THE TEACHING OF MATHEMATICS
2023, Vol. XXVI, 1, pp. 54–55
DOI: 10.57016/TM-KBUI6835

YET ANOTHER ELEMENTARY PROOF OF BRAUER'S THEOREM

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The author would like to dedicate this note to the memory of professor Milosav Marjanović (1931–2023). May he rest in peace.

Abstract. We present another elementary proof of the theorem of Alfred Bauer from 1952 on eigenvalue perturbation.

MathEduc Subject Classification: H65

AMS Subject Classification: 97H60

Key words and phrases: Brauer's theorem; perturbation; eigenvalues and eigenvectors.

In [2] (see also [3]), the authors presented an elementary proof of the theorem of Alfred Bauer (see [1, Theorem 27]) on eigenvalue perturbation.

Here we propose a more direct and elementary proof of this result, which does not depend on the fact that we deal with complex numbers – it works for every field and we feel that it might also be interesting to the readership of this journal. Basically, everything follows from the fact that $(xy^t)w$ is a multiple of x for any w (x, y, w are column vectors from K^n).

Namely, we are going to prove the following theorem.

Theorem 1. *Let K be a field, $A \in \mathbb{M}_n(K)$, x an eigenvector corresponding to the eigenvalue $\lambda \in K$ and $y \in K^n$. Then*

$$\sigma(A + xy^t) = \{\lambda + y^t x\} \cup (\sigma(A) \setminus \{\lambda\}).$$

Proof. Of course, if $x = 0$ we have nothing to do. So, let $x \neq 0$ and $Ax = \lambda x$. Since $(A + xy^t)(x) = \lambda x + x(y^t x) = (\lambda + y^t x)x$, we see that $\lambda + y^t x \in \sigma(A + xy^t)$, and x is a corresponding eigenvector. Let us deal with other elements of spectra in question.

$\subseteq$: Suppose that $\mu \in \sigma(A + xy^t) \setminus \{\lambda + y^t x, \lambda\}$ and let w be a corresponding eigenvector:

$$(A + xy^t)(w) = \mu w. \tag{1}$$

We want to find α such that

$$A(w + \alpha x) = \mu(w + \alpha x). \tag{2}$$

This will show that $\mu \in \sigma(A) \setminus \{\lambda\}$. Equation (2) is equivalent to

$$Aw + \alpha\lambda x = \mu w + \mu\alpha x. \tag{3}$$

From (1) we have $Aw = \mu w - (y^t w)x$, and if we substitute this into (3) we get

$$\mu w - (y^t w)x + \alpha\lambda x = \mu w + \mu\alpha x. \tag{4}$$

Since x is not a zero vector, this is equivalent to $\alpha(\lambda - \mu) = y^t w$ and, since $\mu \neq \lambda$, we can find α as required. Note that if $y^t w = 0$, from (1) we would already get that $\mu \in \sigma(A)$. Also, $w + \alpha x \neq 0$, since w does not correspond to the eigenvalue $\lambda + y^t x$ which it would were it a multiple of x.

The other inclusion is dealt with in much the same way. ■

Acknowledgement. The author was partially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia through the contract 451-03-47/2023-01/200104.

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Received: 30.05.2023
Accepted: 10.06.2023