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GAUSSIAN INTEGRALS DEPENDING ON A QUANTUM PARAMETER IN FINITE DIMENSION

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Abstract. A common theme in mathematics is the evaluation of Gauss integrals. This, coupled with the fact that they are used in different branches of science, makes the topic always actual and interesting. In these notes we shall analyze a particular class of Gaussian integrals that depend on the quantum parameter $\hbar$. Starting from classical results, we will present an overview on methods, examples and analogies regarding the practice of solving quantum Gaussian integrals.

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1. Introduction

Let $\hbar$ be a quantum parameter; an important practice in quantum mechanics, is the evaluation of Gaussian integrals. We will focus our attention on a group of Gaussian integrals that come from quantum mechanics and quantum field theory (QFT). A general Gaussian integral in this class depends on a quantum parameter $k = \frac{1}{\hbar}$ with "$\hbar \to 0$". In our calculations we shall treat $k = \frac{1}{\hbar}$ as a purely formal parameter. Summarising, we will treat integrals of the following form:

$$\int_{\mathbb{R}^n} A\left(v, w\right) e^{-\frac{i}{\hbar}\psi(v,w)}\, dv, \tag{1}$$

where A is the "amplitude", ψ is generally quadratic and $v, w \in \mathbb{R}^n$.

This paper is based on previous discussions in [7,8,16] on different Gaussian integrals of the form (1). A basic text is the work [9] and, for the asymptotic analysis, we treat only the Laplace method described in [17]. In order to get a result about a Gaussian integral, sometimes it is necessary to introduce a special function (a good reference for the properties of special functions is [11]). Instead, on the techniques used, an exhaustive source is the book of Nahin [12].

Regarding the different examples discussed in these pages, they have been inspired by the works [1,6,13,14,15] concerning the geometric quantization.

At the beginning of the paper we will review basic facts concerning Gaussian integrals in 1 and n-dimension. We proceed with different examples coming from geometric quantization.

2. The Gaussian integral in 1-dimension

We start with the Gauss integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{2}$$

The result follows taking the square of the left-hand side and using the polar coordinates:

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy\\ &= \int_0^{2\pi}\int_0^{+\infty} \rho e^{-\rho^2}\,d\rho\,d\vartheta = 2\pi\int_0^{+\infty}\rho e^{-\rho^2}\,d\rho\\ &= \pi\int_0^{+\infty} e^{-s}\,ds = \pi\cdot\lim_{t\to+\infty} -\frac{1}{e^s}\bigg|_0^t = \pi.\end{aligned}$$

In his article [2], Conrad gives eleven proofs of the result (2) using different methods. A quick method consists in using the non elementary formula

$$\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{3}$$

where $\Gamma(s) = \int_0^{+\infty} e^{-x}x^{s-1}\,dx$ (for $s<0$). Using (3), we have that

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \left(2\int_0^{+\infty} e^{-x^2}\,dx\right)^2 = \Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi.$$

A curious reader can read the interesting article [11] where the author treats these integrals as a "puzzle" to solve.

Another Gaussian integral is the following:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}, \tag{4}$$

for every $a>0$. We can modify the original integral in order to obtain different versions. For example adding the term $-bx$ we find that

$$\int_{-\infty}^{+\infty} e^{-ax^2-bx}\,dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}. \tag{5}$$

We can prove the formula completing the square $-ax^2-bx = -\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2+\frac{b^2}{4a}$. In this case:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-ax^2-bx}\,dx &= \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2+\frac{b^2}{4a}}\,dx\\ &= e^{\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2}\,dx = \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}}\int_{-\infty}^{+\infty} e^{-s^2}\,ds = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}.\end{aligned}$$

Alternatives of the Gaussian integral are in the book of Nahin [12], an example is

$$\int_0^{+\infty} x^{2n} e^{-x^2}\, dx = \frac{(2n)!\sqrt{\pi}}{2n!4^n}, \tag{6}$$

for $n \geq 0$, where for $n = 0$ we refind (2) between 0 and $+\infty$. Another version of (6) is the following:

$$\int_{-\infty}^{+\infty} x^n e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)\sqrt{\pi}}{2^{\frac{n}{2}} a^{\frac{n+1}{2}}}, \qquad n = 2, 4, \dots ,$$

where $a > 0$, or this:

$$\int_0^{+\infty} \frac{e^{-px^2} - e^{-qx^2}}{x^2}\, dx = \sqrt{\pi}\,(\sqrt{q} - \sqrt{p})\,,$$

for $q > p \geq 0$.

A complex modification of the Gaussian integral (5) is the following:

$$\int_{-\infty}^{+\infty} e^{-ax^2 - \eta x}\, dx = e^{\frac{\eta^2}{4a}} \sqrt{\frac{\pi}{a}},$$

where $\eta \in \mathbb{C}$ and the integral converges uniformly in any compact region. Now, the integral defines an analytic function that may be evaluated by taking η to be real and then using analytic continuation. To prove this we use the same trick as for the integral (5) and the result is true for all $\eta \in \mathbb{C}$. As observed in [7], when $\eta = i\xi$ we get that

$$\int_{-\infty}^{+\infty} e^{-ax^2 - i\xi x}\, dx = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{\xi^2}{4a}},$$

or, in other words, the Fourier transform of e^{-ax^2} is the Gaussian $e^{-\frac{\xi^2}{4a}}$.

Considering the properties of the Fourier transform we can prove the following result.

Proposition 2.1. *Let m be a positive integer and k be a real positive constant; then*

$$\int_{-\infty}^{+\infty} x^m e^{-i\xi x - \frac{1}{2}kx^2}\, dx = \sqrt{2\pi}\frac{(-i)^m}{k^{m+\frac{1}{2}}} P_m(\xi) e^{-\frac{1}{2k}\xi^2},$$

where $P_m(\xi) = \xi^m + \sum_{j\geq 1} p_{mj}\xi^{m-2j}$ is a monic polynomial in ξ of degree m and parity $(-1)^m$.

Proof. By a following property of the Fourier transform,

$$\mathcal{F}\left(x^m e^{-\frac{1}{2}kx^2}\right) = i^m \frac{d^m}{d\xi^m} \mathcal{F}\left(e^{-\frac{1}{2}kx^2}\right).$$

Now the Fourier transform of $e^{-\frac{1}{2}kx^2}$ is equal to $\sqrt{\frac{2\pi}{k}} e^{-\frac{\xi^2}{2k}}$ and deriving m times $e^{-\frac{1}{2k}\xi^2}$ we find the polynomial P_m where the principal term has coefficient $\frac{(-1)^m}{k^m}$. We must collect this coefficient in order to find the polynomial P_m. ∎

In one dimension we can do much more using special functions as the gamma function and its inductive property $\Gamma(n+1) = n \cdot \Gamma(n)$. For example we have the cubic Gauss integral

$$\int_0^{+\infty} e^{-x^3}\,dx = \Gamma\left(\frac{4}{3}\right).$$

This can be proved by a simple substitution $y = x^3$:

$$\int_0^{+\infty} e^{-x^3}\,dx = \frac{1}{3}\int_0^{+\infty} e^{-y}y^{-\frac{2}{3}}\,dy = \frac{1}{3}\cdot\Gamma\left(\frac{1}{3}\right) = \Gamma\left(\frac{4}{3}\right).$$

This last argument can be generalized for integrals of the form

$$\int_0^{+\infty} e^{-x^m}\,dx = \Gamma\left(\frac{1+m}{m}\right),$$

where $m \geq 1$. We observe that for the case $m = 2$ the gamma function is $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and we recover (2) between 0 and $+\infty$.

Another approach involves the use of the Taylor expansion. In order to explain the method we consider the following integral:

$$\int_{-\infty}^{+\infty} e^{-e^{-x^2}} - 1\,dx.$$

In this case we can work "formally" and write

$$e^{-e^{-x^2}} - 1 = -e^{-x^2} + \frac{1}{2!}e^{-2x^2} - \frac{1}{3!}e^{-3x^2} + \cdots,$$

where we use the McLaurin expansion of $e^x \sim 1 + x + \frac{x^2}{2!} + \cdots$. Now integrating each term we have that

$$\int_{-\infty}^{+\infty} e^{-e^{-x^2}} - 1\,dx = \sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\int_{-\infty}^{+\infty} e^{-kx^2}\,dx = \sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\sqrt{\frac{\pi}{k}}.$$

So we have

$$\int_{-\infty}^{+\infty} e^{-e^{-x^2}} - 1\,dx = \sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\sqrt{\frac{\pi}{k}}.$$

3. The Gaussian integral in n-dimensions

In n dimension we have that

$$\int_{\mathbb{R}^n} e^{-\|x\|^2}\,dx = \sqrt{\pi^n},$$

where $\|x\|^2 = x_1^2 + \cdots + x_n^2$. The result follows observing that $\int_{\mathbb{R}^n} e^{-\|x\|^2}\,dx = \left(\int_{-\infty}^{+\infty} e^{-t^2}dt\right)^n$ and using the polar coordinates:

$$\begin{aligned}\frac{1}{\sqrt{\pi^n}}\left(\int_{-\infty}^{+\infty} e^{-t^2}\,dt\right)^n &= \frac{1}{\sqrt{\pi^n}}\int_0^{+\infty} e^{-\rho^2}\rho^{n-1}c_n\,d\rho \\ &= \frac{c_n}{2}\frac{1}{\sqrt{\pi^n}}\int_0^{+\infty} s^{\frac{n}{2}-1}e^{-s}\,ds = \frac{1}{\sqrt{\pi^n}}\frac{c_n}{2}\Gamma\left(\frac{n}{2}\right) = 1,\end{aligned}$$

where $c_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere. The result has been generalized in [9] for a symmetric positive definite $n \times n$-matrix A:

$$\int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle}\,dx = \frac{\sqrt{\pi^n}}{\sqrt{\det A}}, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$. This result is the analogue of (4) in one dimension. Also, we can find in [9], another result for the Fourier transform. We recall here the theorem.

THEOREM 3.1. (Hörmander) *If A is a non-singular symmetric matrix and* $\operatorname{Re} A \geq 0$ *the Fourier transform of $u(x) = e^{-\frac{1}{2}\langle Ax,x\rangle}$ is a Gaussian function $\widehat{u}(\xi) = (2\pi)^{\frac{n}{2}}(\det B)^{\frac{1}{2}}e^{-\frac{1}{2}\langle B\xi,\xi\rangle}$ where $B = A^{-1}$ and the square root is well defined. If $A = -iA_0$ where A_0 is real, symmetric and non-singular then*

$$\widehat{u}(\xi) = (2\pi)^{\frac{n}{2}}|\det A_0|^{-\frac{1}{2}}e^{\frac{\pi i \operatorname{sgn} A_0}{4} - \frac{1}{2}i\langle A_0^{-1}\xi,\xi\rangle}.$$

In the previous theorem the term $\operatorname{sgn} A_0$ is called the signature of A_0.

The Gaussian integral (7) admits different generalizations, for example we can consider the problem to evaluate the integral

$$I = \int_{\mathbb{R}^n} x_i x_j e^{-\frac{1}{2}x^T\cdot Ax}\,dx, \tag{8}$$

with A a real symmetric $n \times n$ matrix, T denotes the transpose and $\cdot$ the ordinary product of matrices.

A general procedure in order to solve Gauss integrals as (8) consists to introduce a generating function $\mathcal{Z}(J)$ depending by a parameter $J = \begin{pmatrix} J_1 \\ \vdots \\ J_n \end{pmatrix}$, where $\mathcal{Z}(J) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T\cdot Ax + x^T\cdot J}\,dx$. Now we have that

$$I = \left.\frac{\partial^2 \mathcal{Z}(J)}{\partial J_i \partial J_j}\right|_{J=0}.$$

We will use this version of the Wick's theorem. It gives a way to compute the ground-state expectation value of an operator.

THEOREM 3.2. (Wick) *Let us consider J the parameter of the generating function $\mathcal{Z}(J)$; then the expectation value of the state $e^{\frac{1}{2}J^T\cdot A^{-1}J}$ is given by*

$$\frac{\partial^n}{\partial J_{i_1}\cdots\partial J_{i_n}}\left.\left(e^{\frac{1}{2}J^T\cdot A^{-1}J}\right)\right|_{J=0} = \sum \left(A^{-1}\right)_{i_{p_1}i_{p_2}}\cdots\left(A^{-1}\right)_{i_{p_{n-1}}i_{p_n}},$$

where the sum is taken over all pairings: $(i_{p_1}, i_{p_2}), \ldots (i_{p_{n-1}}, i_{p_n})$, of $i_1, \ldots, i_n$. The factors $\left(A^{-1}\right)_{i,j}$ are elements of the inverse matrix A^{-1} (physically denotes the propagator between the space point i and j).

We will use this theorem in the case when $i_1 = i$ and $i_2 = j$. We observe that there is another way to write $\mathcal{Z}(J)$:

$$\mathcal{Z}(J) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x-A^{-1}J)^T A(x-A^{-1}J)+\frac{1}{2}J^T A^{-1} J}.$$

If we substitute $y = x - A^{-1}J$ we have that

$$\mathcal{Z}(J) = e^{\frac{1}{2}J^T A^{-1} J} \int_{\mathbb{R}^n} e^{-\frac{1}{2}y^T A y}\, dy = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} e^{\frac{1}{2}J^T A^{-1} J}.$$

Reconsidering the integral I we have that

$$I = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left.\frac{\partial^2 e^{\frac{1}{2}J^T A^{-1} J}}{\partial J_i \partial J_j}\right|_{J=0}.$$

After calculations:

$$\begin{aligned}
I &= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left.\frac{\partial^2 e^{\frac{1}{2}(A^{-1})_{kl}J^k J^l}}{\partial J_i \partial J_j}\right|_{J=0} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left.\frac{\partial \left[(\frac{1}{2}(A^{-1})_{ai}J^a + \frac{1}{2}(A^{-1})_{ib}J^b) e^{\frac{1}{2}(A^{-1})_{kl}J^k J^l}\right]}{\partial J_j}\right|_{J=0} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left.\frac{\partial \left[((A^{-1})_{ia}J^a) e^{\frac{1}{2}(A^{-1})_{kl}J^k J^l}\right]}{\partial J_j}\right|_{J=0} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left[(A^{-1})_{ij} + (A^{-1})_{ia}J^a \left(\frac{1}{2}(A^{-1})_{bj}J^b + \frac{1}{2}(A^{-1})_{jc}J^c\right)\right] e^{\frac{1}{2}(A^{-1})_{kl}J^k J^l}\Big|_{J=0} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \left[(A^{-1})_{ij} + (A^{-1})_{ia}J^a \cdot (A^{-1})_{bj}J^b\right] e^{\frac{1}{2}(A^{-1})_{kl}J^k J^l}\Big|_{J=0} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} (A^{-1})_{ij}.
\end{aligned}$$

We may call $\mathcal{Z}_0 = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}$ so the result is $I = \mathcal{Z}_0 \left(A^{-1}\right)_{ij}$. This work can be generalized and a good reference is [5].

4. Gaussian integrals depending on a quantum parameter

In this section we study a particular kind of Gaussian integrals deriving from quantum mechanics. The general form of these integrals is the following:

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g(v,w)}\, dv,$$

where the function g is usually quadratic in its variables and $(v, w) \in \mathbb{R}^{2n}$. Here we are interested in particular forms of g.

Proposition 4.1. *We have the following cases:*

1) *if* $g = g_1(v, w) = \omega(v, w) - \frac{i}{2}\|v - w\|^2$ *then*
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_1(v,w)}\, dv = (2\pi\hbar)^{\frac{n}{2}}\, e^{-\frac{1}{2\hbar}\|w\|^2};$$

2) *if* $g = g_2(v, w, u) = -\omega(v, w + u) - \frac{i}{2}\|v\|^2$ *then*
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_2(v,w,u)}\, dv = (2\pi\hbar)^{\frac{n}{2}} e^{-\frac{1}{2\hbar}\|w+u\|^2};$$

3) *if* $g = g_3(v, w, u) = \omega(v, w) + \omega(w, u) - \frac{i}{2}\|v - w\|^2 - \frac{i}{2}\|w - u\|^2$ *then*
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_3(v,w,u)}\, dw = (\pi\hbar)^{\frac{n}{2}} e^{-\frac{i}{\hbar} g_1(v,u)},$$

for all $v, w, u \in \mathbb{R}^n$.

Proof. We start with the first Gaussian integral and we have that
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_1(v,w)}\, dv = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\omega(v,w) - \frac{1}{2\hbar}\|v-w\|^2}\, dv$$
$$= \hbar^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ig\left(J(\beta), \frac{w}{\sqrt{\hbar}}\right) - \frac{1}{2}\|\beta\|^2}\, d\beta = (2\pi\hbar)^{\frac{n}{2}}\, e^{-\frac{1}{2\hbar}\|w\|^2},$$
where $\beta = \frac{1}{\sqrt{\hbar}}(v - w)$ is a new variable used in the integration.

For the second case we proceed in a similar way:
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_2(v,w,u)}\, dv = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\omega(v,w+u) - \frac{1}{2\hbar}\|v\|^2}\, dv$$
$$= \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g(Jv, w+u) - \frac{1}{2}\left\|\frac{v}{\sqrt{\hbar}}\right\|^2}\, dv = \int_{\mathbb{R}^n} e^{-ig\left(J\frac{v}{\sqrt{\hbar}}, \frac{w+u}{\sqrt{\hbar}}\right) - \frac{1}{2}\left\|\frac{v}{\sqrt{\hbar}}\right\|^2}\, dv$$
$$= \hbar^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ig\left(J\beta, \frac{w+u}{\sqrt{\hbar}}\right) - \frac{1}{2}\|\beta\|^2}\, d\beta = (2\pi\hbar)^{\frac{n}{2}} e^{-\frac{1}{2\hbar}\|w+u\|^2},$$
where we have used $\beta = \frac{1}{\sqrt{\hbar}} v$ as a new variable.

In the last case we consider the function $g_3(v, w, u) = \omega\,(v, w) - \frac{i}{2}\|v - w\|^2 + \omega\,(w, u) - \frac{i}{2}\|w - u\|^2$ where $v, w, u \in \mathbb{R}^n$.
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_3(v,w,u)}\, dw = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}[\omega(v,w)+\omega(w,u)] - \frac{1}{2\hbar}\left[\|v-w\|^2 + \|w-u\|^2\right]}\, dw.$$
We set $v - w = t$, so
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_3(v,w,u)}\, dw = e^{-\frac{i}{\hbar}\omega(v,u)} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}[\omega(t,v-u)] - \frac{1}{2\hbar}\left[\|t\|^2 + \|v-u-t\|^2\right]}\, dt.$$
A second variable $\frac{v-u}{2} - t = z$ permits to write
$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} g_3(v,w,u)}\, dw = e^{-\frac{i}{\hbar}\omega(v,u)} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}[\omega(z,u-v)] - \frac{1}{2\hbar}\left[\left\|\frac{v-u}{2} - z\right\|^2 + \left\|\frac{v-u}{2} + z\right\|^2\right]}\, dz.$$

After calculations we find that

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}g_3(v,w,u)}\,dw = e^{-\frac{i}{\hbar}\omega(v,u)-\frac{1}{4\hbar}\|v-u\|^2}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}[\omega(z,u-v)]-\frac{1}{\hbar}\|z\|^2}dz.$$

Now the last integral is an ordinary Gaussian integral of simple estimation:

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}g_3(v,w,u)}\,dw = (\pi\hbar)^{\frac{n}{2}}e^{-\frac{i}{\hbar}\omega(v,u)-\frac{1}{2\hbar}\|v-u\|^2} = (\pi\hbar)^{\frac{n}{2}}e^{-\frac{i}{\hbar}g_1(v,u)}. \quad \blacksquare$$

A variation of the exponent is the function $g_4(v,w) = \omega(v,Jw) - \frac{i}{2}\|v\|^2 - \frac{i}{2}(1+2i\cot\vartheta)\|w\|^2$. This function appears in [13] as the function $\psi_2(v_0, e^{i\vartheta}v_1)$.

PROPOSITION 4.2. *If* $\vartheta \in [0,\pi]$, $v,w \in \mathbb{R}^n$, *then*

$$\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}g_4(v,w)}\,dv\,dw = 2^{\frac{n}{2}}(\hbar\pi)^n\sin^{\frac{n}{2}}\vartheta e^{i\left(\frac{\pi}{2}-\vartheta\right)\frac{n}{2}}.$$

Proof. In this case we have that

$$\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}g_4(v,w)}\,dv\,dw = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(v,Jw)-\frac{1}{2\hbar}\|v\|^2-\frac{1}{2\hbar}(1+2i\cot\vartheta)\|w\|^2}\,dv\,dw$$
$$= (\hbar 2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{1}{2\hbar}(2+2i\cot\vartheta)\|w\|^2}\,dw = \frac{\hbar^n(2\pi)^n}{\sqrt{\det(2+2i\cot\vartheta)I_n}},$$

where I_n is the identity matrix. Observing that $2+2i\cot\vartheta = \frac{2i}{\sin\vartheta}[-i\sin\vartheta+\cos\vartheta]$ we have that

$$\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}g_4(v,w)}\,dv\,dw = \frac{\hbar^n(2\pi)^n}{\sqrt{\det\frac{2ie^{-i\vartheta}}{\sin(\vartheta)}I_n}} = (\hbar\pi)^n 2^{\frac{n}{2}}\sin^{\frac{n}{2}}(\vartheta)e^{i\left(\frac{\pi}{2}-\vartheta\right)\frac{n}{2}}. \quad \blacksquare$$

PROPOSITION 4.3. *If* $g_5(u,v,w) = \frac{1}{2}(\|w-v\|^2+\|v-u\|^2)$ *and* $v,w,u \in \mathbb{R}^n$, *then*

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}g_5(u,v,w)}\,dv = (2\pi\hbar)^{\frac{n}{2}}e^{\frac{i}{2\hbar}\left\|\frac{w-u}{2}\right\|^2}e^{\frac{i\pi n}{4}}.$$

Proof. We have

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\left(\|w-v\|^2+\|v-u\|^2\right)}\,dv = \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\left(\|(w-u)-t\|^2+\|t\|^2\right)}\,dt$$

where we used the substitution $v-u=t$. A second substitution $t = \frac{(w-u)}{2} - z$ gives the integral

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\left(\left\|\frac{(w-u)}{2}+z\right\|^2+\left\|\frac{(w-u)}{2}-z\right\|^2\right)}dz = \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\left(\left\|\frac{(w-u)}{2}\right\|^2+\|z\|^2\right)}dz$$
$$= e^{\frac{i}{2\hbar}\left\|\frac{(w-u)}{2}\right\|^2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\|z\|^2}dz = \hbar^{\frac{n}{2}}e^{\frac{i}{2\hbar}\left\|\frac{(w-u)}{2}\right\|^2}\int_{\mathbb{R}^n} e^{\frac{i}{2}\|q\|^2}dq$$
$$= (2\pi\hbar)^{\frac{n}{2}}e^{\frac{i}{2\hbar}\left\|\frac{(w-u)}{2}\right\|^2}e^{\frac{i\pi n}{4}},$$

where the last substitution was $\frac{1}{\sqrt{\hbar}}z = q$. $\blacksquare$

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