THE TEACHING OF MATHEMATICS
2023, Vol. XXVI, 1, pp. 29–45
DOI: 10.57016/TM-EQGL7427

P PROBLEM AS A MEANS TO ENCOURAGE STUDENTS' CONCEPTUALIZATION OF FRACTIONS

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Abstract. Students connect fractions with pie-charts as they are taught in 3rd grade in Lithuania, but they do not have a deeper understanding of the concept. Word problems could be a tool to find misunderstandings and conceptions of fractions students have. In this study, we use word problems where fractions are shown as a numerical magnitude of a level of liquid in a glass (as an analogy to a number line) and check them with students. Each additional drop of liquid gives a new fraction. This way we raise the discussion in class and find out in which phase of the conception of fractions the students are and what misunderstandings and correct ideas they have about fractions. We did design-based research, conducted the teaching experiment, and analyzed qualitative data. We suggest a lesson plan including word problems that teachers can use to raise a discussion.

MathEduc Subject Classification: F41

AMS Subject Classification: 97F40

Key words and phrases: Measurement; fractions; word problems; conception of fraction; synthetic model.

1. Introduction

1.1. Research problem

Fractions are one of the most challenging topics in primary grades [2, 5, 14]. This is a rubicon—an essential topic, without it students cannot master other issues. Higher-level students cannot succeed in algebra if they do not understand rational numbers [12]. For elementary school students, Simon suggests teaching fractions as a measure [18]. In Lithuania, students in 3rd grade are taught fractions as parts of the whole. We aim to try this presenting method for 3rd-grade students.

We believe that young students have an understanding of what a number is and it is related to quantity or measure, but fractions presented in a part-whole method do not seem the same. This is because students are mostly taught only by a part-whole method [6, 8, 11, 14]. Fractions are introduced in pie-charts in Lithuania textbooks. Pie-charts are a method for dividing a circle into discrete parts. Teaching focused on pie-charts and part-whole methodology reinforces the dominance of whole-number reasoning [12]. G. Vergnaud [22] widely explains how the concept's meaning comes to students. One of his ideas is: "concept's meaning does not come from one situation only but from a variety of situations". We suggest that to understand the concept of fractions, students should meet them in different situations: as part-whole, measurement, ratio, and quotient.

V. V. Davydov [3] represents the idea that "number should be developed as a general concept, such that each new type of number (e.g., rational, irrational) does not require a change in the basic concept" [18]. If a number is presented as a result of measuring it is the basis for natural numbers and also fractions and decimals. Siegler et al. also claim that the general idea for understanding fractions is understanding magnitudes [17]; we will discuss it in the literature review. It could be the suggestion on how to eliminate misunderstandings we are going to talk about.

We aim to use new context—fractions as measurements for students who were taught fractions as part-whole. Word problem is a tool to present new concepts not directly but in a clearer context for students. We expect word problems to initiate a discussion about fractions, to show students' conception level about fractions, and to raise new questions that would help to find out what misunderstandings 3rd-grade students have.

1.2. Research focus

It is not difficult to notice that students really have misunderstandings about fractions. M. A. Simon et. al. [18] represent three difficulties about fractions that students have: (1) Lack of fractions as quantity, (2) Understanding a fraction as two numbers, (3) Limited part-whole concept. We also expected students to have other misunderstandings, as (4^*) Determine fraction magnitude by natural number rules, for example, the fraction with larger numbers is greater [4], (5^*) Understand fractions with different denominators as separate sets of objects, (6^*) Do not relate fractions magnitudes with natural numbers magnitudes. We can expect students to have synthetic models in their ideas.

Definition 1. Students trying to assimilate new information into their existing framework, is called a synthetic model [7].

We suggest trying a new method for presenting fractions and exploring what synthetic models 3rd-grade students have. We decided to use word problems as a tool to provoke a discourse that could disclose synthetic models students in concrete 3rd class has. X. Vamvakoussi et. al. [19–21] suggest teaching fractions by measurements for elementary school students and exemplified this through conceptual change theory. We have the study with primary school (3rd grade) students, this is the time when fractions are introduced to students in Lithuania.

To do that we use word problems about liquid. We expect students to discuss unusual situations where fractions are used. Those situations can be created by specific word problems—P problems. Word problems can be divided into two types: S problems (Standard problems) and P problems (problematic problems). This classification was used by Vershaffel et al. [23]. They describe these problems as standard and problematic. We specified the following definitions definitions [9]:

Definition 2. We say that the WP is an S problem if it has the optimal information needed to answer the question.

Definition 3. We say that the word problem is a P problem if it has either more than needed sufficient information or has less than needed information to answer the question.

P problems by themselves have reasoning requests to find needed information. We will use P problems to stimulate students' investigation of questions about fractions. Word problems play different goals in mathematics education. One of them is to assist in the development of new mathematical concepts and skills [23]. We will create a P problem and will test this role of word problems. Also, we will use this P problem to test students' conceptions of fractions.

1.3. Research aim and research questions

Our aim is to use P problems to raise discussion that would help to see how students understand fractions. We also present a lesson plan we improved during the study. These results can be used by teachers and other researchers for further inquiry. Our research questions are:

Q1: Could the P problem be a tool to examine students' degree of conceptualization of fractions?

Q2: What misunderstandings do students have about fractions?

Below we examine the theoretical background and describe the experiment with 3rd-grade students.

2. Literature review

2.1. Research on fractions for 3rd grade

Teaching fractions in primary school has not gained much attention in research. We looked through research about fractions in primary school to see what methods students are taught about fractions and what methods researchers suggest. There are authors that suggest presenting fractions for primary school students using the Realistic mathematics education ideas of guided reinvention and creating lesson plans for teaching and learning fractions in primary grades. In Keijzer's Ph.D. thesis [8] "fractions are presented as folded bars and numbers on the number line, fractions are presented as (single) numbers between integers". Sari [14] constructs learning trajectory of 6 lessons for 3rd-grade students with the following steps: 1. Constructing the meaning of fair sharing; 2. Producing simple fractions as a result of fair sharing; 3. Using fractions as a unit of measurement; 4. Building the relation among fractions. The author uses the part-whole method and fraction line to reach these four steps.

There is an important and interesting paper by Moss et al. [13] where authors use percentages to teach fractions, they had an experiment with 4 graders and in exercises they used beakers of water and teach percentages on them. Then they go from percentages to decimals, and after that to fractions. Students after that experiment showed a deeper understanding of rational numbers than the control group students.

Also, there are other research papers that show ways how young students learn about fractions: part-whole, operator, ratio, number line [10, 11, 16].

Gunderson et al. [6] experiment shows that teaching fractions on a number line lead to better results than with area figures. The authors say that "results from this experimental training study provide causal evidence of a connection between fractions on number line estimation and fraction magnitude concepts."

We can see that there is a gap in research about fractions as measurements in primary school.

2.2. From object to process

A. Sfard [15] explains that a concept "can be conceived in two fundamentally different ways: structurally—as objects, and operationally—as processes". Based on this theoretical framework we can explain that fractions can be understood as processes, e.g.: dividing whole and just later becoming objects. When students begin to learn fractions they should go through three phases: interiorization, condensation, and reification, to understand fractions as objects [15]. A. Sfard explained all three phases:

1. Iteriorization: "a learner gets acquainted with the processes which will eventually give rise to a new concept";
2. Condensation "is a period of 'squeezing' lengthy sequences of operations into more manageable units";
3. Reification "is defined as an ontological shift—a sudden ability to see something familiar in a totally new light".

We describe all three phases of the conceptualization of fractions in the research methodology part.

2.3. Conceptual change theory

Difficulties in learning fractions can be explained by the conceptual change theory. X. Vamvakoussi et. al. studied this topic from such a perspective [14, 24]. Empirical findings show that children from 4 to 5 years already have an understanding that a number is a counting number and represents quantity [24]. And this understanding is culturally supported. They can already count on and conduct some solutions. "It appears that children form an initial concept of number which allows them to deal with number-related tasks long before they are exposed to formal instruction in mathematics" [24]. It is the initial concept of numbers and it is very similar to the natural number concept. During the first years at school, students use counting numbers a lot and it makes the initial concept even stronger. After that, when they become familiar with the fraction concept, if we expect them to use fractions as measures or quantities they should restructure the initial concept of the number [24]. One can see the main changes in the initial concept of numbers and fractions in Table 1.

Table 1. Differences between initial concept of number and fraction

The initial concept of number	Fraction
1, 2, 3, ...	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$
Counting objects	Counting parts of objects
Multiplication makes bigger, division smaller	Multiplication and division make both bigger or smaller
Longer number is bigger	Longer number can be bigger or smaller
Each number has a unique numeral	One number can be written in different symbols.

Source: [20] and our additions

In the first years of school, students have experience only with natural numbers, which affects the interpretation of what the word "number" means. Students know that the "number" is a natural number. When you add or multiply natural numbers you get a bigger number. When you subtract or divide you get a smaller natural number. Different natural numbers have different numerals and each natural number has just one numeral. Longer natural numbers are bigger. X. Vamvakoussi et al. [20] discuss widely how "the broadening of numbers to include non-natural numbers essentially changes what counts as a number and how it behaves". The initial number concept and the rational number concept are so different that the students create synthetic models. Synthetic models can be a starting position to create a correct perception of a concept, this is the reason why they are important to find, understand, and be analyzed.

2.3.1. Using number line. Different authors [18, 20] offer to use the experiences students have. They suggest that students could gain new information about rational numbers without denying what was known about natural numbers. The number line could be the base where natural and non-natural numbers are included. H. Wu [25] uses the concept of a number as a point on the number line. Then fraction represents a number and it is related to natural numbers. Wu explains: "once you accept this definition, you can use logical reasoning to explain all other meanings of this concept". Also with this definition, you can use mathematical operations in the same way for fractions and whole numbers. Here he uses the following definition of fraction:

Definition 4. The fractions are the points on the number line defined in the following manner: Fixing a whole number $n > 0$, we divide the unit segment into n parts of equal length. Then the first division point to the right of 0 will be denoted by $\dfrac{1}{n}$. The multiples of $\dfrac{1}{n}$ then form an equispaced sequence associated with n. The totality of all the points in these sequences as n runs through $1, 2, 3, \ldots$ is by definition the collection of all the fractions.

If the fraction is the point on the number line, natural numbers can be also found on the same number line. And we can compare fractions and natural numbers: "$a < b$ means exactly that the point a on the number line is to the left of b" [25].

We did not find in literature where word problems are considered as a tool to find misunderstandings. Our idea is to connect all presented theoretical ideas and to use the P problem as a tool to check what conception of fractions students have.

3. Research methodology

3.1. Theoretical framework

Interpreting A. Sfard [15] theoretical framework we will separate fraction understanding as objects, and as processes. We assume that 3rd grade students have an understanding of natural numbers as objects. Students have learned fractions at school so they begin to have some understanding of fractions. We hypothetically guess that they do not have an understanding of fractions as objects, but they have an understanding of fractions as processes. Using this theoretical framework we expect to find misunderstandings students have about fractions and to indicate that even after lessons about fractions students still do not have an understanding of fractions as objects. By A. Sfard [15] we can identify three phases of the conception of fractions 3-grade students could reach.

1. Interiorization—when the whole (usually a circle) is divided into parts and some parts are colored, students write down what fraction it is depicted. Or vice versa, the whole is divided into parts, and students' color parts depend on a given fraction.
2. Condensation—when some figure and fraction are given to students and they divide the figure by themselves into a necessary number of parts and color it.
3. Reification—mark fractions between other numbers (and fractions) and make some operations with fractions.

When a student reaches the reification phase we can say he/she understands fractions as objects. We submit the P problem (Exercise I) to students to check if they already have reached the reification phase. After that using exercises (we expected they have not reached reification phase) we check if they reached the condensation (Exercise II) and interiorization (Exercise III) phases. We are also going to find some synthetic models in this process.

3.2. General background

We wanted to create activities that would help to expose synthetic models of fractions students have and to show their degree of conceptualization of fractions. We choose the Design-Based Research (DBR) genre for this. The DBR method allows the lesson environment to change continuously [1]. In this method, educational ideas are formulated in the design but can be disorganized during empirical testing. The design research "consists of cycles of three phases each: preparation and design, teaching experiment, and retrospective analysis" [1]. Below we present two cycles of our teaching experiment in all phases.

3.3. Preparation and design

Preparation consisted of a literature review, a 3rd-grade textbooks survey, and also searching how to use the measurement for presenting fractions. We decided to present fractions not as a part-whole, but by measuring something continuous, like water, time, age, and height. We chose a continuous object to show how fractions are related to natural numbers and for future topics to be able to show that there are many fractions. We believe it will raise a discussion and will help to find out misunderstandings students have. We decided to present fractions as liquid quantities in the glass. The reasons why we choose liquid in glass are:

- It is similar to the number line when we have a ruler on the glass;
- you can choose the length of the unit;
- one more drop of liquid gives the new number;
- you can imagine the glass as high as you want;
- it is easy to understand for younger students, also they can pour it in and try to measure themselves.

We predicted that it would connect natural numbers and fractions and raise new questions. We use P problems to raise a discussion to help them to find some ideas about fractions by themselves.

3.4. Teaching experiment

We had 13 remote lessons with different classes and changed the lesson plan after some of them. Lessons, where students interacted, were recorded and we used open questions and interviews. The researcher was the teacher of the lessons, and the usual class teachers observed the lessons. Lessons were conducted online, with Zoom/Teams (everything occurred during the COVID-19 lockdown period). A total of 298 third graders (8–9 years old) participated in this study. They came from 14 classes located in different parts of Lithuania. All students studied from the same textbook, and all of them knew the following concepts: half, third, part, multiplication, and division of natural numbers, and already studied fractions in the part-whole method.

3.5. Retrospective analysis

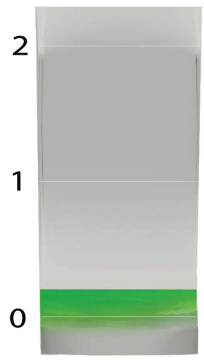
The collected data include video recordings from the lessons, questionnaire answers, interview recordings, and students' graphical representations of fractions. After each lesson, we reflected and decided whether we should change something in the lesson or not.

4. Experiment

4.1. First cycle of experiment

4.1.1. Phases 1 and 2. Design and experiment. The main word problem (Exercise I) in the lesson is the P problem. It has a text story, and question, but not all the needed information. Information is given visually and students have to process the given information to obtain numerical values from it. Defined as a P problem [2], it has less than needed information to answer the question. The word problem asks students to write how much liquid is in a graded glass when the quantity of liquid is below the first marking. The aim of this P problem is to see whether students reached the reification phase, what fractions students guess, and from guesses to see how far off are the guesses from the fraction magnitude.

Exercise I. You created an invisibility potion. You have a glass with a ruler on it. One unit shows how much potion one person needs to become invisible. You made this much potion on the first day.

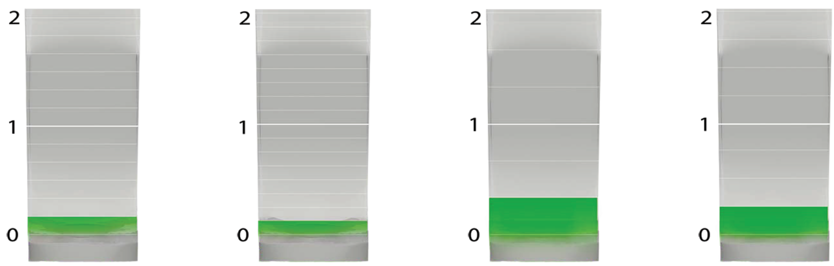


Write down how much you made on the first day.

All students already knew concepts: half, one-third, quarter, and fraction. They were guessing and together we named the fraction.

We used Exercise II and Exercise III to see if they are at the interiorization and condensation phases.

Exercise II. Say how much of the liquid portion is in the following glasses



The last exercise was the opposite: draw glasses when fractions are given.

Exercise III. Draw glasses and mark $\frac{1}{4}$, $\frac{1}{12}$, $\frac{7}{8}$, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{2}{2}$ in them.

After the lesson, some students had an interview.

4.1.2. Retrospective. After the first lesson, we decided to make some improvements. We decided to use two fractions with a small difference in Exercise I. It allows us to show that a small difference in liquid levels gives a new fraction. We changed fraction of Exercise I to two fractions: $\frac{1}{3}$ and $\frac{3}{7}$. The fraction $\frac{3}{7}$ was chosen not by chance, it is not a usual fraction and we wanted the students to speculate so that we could notice how they were able to relate fractions and liquid levels in the glass. In Exercise III we decided to ask them to mark fractions in glasses they drew before because otherwise, they would draw glass and mark full glass as 1.

4.2. Second cycle of experiment

After the first cycle, we found what we wanted to do differently. For the next classes we wanted to stress that when the amount of potion in the glass changes just a little bit, it is a new, different fraction. Also, we expected to come up with a discussion that in the same glass we can mark fractions with different denominators. In Exercise I we changed the amount of the potion to $\frac{1}{3}$ and $\frac{3}{7}$. $\frac{3}{7}$ is bigger than $\frac{1}{3}$, but less than $\frac{1}{2}$, we expected it would be hard for students to answer. We expected Exercise I to show students' understanding of fraction magnitudes. During the interview, students were asked to mark fractions in a graded glass.

5. Results

This section presents results obtained by the analysis of lessons. We will review the tasks completed by the students and the interviews and we will summarize the results.

5.1. Experiment results

Exercise I: Reification phase. In Exercise I students were asked to answer how many potions are in glass (it was $\frac{1}{3}$ and $\frac{3}{7}$). It is natural that students answered $\frac{1}{3}$ correctly faster. Further, the distribution of answers is narrow. They guessed fractions as: $\frac{1}{2}$, 0.3, $\frac{1}{4}$, $\frac{3}{2}$, $\frac{1}{5}$. With fraction $\frac{3}{7}$, students guessed even numbers bigger than 1, they did not interpret that fractions $\frac{2}{2}$, $\frac{2}{1}$, $\frac{3}{2}$ are greater than 1. This shows that they know fractions but do not relate them with magnitudes, especially when they are not usual fractions. After every guess, the class and the teacher together noted where that fraction should be marked. Students came to some correct ideas in that process:

- The bigger the denominator, the less the fraction.

- Fraction $\frac{2}{2}$ is equal to $\frac{1}{1}$ and it is equal to 1.
- Fraction $\frac{1}{0}$ cannot be marked on a glass.

These P problems raise questions that help students to figure out some characteristics of fractions by themselves.

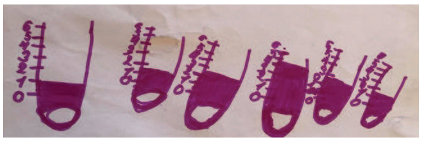
Exercise II: Interiorization phase. Exercise II was easy for almost all students, they answered fast and correctly how much of the liquid portion is in the four glasses. From images of glasses, they wrote down fractions that were depicted. It means that most of the students have reached the interiorization phase. Students have already heard about fractions as pie-charts, this is the reason we expected they can reach the interiorization phase.

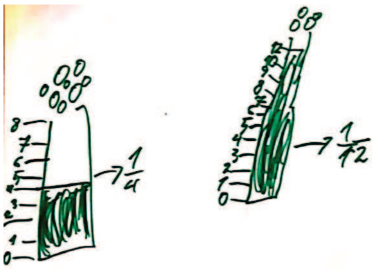
Exercise III: Condensation phase. In Exercise III students were asked to mark different fractions in glasses. We can see from the students' paintings that few students reached the condensation phase, but most of the students were at the interiorization phase of the conception of fractions.

We noticed that almost all students could easily draw a cylinder glass with sections and assign natural numbers to those sections. From their paintings, we can see some synthetic models. We found four different types of paintings.

1. Some students just left empty glasses or painted all glasses completely. We hypothesize that this means that they do not know what fractions are or do not understand the task.
2. There were quite unusual pictures (Figure 1), that showed that not all students could mark all fractions (only common or simpler fractions). We can see that Student 1 marked all fractions greater or equal to 1. Student 2 marked $\frac{1}{4}$ as 4, and $\frac{1}{12}$ as 12. Student 3 marked $\frac{1}{12}$ as 1 (but graded to 12 or 13) small sections: marked the numerator and divided that space to the denominator number of sections. It would be correct if he would color one section. These examples help us to mark what we have to do when teaching fractions:
   - Clarify what the numerator and denominator "does".
   - Clarify fractions' relationship with the number 1 and other natural numbers.
   - Talk about various fractions, not just as simple as $\frac{1}{3}$ and $\frac{2}{5}$, but also such as $\frac{7}{12}$ and $\frac{1}{79}$.

Student 1 Student 2





Student 3

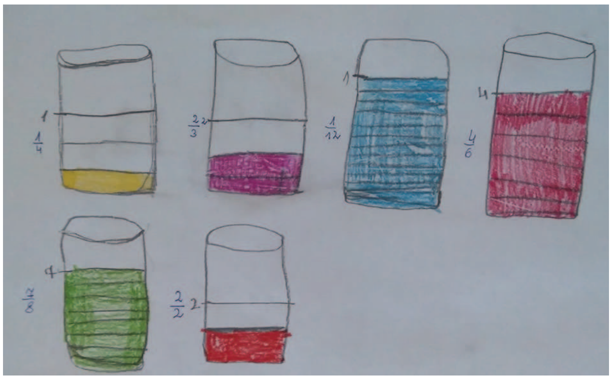


Fig. 1. Unusual graphical representations of fractions on glass

3. In Figures 2a and 2b we can see that some students connect fractions with part of a whole but not with natural numbers. Students mark $\frac{1}{4}$ of all glass, not $\frac{1}{4}$ as a number (some colored parts are even raised, and left as empty space).

Fig. 2a. Graphical representations where fractions are as part of whole (I)

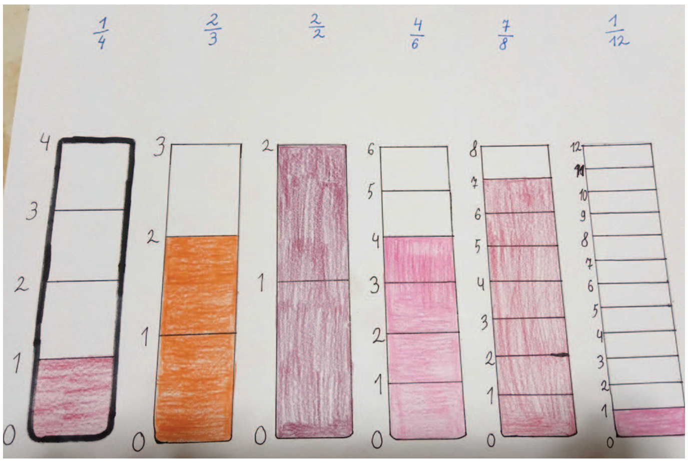


Fig. 2b. Graphical representations where fractions are as part of whole (II)

4. We also had some paintings with a fair understanding of fractions as shown in Figures 3 and 4.

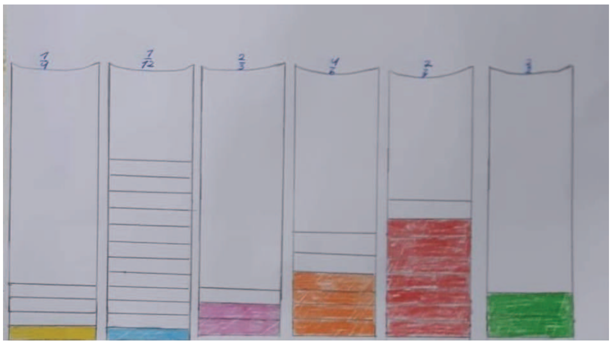


Fig. 3. Graphical representation with correct marking, but without number 1

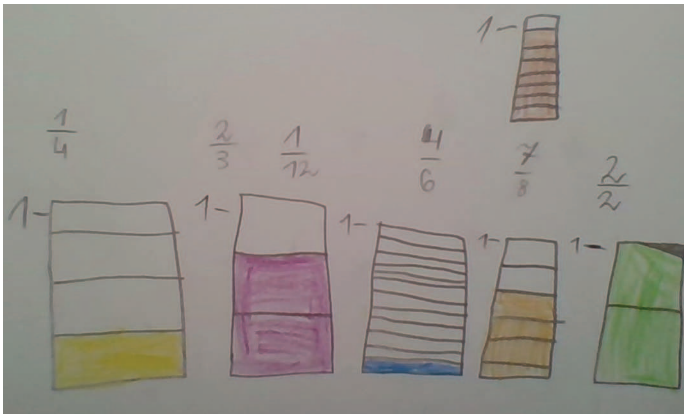


Fig. 4. Graphical representation with correct marking

5.2. Interview: synthetic models finding

During the interviews, we tried to understand how students perceive fractions as a concept, and what this concept means for students. A third-grade student from our study was asked "Is $\dfrac{1}{3}$ a number?" and he said: "A number is a number, not a fraction." This shows that in some students' minds, there is no relation between fractions and natural numbers. Through the episodes from the interviews, we will show the synthetic models we recognized. Students' names are changed.

1. Students do not connect fractions with different denominators, we can see it in Table 2.

Table 2. Interview extract (1)

Interviewer	Toma
How many numbers we can mark in the glass between 0 and 1?	We can mark $\frac{1}{1}$.
Could we mark $\frac{1}{2}$?	Yes, in another glass if we would draw more marks.

2. Some students have a strong understanding of fractions as decimal numbers, we can see such an example in Table 3. Kajus knows decimals and acts with fractions the same way as with decimals.

Table 3. Interview extract (2)

Interviewer	Kajus
Came up with a fraction.	2 point 3
Could you mark it in a number line?	⟨Marks it as 2.3 quite exactly.⟩
Could you mark $\frac{1}{5}$?	⟨Marks it as 1.5⟩
⟨Together with the teacher found out that it is one point five.⟩	
⟨Then asked where should be $\frac{1}{5}$ he marks it as 5.1⟩	

3. Associating fractions with pie-charts properties, we can see in Table 4. Saulė uses geometry knowledge to think about fractions.

Table 4. Interview extract (3)

Interviewer	Saulė
How many different fractions you can think of?	I was thinking maybe 40, but you cannot divide a circle into 40 parts, but it should be possible to divide it into 35 parts, so I think 35.

4. Marking the fractions in number line at denominator magnitude: $\frac{1}{2}$ as 2, $\frac{1}{3}$ as 3. Or even $\frac{2}{5}$ as two points: 2 and 5, we can see an example of it in Table 5.

Table 5. Interview extract (4)

Interviewer	Ben
Came up with a fraction.	$\frac{1}{5}$
Could you mark it on the number line?	⟨Marks at number 1.5.⟩
Came up with another fraction.	$\frac{1}{2}$
Could you mark it on the number line?	⟨Marks at number 2.1.⟩
Could we find more fractions between these two?	Yes, but not many.

We came up with some synthetic models that bother to understand fractions correctly. Teaching content should be designed to avoid such synthetic models.

6. Discussion

We introduced a P problem (word problem with less than needed information) to 13 classes of students. We checked at which phase of the conception of fractions students are. The students' answers provide information for formative assessment of reaching the phases of the conception of fractions. Phases of conception were checked for the whole class, not for students individually, it was done to raise discussion in class using the P problem.

P problem about the liquid in glass raised the following synthetic models:

1. Students understand fractions with different denominators as separate sets of objects.
2. Some students have a strong interpretation of fractions as decimal points, as $\frac{1}{5} = 1.5$.
3. Students associate fractions with pie-charts properties;
4. Students mark the fractions in number line at denominator magnitude: $\frac{1}{2}$ as 2, $\frac{1}{3}$ as 3, or $\frac{2}{5}$ as two points: 2 and 5.
5. Students do not notice that fractions $\frac{2}{2}$, $\frac{2}{1}$, $\frac{3}{2}$ are not less than 1.

Students came to the following ideas by themselves:

1. The bigger the denominator, the less the fraction.
2. Fraction $\frac{2}{2}$ is equal to $\frac{1}{1}$ and it is equal to 1.
3. Fraction $\frac{1}{0}$ cannot be marked on a glass.

Experiment results showed that the P problem is a tool to see students' conception of fractions, further research is needed to see if it is a better tool than a standard word problem. We used the P problem for a whole class discussion, it also could be tried using for the individual students' conception of fractions. To fill a gap in current research, we conducted a study with younger students and presented fractions as measurements for 3rd-grade students.

Based on A. Sfard theory [15] we found that 3rd-grade students have not reached the reification phase and some of them are at condensation, some at the interiorization phase, they still have to go from fraction as an object to a fraction as a process. Further research is needed to study how these phases can be reached.

Based on the conceptual change theory [20, 21] and the findings of this research, a deeper understanding of fractions is necessary to effectively integrate this mathematical concept among students.

One potential approach to achieve this goal is to employ the use of a number line [25], which was utilized in this study by connecting it with water to assess students' comprehension of fractions. A more extended duration of instruction is required to determine the feasibility and efficiency of this method for teaching fractions.

7. Conclusion

The study described an alternative didactic approach to teaching the concept of fractions in the context of measurements. This approach provides information for students' formative assessment.

We can see that this method raised discussion between students. We propose to give more attention to P problems as a tool to raise discussion in lessons and to look deeper into synthetic models students have.

We created and improved an alternative lesson plan with P problems for fractions that teachers and researchers can use for further inquiry. Quantitative verification of effects should be presented in the future.

Acknowledgements

1. The datasets generated during and/or analyzed during the current study are available from the corresponding author upon reasonable request.
2. Thanks to all students and teachers who participated in the study.
3. The author expresses gratitude to the referee whose suggestions helped in improving the paper.

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Received: 20.02.2023, in revised form 17.04.2023

Accepted: 18.04.2023