{max}^{q-1}y_t^q. \tag{16}$$

In the next step, we introduce the change of variables (14),

$$y_t=z_t\left(\frac{A+1-\delta}{A\alpha Y_{max}^{q-1}}\right)^{\frac{1}{q-1}},$$

and we obtain the generalized logistic equation

$$z_{t+1}=(A+1-\delta)z_t(1-z_t^{q-1}),$$

if $A+1-\delta>0$ and $\left(\frac{A\alpha Y_{max}^{q-1}}{A+1-\delta}\right)^{\frac{1}{q-1}}\leq 1$. ■

The advantages of implementation of this theorem in courses for economists are fourfold: 1) the model presented in the Theorem 1 is more straightforward than the AKL model since we take into account only one production factor, i.e., capital; 2) the model presented in the Theorem 1 is more realistic than AK and AKL models since it shows the relationship between real income, private household consumption and investment and how the stock of capital changes over time; 3) students can identify how they can apply straightforward mathematical procedures to prove a theorem; 4) students will become more aware of the application of difference equations in economics.

Our following result is a generalization of the AK model with government spending presented by Theorem 2.

THEOREM 2. *The model*

$$\begin{aligned} Y_t &= AK_t, \quad A > 0, \\ G_t &= \alpha Y_t^q, \quad \alpha > 0, q > 1, \\ C_t &= \beta Y_t + \gamma K_t, \quad \beta, \gamma > 0, \\ Y_t &= C_t + I_t + G_t, \\ \Delta K_t &= I_t - \delta K_t, \quad \delta > 0, \end{aligned} \tag{17}$$

where G_t is government spending in period t, and all other variables are the same as in the previous theorem, can be reduced to the generalized logistic equation if $A + 1 - \delta - A\beta - \gamma > 0$ *and* $\left(\dfrac{A\alpha Y_{max}^{q-1}}{A+1-\delta-A\beta-\gamma}\right)^{\frac{1}{q-1}} \leq 1.$

Proof. By applying similar substitution as in the Theorem 1, we obtain:

$$Y_{t+1} = (A + 1 - \delta - A\beta - \gamma)Y_t - A\alpha Y_t^q. \tag{18}$$

This form (18) is the same as (12), and it is possible to transform it into a generalized logistic map under appropriate conditions. The rest of the proof is analogous to the proof of the Theorem 1. ■

An additional advantage of implementing the Theorem 2 in economic courses, apart from those mentioned for Theorem 1, is that in this AK model, we added government consumption and the relationships between real income, private household consumption, investment, and government consumption.

In [10], a similar model was presented:

$$\begin{aligned} Y_t &= AK_t, \quad A > 0, \\ G_t &= \alpha Y_t^2, \quad \alpha > 0, \\ C_t &= \beta Y_t + \gamma K_t, \quad \beta, \gamma > 0, \\ Y_t &= C_t + I_t + G_t, \\ \Delta K_t &= I_t - \delta K_t, \quad \delta > 0. \end{aligned} \tag{19}$$

Jablanović's model can be reduced to the logistic equation. On the other hand, our model (where there is the power q in the formula showing relationship between government consumption and real income) can be reduced to the generalized logistic equation.

In the mainstream macroeconomic literature (e.g., [3, 4, 5, 18, 25]), the relationship between private household consumption and real income is linear, as well as the relationship between government consumption and real income. On the other hand, Grossman [8] analyzed the non-linear relationship between government consumption and economic growth. Moreover, Jablanović [10] introduced the model with a quadratic relationship between mentioned variables, while in our theorems, we used the exponent $q > 1$.

Parameter q in our two models does not fundamentally complicate them but gives us additional freedom of choosing q to obtain a better fitting in model validation. Presented theorems are relatively simple so that students can apply their knowledge from Mathematics to Economics courses. By modifying the AK model, mathematical elegance, parsimony, and empirical accuracy are obtained, allowing students to grasp the concepts quickly.

Finally, the modified simple chaotic AK model with the increasing returns can not be easily transformed into the generalized logistic map. It remains an open question. The model is as follows:

$$
\begin{aligned}
Y_t &= AK_t^q, \quad A > 0, q > 1, \\
K_{t+1} &= I_t + (1-\delta)K_t, \quad \delta > 0, \\
I_t &= aY_t - bK_t, \quad a > 0, b > 0,
\end{aligned}
\tag{20}
$$

where a and b are coefficients of investment function and all other variables are the same as in the previous two models. In [10], a similar model is presented, but with $Y_t = AK_t^2$ instead of $Y_t = AK_t^q$, $q > 1$. From previous relations we obtain $K_t = \dfrac{Y_t^{1/q}}{A^{1/q}}$ and further

$$Y_{t+1}^{1/q} = aA^{1/q}Y_t + (1-\delta-b)Y_t^{1/q}.$$

By introducing the new variable $Z_t = Y_t^{1/q}$, we finally obtain

$$Z_{t+1} = (1-\delta-b)Z_t + aA^{1/q}Z_t^q. \tag{21}$$

The form (21) is almost generalized logistic map.

The importance of this model for students lies in the presence of the increasing returns to scale, which is closer to reality than constant returns to scale.

5. Empirical applications of the model

In this section, we choose one of the modifications of the AK model (the model from Theorem 1) reduced to the generalized logistic map and estimate the parameters of the model based on empirical data. The goal is to analyze the advantages of the model reduced to the generalized logistic map compared to the model reduced to the standard logistic map.

The general regression model is

$$Y_i = h(X_i^{(1)}, X_i^{(2)}, \ldots, X_i^{(m)}; \theta_1, \theta_2, \ldots, \theta_p),$$

where h is an appropriate function that depends on the predictor variables and parameters, which can be combined into vectors

$$X_i = (X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(m)})^T, \quad \theta = (\theta_1, \theta_2, \dots, \theta_p)^T.$$

In non-linear regression, function h is non-linear in parameters. Often, such a function is derived from theory. In principle, there are unlimited possibilities for describing the deterministic part of the model. Non-linear regression is a highly flexible tool for analysis that can fit almost any data [23]. After fitting an appropriate model, it is essential to determine how well the model describes the data. In the linear regression models, both the coefficient of determination and the standard error of the regression show how well the regression line fits the data. However, the coefficient of determination as a goodness-of-fit measure is invalid for non-linear regression. Spiess and Neumeyer [28] show that in non-linear regression: (1) coefficient of determination is consistently high for both excellent and appalling models, (2) coefficient of determination will not rise for better models all the time, and (3) coefficient of determination, as an indicator of the better model, will lead to the proper decision only in $28 - 43\%$ of the time. Therefore, in the case of non-linear regression, other goodness-of-fit measures have to be used, such as the standard error of the regression and the confidence interval.

We analyzed and compared two models: 1) model reduced to the standard logistic map, and 2) model reduced to the generalized logistic map. The first model is obtained from equation (10), where we normalized the variable Y_t, dividing it by Y_{max}, to get the equation:

$$y_{t+1} = ay_t - by_t^2, \quad a, b > 0.$$

It can be transformed into the logistic equation in standard form.

The second model is obtained from the equation (12), where we normalized the variable Y_t, dividing it by Y_{max}, to get the equation:

$$y_{t+1} = ay_t - by_t^c, \quad a, b > 0, c > 1,$$

which can be transformed into the generalized logistic equation in the standard form.

The following non-linear regressions based on the previously defined mathematical models are estimated to test which model fits the data better. Information about country GDP growth rates is obtained from the World Bank data set for 1999–2018[2]. The GDP growth rates of China, India, Russia, Ethiopia, Qatar, Turkey, and Panama are analyzed. These countries are selected because they have shown the most sustained and robust GDP growth over the past two decades, according to the IMF. The estimation of the parameters and standard errors for the first and the second model, as well as 95% confidence intervals, are presented in Table 1.

[2]Since this is illustrative example, it is not necessary to have longer time series (till 2021)

Table 1: Parameters estimation and confidence intervals of two compared models

	Parameter	Estimate	Std. Error	95% Confidence interval	
				Lower	Upper
China					
First model:	a	1.698	0.546	0.723	2.011
	b	0.352	0.639	-0.215	0.880
Second model:	a	0.825	0.239	0.466	1.036
	b	0.122	0.332	-0.258	0.322
	c	1.087	0.658	0.551	1.325
India					
First model:	a	2.012	0.669	1.452	3.124
	b	0.836	0.511	-0.123	1.677
Second model:	a	1.233	0.547	0.421	2.121
	b	0.322	0.299	-0.399	0.973
	c	1.569	0.365	0.523	2.556
Russia					
First model:	a	1.024	0.588	0.478	1.804
	b	0.682	0.369	-0.268	1.418
Second model:	a	0.703	0.403	0.312	1.003
	b	0.499	0.355	-0.168	1.055
	c	2.145	0.879	0.301	3.807
Ethiopia					
First model:	a	1.484	0.458	0.632	2.124
	b	0.915	0.339	0.122	1.866
Second model:	a	0.953	0.323	0.114	1.752
	b	0.552	0.215	-0.308	1.223
	c	3.121	0.699	2.117	4.552
Qatar					
First model:	a	1.225	0.537	0.527	2.055
	b	0.589	0.412	0.014	1.127
Second model:	a	0.998	0.507	0.147	2.027
	b	0.326	0.366	-0.286	1.098
	c	2.011	0.599	1.135	3.122
Turkey					
First model:	a	1.225	0.601	0.633	2.015
	b	0.547	0.459	-0.122	1.296
Second model:	a	0.699	0.588	0.054	1.366
	b	0.469	0.321	-0.218	1.145
	c	2.002	0.703	1.117	3.152
Panama					
First model:	a	1.024	0.589	0.588	1.626
	b	0.611	0.322	-0.032	1.158
Second model:	a	0.778	0.571	0.054	1.366
	b	0.556	0.299	0.012	1.025
	c	3.018	0.638	2.068	4.214

Source: Authors' calculation

The results indicate that the better model is the one reduced to the generalized logistic map since standard errors are smaller and confidence intervals are narrower. Furthermore, all of the estimated parameters have the expected sign. These results confirm that long-term economic growth can be better explained by introducing the dynamics into the system. The empirical validation of the mathematical model developed in this paper supports our theoretical analysis. This empirical analysis can be good illustration for students how knowledge from Statistics can be applied to Economics. Additionally, at the master level of studies, students can go one step further: from the estimated coefficients presented in previous table, they can analyze the stability of the steady state of the models.

6. Conclusion

In this paper, we illustrated one interesting application of the difference equations in Economics in a frame of the well-known model-AK model. The Solow model is firstly presented in the paper, then its simplification in the form of the AK model, and finally the modifications of the AK model which are more realistic than the simple AK model: modified AK model, AK model and the government, and simple chaotic AK model with increasing returns. Logistic and generalized logistic equations are frequently used in the mathematical modeling of the non-linear dynamics of complex economic systems, such as economic growth systems. Using this methodology, we reduced the modified AK model and the AK model and the government to the generalized logistic equation. The empirical data confirm the advantage of the reduced forms of the modified AK model over the simple one. We chose one of the modifications of the AK model reduced to the generalized logistic map in the Theorem 1 and estimated parameters of the model using empirical data set on the GDP growth rates of the group of countries for the period 1999–2018.

The results showed that the generalized logistic map is much better than the standard logistic map: we obtained smaller standard errors and narrower confidence intervals of the parameter estimations, which is our main contribution to the existing literature. The methodological-didactic contribution of our paper lies in the possibility of including analyzed theorems in the syllabus of economic courses for students of both economic and non-economic majors. In such a way, students will be more aware of the importance of the use of mathematics in studying economics. Mathematical operations used in theorems and their proofs are not complicated, which is expected to increase students' eagerness to accept mathematical models in courses. Modified versions of the AK model used in the paper are more realistic than the general AKL and general AK models. The aim of the applications of the theorem on the real data is to demonstrate to students how theoretical statistical knowledge can be applied to real data set. Also, the analysis of the stability of steady state can be included in the syllabus at a higher level of studies. Such analysis aims to predict the growth rates of economies and mitigate the consequences of the crises caused by unexpected adverse factors that impact economic growth. Nowadays, these aims become even more important when we witness the global economic crisis influenced by Covid-19 and Ukrainean war.

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