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SMOOTHNESS OF THE SIGNED DISTANCE FUNCTION: A SIMPLE PROOF

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Abstract. The paper is of pedagogical nature and is aimed mainly at students. It presents a detailed proof of the well-known fact that if the boundary of an open set in $\mathbb{R}^n$ is of class C^k, $k \geq 2$, so is the signed distance to the boundary function. This function plays an important role in problems of Analysis and Geometry. The presented proof could give a teacher a good opportunity to discuss important theorems in Calculus.

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1. Introduction

Many years ago, when I was studying Hörmander's book [4] on several complex variables, I saw in the book the claim that if Ω is an open set in $\mathbb{R}^n$ with C^2-boundary, the function

$$\delta(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega), & x \in \Omega \\ 0, & x \in \partial\Omega \\ -\operatorname{dist}(x, \partial\Omega), & x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases} \tag{1}$$

is also C^2-smooth near the boundary of Ω. In fact, a similar statement holds true for a C^k-boundary:

THEOREM 1. *If Ω is an open set in $\mathbb{R}^n$ with C^k-smooth boundary $\partial\Omega$ and $k \geq 2$, the signed distance function δ is C^k-smooth in a neighborhood of $\partial\Omega$.*

No proof or reference was given in [4], only a remark in parentheses on page 50: "This follows from the implicit function theorem". Then I spent some time to find a proof of the claim. This proof uses only standard facts of Calculus and is presented below in great detail in order to be easily understandable and instructive for undergraduate students (I have recently noticed that it has a non-empty overlap with that in [6]). I hope it would give a teacher a good opportunity to discuss important theorems in Calculus.

Several years later, I learned that a proof can be found in the Gilbarg and Trudinger book [3] on partial differential equations, but this book was not available for me at that time. Their proof can be seen in [3, Appendix: Boundary Curvatures and the Distance Function]. It involves the notion of principal curvatures of $\partial\Omega$ and makes use of the fact that at each point $y_0 \in \partial\Omega$ there exists a ball B depending on y_0 such that $\overline{B} \cap (\mathbb{R}^n \setminus \Omega) = y_0$ and the radii of the balls B are bounded from below by a positive constant. Other proofs are due to Krantz and Parks [5] (see also [6, Theorem 1.2.6] and Foote [2]. In both papers, it is shown that Theorem 1 can be extended to the class C^1 under the additional assumption that $\partial\Omega$ is of positive reach, a notion introduced in [1]. The latter assumption cannot be omitted as shown in [5]. A subset M of $\mathbb{R}^n$ is of positive reach if it admits a neighborhood U such that for every $x \in U$ there is a unique point $y(x) \in M$ for which $\operatorname{dist}(x, M) = |x - y(x)|$. If $\partial\Omega$ is C^2-smooth, then it is of positive reach, as shown in [1] (see also Remark 2 below).

2. Preliminaries

2.1. Distance between a point and a set in $\mathbb{R}^n$

We recall two basic properties (with their standard proofs) of the distance from a point to a set in $\mathbb{R}^n$.

If M is a subset of $\mathbb{R}^n$ and x a point in $\mathbb{R}^n$, the distance between x and M is defined as

$$d(x, M) = \inf\{|x - u| : \ u \in M\}.$$

Proposition 1. (*a*) *For every x and y in $\mathbb{R}^n$, $|d(x, M) - d(y, M)| \leq |x - y|$.*

(*b*) *If M is closed, for every $x \in \mathbb{R}^n$ there is a point $v \in M$ such that $d(x, M) = |x - v|$.*

Proof. (*a*) By the triangle inequality, if $u \in M$, $d(x, M) \leq |x - u| \leq |x - y| + |y - u|$, thus $d(x, M) - |x - y| \leq |y - u|$. Hence $d(x, M) - |x - y| \leq \inf\{|y - u| : \ u \in M\} = d(y, M)$. Similarly, $d(y, M) - |x - y| \leq d(x, M)$.

(*b*) For every $n = 1, 2, \ldots$, the set $K_n = M \cap \{z \in \mathbb{R}^n : \ [z - x| \leq d(x, M) + \frac{1}{n}\}$ is closed and bounded, hence compact. Moreover, $K_1 \supset K_2 \supset K_3 \supset \cdots$. By Cantor's intersection theorem, the intersection of all K_n's is not empty. Let $v \in \bigcap\{K_n : \ n = 1, 2, \ldots\}$. Then $d(x, M) \leq |x - v| \leq d(x, M) + \frac{1}{n}$. Therefore $d(x, M) = |x - v|$. ∎

Corollary 1. *If M is closed and $d(x, M) = 0$, then $x \in M$.*

2.2. Smooth boundary of an open set in $\mathbb{R}^n$

Recall that a subset M of $\mathbb{R}^n$ is said to be C^k-smooth at a point $a \in M$, $k \geq 1$, if there are a neighborhood U of the point a and a C^k-smooth function ρ in U such that $M \cap U = \{x \in U : \ \rho(x) = 0\}$ and $\operatorname{grad}\rho = \left(\dfrac{\partial\rho}{\partial x_1}, \ldots, \dfrac{\partial\rho}{\partial x_n}\right) \neq 0$ at every point of U. A function ρ with this property is called a defining function of M at the point a.

A set M is called a C^k-hypersurface if it is C^k-smooth at all of its points.

In these definitions and henceforth, by a C^k-smooth function we mean, as usual, a function with continuous partial derivatives up to order k for $0 \leq k \leq \infty$, or a real analytic function.

The next important property of the defining functions is not often mentioned in the textbooks although its proof is not difficult.

PROPOSITION 2. *Let ρ and $\widetilde{\rho}$ be two C^k-defining functions of M at a point $a \in M$. Then there are a neighborhood W of a and a nowhere vanishing C^{k-1}-smooth function h in W such that $\widetilde{\rho}(x) = h(x)\rho(x)$ for $x \in W$.*

Proof. Reordering the standard coordinates of $\mathbb{R}^n$, we may assume that $\dfrac{\partial\rho}{\partial x_1}(a) \neq 0$. Let V' be a neighborhood of a contained in the domains of ρ and $\widetilde{\rho}$, and consider the C^k-smooth map $F(x) = (\rho(x), x_2, \dots, x_n)$, $x \in V'$. The Jacobian of this map at the point a is $\dfrac{\partial\rho}{\partial x_1}(a) \neq 0$. Hence, by the inverse function theorem, there is a neighborhood $V \subset V'$ of a such that $F\,|\,V$ is a C^k-diffeomorphism of V onto a ball B in $\mathbb{R}^n$ with center at the point $F(a) = (0, a_2, \dots a_n)$. Clearly, $F(M \cap V) = \{\xi = (\xi_1, \dots, \xi_n) \in B : \ \xi_1 = 0\}$. Setting $\sigma = \rho \circ F^{-1}$, $\widetilde{\sigma} = \widetilde{\rho} \circ F^{-1}$, we have $\sigma(\xi) = \xi_1$, $\widetilde{\sigma}(0, \xi_2, \dots, \xi_n) = 0$ for $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in B$. Then

$$\widetilde{\sigma}(\xi) = \widetilde{\sigma}(\xi_1, \xi_2, \dots, \xi_n) - \widetilde{\sigma}(0, \xi_2, \dots, \xi_n) = \int_0^1 \frac{d}{dt}\widetilde{\sigma}(t\xi_1, \xi_2, \dots, \xi_n)\, dt$$
$$= \xi_1 \int_0^1 \frac{\partial\widetilde{\sigma}}{\partial\xi_1}(t\xi_1, \xi_2, \dots, \xi_n)\, dt.$$

Letting

$$g(\xi) = \int_0^1 \frac{\partial\widetilde{\sigma}}{\partial\xi_1}(t\xi_1, \xi_2, \dots, \xi_n)\, dt, \quad \xi \in B,$$

we get a C^{k-1}-smooth function in B for which $\widetilde{\sigma}(\xi) = \sigma(\xi).g(\xi)$. Hence, $h = g \circ F$ is a C^{k-1}-smooth function in V such that $\widetilde{\rho} = h.\rho$ in V. Since $\rho(a) = 0$, clearly $(\operatorname{grad}\widetilde{\rho})(a) = h(a)(\operatorname{grad}\rho)(a)$, hence $h(a) \neq 0$. Thus, h does not vanish in a neighborhood W of the point a. ■

REMARK 1. For another proof, see [6, Proposition 1.2.3].

The following useful statement is well-known (and easy to prove), but I am not able to give a reference.

PROPOSITION 3. *Let Ω be an open set in $\mathbb{R}^n$. Let ρ be a C^k-smooth defining function of the boundary $\partial\Omega$ at a point $a \in \partial\Omega$ such that $\dfrac{\partial\rho}{\partial x_1}(a) \neq 0$, $k \geq 1$. Then, there is a neighborhood V of a such that the restriction to V of the map $F(x) = (\rho(x), x_2, \dots, x_n)$ is a C^k-diffeomorphism of V onto a ball B in $\mathbb{R}^n$. Any such neighborhood V has the following properties:*

(i) $F(V \cap \partial\Omega) = \{\xi \in B : \ \xi_1 = 0\}$;

(ii) $\partial\Omega$ divides V into two pieces, i.e. $V \setminus \partial\Omega$ has two connected components V_1 and V_2;

(*iii*) *One of the open connected sets* V_1 *and* V_2*, say* V_1*, is given by* $\{x \in B : \ \rho(x) > 0\}$*, the other one is* $\{x \in B : \ \rho(x) < 0\}$*;*

(*iv*) *One of the sets* V_1 *and* V_2 *lies in* Ω*, the other one lies in* $\mathbb{R}^n \setminus \overline{\Omega}$*.*

Proof. By the inverse function theorem, F is a diffeomorphism of a neighborhood V of a onto a ball B. Property (i) is obvious. Properties (ii) and (iii) are also obvious with $V_1 = F^{-1}(\{\xi \in B : \ \xi_1 > 0\})$ and $V_2 = F^{-1}(\{\xi \in B : \ \xi_1 < 0\})$. To show (iv), note that $V = V_1 \cup V_2 \cup (V \cap \partial\Omega$, hence $V \cap \Omega = (V_1 \cap \Omega) \cup (V_2 \cap \Omega)$. The intersection $V \cap \Omega$ is not empty, thus $V_1 \cap \Omega \neq \emptyset$, or $V_2 \cap \Omega \neq \emptyset$. Suppose that $V_1 \cap \Omega \neq \emptyset$. Then $V_1 \subset \Omega$, otherwise $V_1 \cap (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$ since V_1 does not intersect the boundary $\partial\Omega$; thus the connected open set V_1 would be the union of the disjoint non-empty open sets $V_1 \cap \Omega$ and $V_1 \cap (\mathbb{R}^n \setminus \overline{\Omega})$, which is a contradiction. Now, since $V_1 \subset \Omega$, we have $V_2 \cap (\mathbb{R}^n \setminus \overline{\Omega}) = V \cap (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$. This implies $V_2 \subset \mathbb{R}^n \setminus \overline{\Omega}$. ■

3. Proof of the theorem

We show first that the function $\varphi(x) = d(x, \partial\Omega)^2$ is of class C^k in a neighborhood of $\partial\Omega$. Clearly, it is enough to show this in a neighborhood of every point $a \in \partial\Omega$. Let the hypersurface $\partial\Omega$ be defined at the point a by a C^k-function ρ in a ball $B_r(a)$ with center at a and radius $r > 0$: $\partial\Omega \cap B_r(a) = \{x \in B_r(a) : \ \rho(x) = 0\}$, $\operatorname{grad}\rho \neq 0$ in $B_r(a)$. Reordering the variables, replacing ρ with $-\rho$, and shrinking the ball $B_r(a)$, if necessary, we may suppose that

$$\frac{\partial \rho}{\partial x_1} > 0 \quad \text{in } B_r(a). \tag{2}$$

I. Fix a point $x^0 = (x_1^0, \ldots, x_n^0)$ in the ball $B_{r/2}(a)$. Then $d(x^0, \partial\Omega) \leq |x^0 - a| < \dfrac{r}{2}$. Since the set $\partial\Omega$ is closed, there is a point $y^0 \in \partial\Omega$ such that $|x^0 - y^0| = d(x^0, \partial\Omega)$. We have $|y^0 - a| \leq [y^0 - x^0| + |x^0 - a| < \dfrac{r}{2} + \dfrac{r}{2} = r$, thus $y^0 \in \partial\Omega \cap B_r(a)$. Now, note that

$$\begin{aligned} \inf\{|x^0 - y|^2 : \ y \in \partial\Omega \cap B_r(a)\} &\geq \inf\{|x^0 - y|^2 : \ y \in \partial\Omega\} = d(x^0, \partial\Omega)^2 \\ &= |x^0 - y^0|^2 \geq \inf\{|x^0 - y|^2 : \ y \in \partial\Omega \cap B_r(a)\}. \end{aligned}$$

Hence, $\inf\{|x^0 - y|^2 : \ y \in \partial\Omega \cap B_r(a)\} = |x^0 - y^0|^2$. This means that the function $f(y) = \sum\limits_{k=1}^{n} |y_k - x_k^0|^2$, $y = (y_1, \ldots, y_n) \in \partial\Omega \cap B_r(a)$, has minimum equal to $|x^0 - y^0|^2$ at the point y^0. By the Lagrange multiplier theorem for finding conditional extrema, there is a constant $\lambda \in \mathbb{R}$ such that $\dfrac{\partial f}{\partial y_\nu}(y^0) = \lambda \dfrac{\partial \rho}{\partial y_\nu}(y^0)$, $\nu = 1, \ldots, n$, or equivalently $y^0 - x^0 = \dfrac{\lambda}{2}(\operatorname{grad}\rho)(y^0)$. (Hence the vector $y^0 - x^0$ lies on the normal to $\partial\Omega$ passing through x^0). Thus (x^0, y^0) is a solution of the system $\rho(y) = 0, \quad y - x = \dfrac{\lambda}{2}(\operatorname{grad}\rho)(y)$, where $\lambda = 2(y_1^0 - x_1^0)(\dfrac{\partial \rho}{\partial y_1}(y^0))^{-1}$. This suggests the following considerations.

II. It is convenient to set

$$\psi_\nu(y) = \frac{\partial\rho}{\partial x_\nu}(y)\Big(\frac{\partial\rho}{\partial x_1}(y)\Big)^{-1}, \quad y \in B_r(a),\ \nu = 1,\dots,n.$$

Now, we consider in $B_r(a) \times B_r(a)$ the system

$$\begin{aligned} &\rho(y_1, y_2, \dots, y_n) = 0 \\ &y_1 - x_2 - (y_1 - x_1)\psi_2(y) = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &y_n - x_n - (y_1 - x_1)\psi_n(y) = 0. \end{aligned} \tag{3}$$

Clearly, all functions involved in this system are of class C^{k-1}. The point $(x = a, y = a)$ is a trivial solution of this system. Its Jacobian with respect to $y_1,\dots,y_n$ at this point is equal to

$$\frac{\partial\rho}{\partial y_1}(a) + \frac{\partial\rho}{\partial y_2}(a)\psi_2(a) + \dots + \frac{\partial\rho}{\partial y_n}(a)\psi_n(a) = \Big(\frac{\partial\rho}{\partial y_1}(a)\Big)^{-1}\sum_{\nu=1}^{n}\Big(\frac{\partial\rho}{\partial y_\nu}(a)\Big)^2 \neq 0,$$

By the implicit function theorem, there exist open neighborhoods X and $Y \subset B_r(a)$ of the point a, and (unique) C^{k-1}-smooth functions $y_1(x),\dots,y_n(x)$ defined on X such that

(i) $y(x) = (y_1(x),\dots,y_n(x)) \in Y$ for every $x \in X$;

(ii) A point $(x,y) \in X \times Y$ satisfies the system (3) if and only if $y = y(x)$.

Take a ball $B_R(a) \subset X \cap Y$. As in part **I** of the proof, for $x \in B_{R/2}$ there is a point $y \in B_R(a) \cap \partial\Omega$ such that $|x-y| = d(x,\partial\Omega)$. Moreover, by part **I**, the point (x,y) satisfies the system (3). Therefore $y = y(x)$. In particular, if $x \in B_{R/2} \cap \partial\Omega$, then $y(x) = x$.

Remark 2. This gives a proof of the result cited above that if $\partial\Omega$ is at least of class C^2, then it is of positive reach.

Let us note that, as it is easy to check,

$$(X \cap Y) \cap \partial\Omega = \{x \in X \cap Y :\ x = y(x)\} = \{x \in X \cap Y :\ x_1 = y_1(x)\}.$$

Moreover, $\dfrac{\partial(y_1(x) - x_1)}{\partial x_1}(a) = \dfrac{\partial y_1}{\partial x_1}(a) - 1 \neq 0$. Otherwise, it would follow from (3) that $\dfrac{\partial y_\nu}{\partial x_1}(a) = 0$ for $\nu = 2,3,\dots,n$. Then, differentiating the identity $\rho(y_1, y_2, .., y_n) = 0$ of (3), we would have $\dfrac{\partial\rho}{\partial x_1}(a)\dfrac{\partial y_1}{\partial x_1}(a) = 0$, hence $\dfrac{\partial\rho}{\partial x_1}(a) = 0$, a contradiction. Thus, $y_1(x) - x_1$ is a C^{k-1}-smooth defining function of $\partial\Omega$ at the point a.

III. Now, we are ready to prove that the function $\varphi(x) = d(x,\partial\Omega)^2$ is of class C^k. Clearly, it is enough to show that its first partial derivatives are C^{k-1}-smooth in the ball $B_{R/2}(a)$. On this ball,

$$\varphi(x) = |x - y(x)|^2 = \sum_{\nu=1}^{n}(y_\nu(x) - x_\nu)^2,$$

thus

$$\frac{\partial\varphi}{\partial x_\mu}(x) = 2\sum_{\nu=1}^{n}(y_\nu(x)-x_\nu)\frac{\partial y_\nu}{\partial x_\mu}(x) - 2(y_\mu(x)-x_\mu), \quad \mu = 1,\dots,n.$$

The first summand at the right-hand side vanishes, hence the function

$$\frac{\partial\varphi}{\partial x_\mu}(x) = -2(y_\mu(x)-x_\mu) \tag{4}$$

is C^{k-1}-smooth. Indeed, according to (3),

$$\sum_{\nu=1}^{n}(y_\nu(x)-x_\nu)\frac{\partial y_\nu}{\partial x_\mu}(x) = (y_1(x)-x_1)\Big(\frac{\partial\rho}{\partial y_1}(y(x))\Big)^{-1}\sum_{\nu=1}^{n}\frac{\partial\rho}{\partial y_\nu}(y(x))\frac{\partial y_\nu}{\partial x_\mu}(x) = 0$$

since $\displaystyle\sum_{\nu=1}^{n}\frac{\partial\rho}{\partial y_\nu}(y(x))\frac{\partial y_\nu}{\partial x_\mu}(x) = 0$ which follows from the identity $\rho(y_1(x),\dots,y_n(x)) = 0$.

IV. In this part of the proof, we show that the signed distance function δ admits first order partial derivatives, which are C^{k-1}-smooth functions. Set $B = B_{R/2}(a)$ for short. The function $\varphi(x) = d(x,\partial\Omega)^2$ is C^k-smooth on B and does not vanish on $B\setminus\partial\Omega$. Hence the function $d(x,\partial\Omega) = \sqrt{\varphi(x)}$ is also C^k-smooth on $B\setminus\partial\Omega$. By (3), for every $x\in B$

$$\begin{aligned}\sqrt{\varphi(x)} &= \sqrt{\sum_{\nu=1}^{n}(y_\nu(x)-x_\nu)^2} = |y_1(x)-x_1|\sqrt{\sum_{\nu=1}^{n}\psi_\nu(x)^2}\\ &= |y_1(x)-x_1|\Big|\frac{\partial\rho}{\partial x_1}(x)\Big|^{-1}\sqrt{\sum_{\nu=1}^{n}\Big(\frac{\partial\rho}{\partial x_\nu}(x)\Big)^2}\\ &= |y_1(x)-x_1|\Big(\frac{\partial\rho}{\partial x_1}(x)\Big)^{-1}|(\operatorname{grad}\rho)(x)|\end{aligned}$$

in view of (2). Identities (3) and (4) give

$$\frac{\partial\varphi}{\partial x_\mu}(x) = -2(y_\mu(x)-x_\mu) = -2(y_1(x)-x_1)\frac{\partial\rho}{\partial x_\mu}(y(x))\Big(\frac{\partial\rho}{\partial x_1}(y(x))\Big)^{-1},\ x\in B_{R/2}(a).$$

Hence, for $x\in B\setminus\partial\Omega$,

$$\begin{aligned}\frac{\partial\delta}{\partial x_\mu}(x) &= \pm\frac{1}{2\sqrt{\varphi(x)}}\frac{\partial\varphi}{\partial x_\mu}(x)\\ &= \pm\frac{x_1-y_1(x)}{|x_1-y_1(x)|}\frac{1}{|(\operatorname{grad}\rho)(x)|}\frac{\partial\rho}{\partial x_1}(x)\frac{\partial\rho}{\partial x_\mu}(y(x))\Big(\frac{\partial\rho}{\partial x_1}(y(x))\Big)^{-1},\end{aligned}$$

where the sign is plus if $x\in\Omega$ and minus if $x\in\mathbb{R}^n\setminus\overline{\Omega}$. By Proposition 3, there is a neighborhood $V\subset B$ of the point a such that the map $F(x) = (\rho(x),x_2,\dots,x_n)$

is a C^k-diffeomeorphism of V onto a ball D in $\mathbb{R}^n$ and V possesses the properties (i)–(iv). In the notation of this propostion, to be specific, let $V_1 = \{x \in V :\ \rho(x) > 0\} \subset \Omega$ and $V_2 = \{x \in V :\ \rho(x) < 0\} \subset \mathbb{R}^n \setminus \overline{\Omega}$. The functions $\rho(x)$ and $y_1(x) - x_1$ are two C^{k-1}-defining functions of $\partial\Omega$ at the point a. Hence, by Proposition 2, there is a nowhere vanishing continous function h in a neighborhood of a such that $\rho(x) = h(x)(y_1(x) - x_1)$. Shrinking V, we may assume that this neighbourhood is V, thus h has a constant sign in V, say $h(x) > 0$. Then for $x \in V_1 \cup V_2$

$$\frac{\partial\delta}{\partial x_\mu}(x) = \frac{1}{|(\operatorname{grad}\rho)(x)|}\frac{\partial\rho}{\partial x_1}(x)\frac{\partial\rho}{\partial x_\mu}(y(x))\Big(\frac{\partial\rho}{\partial x_1}(y(x))\Big)^{-1}. \tag{5}$$

The function $\tau = \delta \circ F^{-1}$ is C^k-smooth on the set $D \setminus \{\xi \in \mathbb{R}^n :\ \ \xi_1 = 0\}$. Differentiating the identity $\tau(\rho(x), x_2, \ldots, x_n) = \delta(x)$, it follows from (5) that if $\xi \in D$, $\xi_1 \neq 0$, and $x = F^{-1}(\xi)$,

$$\frac{1}{|(\operatorname{grad}\rho)(x)|}\frac{\partial\rho}{\partial x_1}(x) = \frac{\partial\tau}{\partial\xi_1}(\xi)\frac{\partial\rho}{\partial x_1}(x).$$

Hence,

$$\frac{\partial\tau}{\partial\xi_1}(\xi) = \frac{1}{|(\operatorname{grad}\rho)(F^{-1}(\xi))|}. \tag{6}$$

Therefore, for $\eta = (0, \eta_2, \ldots, \eta_n) \in D$,

$$\lim_{\xi\to\eta,\xi_1>0}\frac{\partial\tau}{\partial\xi_1}(\xi) = \lim_{\xi\to\eta,\xi_1<0}\frac{\partial\tau}{\partial\xi_1}(\xi) = \frac{1}{|(\operatorname{grad}\rho)(F^{-1}(\eta))|}.$$

Then, by L'Hôpital's rule,

$$\lim_{\xi\to\eta,\xi_1\neq 0}\frac{\tau(\xi) - \tau(\eta)}{\xi_1} = \lim_{\xi\to\eta,\xi_1\neq 0}\frac{\partial\tau}{\partial\xi_1}(\xi) = \frac{1}{|(\operatorname{grad}\rho)(F^{-1}(\eta))|}.$$

Thus,

$$\frac{\partial\tau}{\partial\xi_1}(\eta) = \frac{1}{|(\operatorname{grad}\rho)(F^{-1}(\eta))|}.$$

Therefore, identity (6) holds on the whole set D. In particular, the function $\dfrac{\partial\tau}{\partial\xi_1}$ is C^{k-1}-smooth. Hence, the function $\dfrac{\partial\delta}{\partial\xi_1}$ is also C^{k-1}-smooth.

Clearly, $\tau(0, \xi_2, \ldots, \xi_n) = 0$, hence if $\eta = (0, \eta_2, \ldots, \eta_n) \in D$,

$$\frac{\partial\tau}{\partial\xi_\mu}(\eta) = 0, \quad \mu = 2, 3, \ldots, n.$$

Then, if $x \in V \cap \partial\Omega$ and $\eta = F(x)$,

$$\frac{\partial\delta}{\partial x_\mu}(x) = \frac{\partial\tau}{\partial\xi_1}(\eta)\frac{\partial\rho}{\partial x_\mu}(x) + \frac{\partial\tau}{\partial\xi_\mu}(\eta) = \frac{1}{|(\operatorname{grad}\rho)(x)|}\frac{\partial\rho}{\partial x_\mu}(x), \quad \mu = 2, 3, \ldots, n.$$

We have $y(x) = x$ since $x \in \partial\Omega$, thus the right-hand side of the latter identity coincides with right-hand side of the identity (5). In this way, identity (5) holds on the whole set V. In particular, the derivatives $\dfrac{\partial \delta}{\partial x_\mu}$, $\mu = 2, 3, \ldots, n$, are C^{k-1}-smooth.

This proves that δ is a C^k-smooth function.

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